RAIRO-Oper. Res. 52 (2018) 713–724
https://doi.org/10.1051/ro/2017087

**RAIRO Operations Research**
www.rairo-ro.org

# A NOTE ON RELIABILITY ANALYSIS OF AN $N$-POLICY UNRELIABLE $M^X/\binom{G1}{G2}/1$ QUEUE WITH OPTIONAL REPEATED SERVICE

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**Abstract.** This article deals with analyse the system reliability function of an unreliable $M^X/\binom{G_1}{G_2}/1$ queue with optional repeated service under $N$ policy, where server turned on the system when batches of $N(\geq 1)$ units accumulated in the queue.

**Mathematics Subject Classification.** 60K25, 90B22

Received March 31, 2017. Accepted November 30, 2017.

## 1. Introduction

Madan *et al.* [1] investigated the queuing system with two types of general heterogeneous service for batch arrival queue by introducing the concept of re-service. In such a system, the server provides two types of general heterogeneous services and an arriving customer can choose either type of service before its service start. However, if a customer is not satisfied by the service provided by the server then it may repeat the same type of service once again. Among some other related works in this area, we should mention the papers by Madan *et al.* [2], Al-khedhairi and Tadj [3], Tadj and Ke [4] and Baruah *et al.* [5].

The reliability function for an unreliable server queuing system was first investigated by Li *et al.* [6]. Tang [7] investigated similar type of model for batch arrival queue. Consequently, Wang [8] studied such a model for two phase of service.

The batch arrival $N$ policy queue was first studied by Lee and Sirinivasan [9], where the server remains idle till the queue size becomes $N(\geq 1)$, *i.e.* batches of size $N$ accumulated in the system. Later, Lee *et al.* [10] had made an extensive analysis on this model through different techniques. Recently, Choudhury *et al.* [11] have investigated a similar type of model for unreliable server with two phase of service. More recently, Choudhury and Tadj [12] generalized this model for Bernoulli vacation schedule.

Although several aspects has been discussed on queuing model under $N$ policy, however no works has been done on reliability function for such type of model. Thus in this present paper we proposed to analyse reliability function to an unreliable $M^X/G/1$ queue provides two types of general heterogeneous service with optional repeated service under $N$ policy.

*Keywords and phrases:* Two types of service, repeated service, elapsed time, $N$ policy and reliability function.

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From the practical point of view one may encounter utility of such queuing situation in production manufacturing system, maintenance and quality control in industrial organization, inventory system, etc.

For example, consider a production system in which the production does not start until some specified number of raw materials, say N, are accumulated during an idle period. We assume that the production system produces two different types of product and the raw materials for either type of product arrive in batches. It may so happen that the production process could be interrupted due to some unpredictable events (breakdowns) in the system. Moreover, re-production of some produced item may be necessary if the item does not satisfy a desired quality level on a quick on the spot inspection on completion of a production. Hence, it is worthwhile to investigate such a system from the reliability point of view.

The remaining part of the paper is organized as follows. In Section 2 we briefly describe the mathematical model, Section 3 deals with prerequisite definition for the model, in Section 4 we derive LST of the system reliability function. Finally, numerical illustration of the system reliability measures MTFF is done in Section 5.

## 2. Mathematical model

We consider an $M^X/G/1$ queuing system with two types of general heterogeneous service and optional repeated service in which an unreliable server operates an $N$-policy, where the number of individual primary units arrive to the system according to compound Poisson process with arrival rate $\lambda$. The size of successive arriving batches are i.i.d. random variables $X_1, X_2, \dots,$ distributed with probability mass function $c_n = Prob\{X = n\}; n \geq 1$, PGF $c(z) = E[z^X]$, and finite factorial moments $c_{[k]} = E[X(X-1)\cdots(X-k+1)]$. The server is turned off each time until the queue size (including one being served, if any) becomes $N$ (threshold). As soon as the queue size exceeds $N(\geq 1)$, the server is turned on and begin to serve all the arriving units. The server provides two types of general heterogeneous service to each unit on first come first served (FCFS) basis, before its service starts, each unit has the option to select either type of service. *i.e.*, each unit can select either first type of service (FTS) denoted by $S_1$ with probability $p_1$ or second type of service (STS) denoted by $S_2$ with probability $p_2$, where $p_1 + p_2 = 1$. Thus the time required by a unit to complete the service is given by,

$$S = \begin{cases} S_1, & \text{with probability } p_1 \\ S_2, & \text{with probability } p_2 \end{cases}$$

The service time random variable $S_i (i = 1, 2)$ (respectively S) of $i$th type of service (respective total service time) are assumed to follow general law of probability with distribution function (d.f.) $S_i(x)$ (respectively S(x)), Laplace Stieltjes Transform (LST) $S_i^*(\theta) = E[e^{-\theta S_i}]$ (respectively $S^*(\theta) = E[e^{-\theta S}]$ and finite $k$th moments $s_i^{(k)}, i = 1, 2$ (respectively $s^k)(k \geq 1)$.

More specifically, the LST of the total service time after the choice of a service is given by

$$S^*(\theta) = p_1 S_1^*(\theta) + p_2 S_2^*(\theta). \tag{2.1}$$

As soon as either type of service completed by an unit, such an unit has further option to repeat the same type of service denoted by $B_i$ once again with probability $q_i$ or leave the system with probability $(1 - q_i)$, for $i = 1, 2$ . Thus the total service time required to a unit to complete the $i$th type of service which may be called modified service time $(i = 1, 2)$ is given by

$$S_i = \begin{cases} S_i + B_i, & \text{with probability } q_i \\ S_i, & \text{with probability} (1 - q_i) \end{cases}$$

Also it is assumed that repeated service time random variable follows general distribution law with probability distribution function $B_i(x)$, LST $B_i^*(\theta) = E[e^{-\theta B_i}]$ and finite $k$th moments $b_i^{(k)}$ for $i = 1, 2$.

Clearly, the LST $S_i^*(\theta)$ of $S_i$ for $i = 1, 2$ is

$$S_i^*(\theta) = (1 - q_i)S_i^*(\theta) + q_i S_i^*(\theta)B_i^*(\theta). \tag{2.2}$$

Now utilizing equation (2.2) in (2.1) for i = 1,2, we get the LST of the modified service time is given by

$$S^*(\theta) = \{(1 - q_1) + q_1 B_1^*(\theta)\}p_1 S_1^*(\theta) + \{(1 - q_2) + q_2 B_2^*(\theta)\}p_2 S_2^*(\theta). \tag{2.3}$$

This type of queuing model is known as two type of heterogeneous service queue with optional repeated service discipline with N- policy. In Kendal's notation our model is denoted by $M^X / \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} /1(UR)/N - Policy$ repeated service queue, where UR represents unreliable server. While the server is working with any type of service or repeated service, it may breakdown at any time for a short interval of time. The breakdown *i.e.*, server's life times are generated by exogenous Poisson process with rates $\alpha_1$ for FTS or FTRS (*i.e.*, first type of repeated service) and $\alpha_2$ for STS or STRS (*i.e.*, second type of repeated service).

Further we assume that arrival process, service time, repeated service time and server's life time random variables are mutually independent of each other.

## 3. Prerequisite definition

Let $N_Q(t)$ be the queue size (including the one being served, if any) at time t, $S_i^0(t)$ and $B_i^0(t)$ be the elapsed service time and elapsed repeated service time of the customer for the $i$th types of service at time t for $i = 1, 2$ denoting type 1 and type 2 service respectively.

Also let us consider the following random variable:

$$Y(t) = \begin{cases} 0, & \text{if the server is idle at time } t \\ 1, & \text{if the server is busy with type 1 service at time } t \\ 2, & \text{if the server is busy with type 2 service at time } t \\ 3, & \text{if the server is busy with repeating type 1 service at time } t \\ 4, & \text{if the server is busy with repeating type 2 service at time } t \end{cases}$$

We now introduce the supplementary variables $S_i^0(t), B_i^0(t)$ for $i = 1, 2$ in order to obtain a bivariate Markov process $\{N_Q(t), Y(t)\}$.Let us now define the following probabilities.

$$U_n(t) = Pr\{N_Q(t) = n, Y(t) = 0\}; n = 0, 1, 2, \ldots, N - 1.$$

and for $n \geq 1$

$$P_{n,1}(x;t)\mathrm{d}x = Pr\{N_Q(t) = n, Y(t) = 1; x < S_1^0(t) \leq x + \mathrm{d}x\}; x > 0$$

$$P_{n,2}(x;t)\mathrm{d}x = Pr\{N_Q(t) = n, Y(t) = 2; x < S_2^0(t) \leq x + \mathrm{d}x\}; x > 0$$

$$Q_{n,1}(x;t)\mathrm{d}x = Pr\{N_Q(t) = n, Y(t) = 3; x < B_1^0(t) \leq x + \mathrm{d}x\}; x > 0$$

$$Q_{n,2}(x;t)\mathrm{d}x = Pr\{N_Q(t) = n, Y(t) = 4; x < B_2^0(t) \leq x + \mathrm{d}x\}; x > 0.$$

Further, it is assumed that $S_i(0) = 0, S_i(\infty) = 1, B_i(0) = 0, B_i(\infty) = 1$, for $i = 1, 2$; $S_i(x)$ is continuous at x = 0, $B_i(x)$ is continuous at x = 0, for $i = 1, 2$ so that

$$\mu_i(x)\mathrm{d}x = \frac{dS_i(x)}{1 - S_i(x)}, \eta_i(x)\mathrm{d}x = \frac{dB_i(x)}{1 - B_i(x)},$$

are the first order differential (hazard rate) function of $S_i$ and $B_i$ respectively for $i = 1, 2$.

## 4. Reliability function

Our aim in this paper is to derive the system reliability function. Let $\pi$ be the time to the first failure of the server, and then the reliability function of the server is $R(t) = P(\pi > t)$.

We obtain the reliability of the server by considering the failure state of the server is the absorbing state, which yields a new system. In order to derive the LST of reliability function of this system, the Kolmogorov forward equations (*e.g.* see Cox [13]) can be written as follows:

$$\frac{\partial U_n(t)}{\partial t} + \lambda U_n(t) = \delta_{n,0} \sum_{i=1}^{2} \left[(1 - q_i) \int_0^\infty P_{n+1,i}(x;t)\mu_i(x)\mathrm{d}x + \int_0^\infty Q_{n+1,i}(x;t)\eta_i(x)\mathrm{d}x\right] + (1 - \delta_{n,0})\lambda \sum_{k=1}^{n} c_k U_{n-k}(t); n = 0, 1, 2, \ldots, N - 1 \quad (4.1)$$

and for $i = 1, 2$, and $n \geq 1$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) P_{n,i}(x;t) + [\lambda + \alpha_i + \mu_i(x)]P_{n,i}(x;t) = \lambda \sum_{k=1}^{n} c_k \ P_{n-k,i}(x;t) \quad (4.2)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) Q_{n,i}(x;t) + [\lambda + \alpha_i + \eta_i(x)]Q_{n,i}(x;t) = \lambda \sum_{k=1}^{n} c_k \ Q_{n-k,i}(x;t), \quad (4.3)$$

where $\delta_{i,j}$ is the Knocker's delta function.

The boundary conditions at x = 0 for $i = 1, 2$; to solve the above equations are

$$P_{n,i}(0;t) = (1 - q_1)p_i \int_0^\infty P_{n+1,1}(x;t)\mu_1(x)\mathrm{d}x + (1 - q_2)p_i \int_0^\infty P_{n+1,2}(x;t)\mu_2(x)\mathrm{d}x + p_i \int_0^\infty Q_{n+1,1}(x;t)\eta_1(x)\mathrm{d}x + p_i \int_0^\infty Q_{n+1,2}(x;t)\eta_2(x)\mathrm{d}x; \quad 1 \leq n \leq N - 1 \quad (4.4)$$

$$P_{n,i}(0;t) = (1 - q_1)p_i \int_0^\infty P_{n+1,1}(x;t)\mu_1(x)\mathrm{d}x + (1 - q_2)p_i \int_0^\infty P_{n+1,2}(x;t)\mu_2(x)\mathrm{d}x + p_i \int_0^\infty Q_{n+1,1}(x;t)\eta_1(x)\mathrm{d}x + p_i \int_0^\infty Q_{n+1,2}(x;t)\eta_2(x)\mathrm{d}x + \lambda \sum_{k=0}^{N-1} c_{n-k} U_k(t); \quad n \geq N \quad (4.5)$$

$$Q_{n,i}(0;t) = q_i \int_0^\infty P_{n,i}(x;t)\mu_i(x)\mathrm{d}x; \quad n \geq 1 \tag{4.6}$$

and the initial condition is $U_n(0) = \delta_{n,0},\ n \geq 0$.

We now introduce the following LSTs of probability generating functions (PGFs) for $|z| < 1$ and $i = 1, 2$ to solve the above system of equations:

$$P_i^*(x,\theta;z) = \sum_{n=1}^{\infty} z^n P_{n,i}^*(x;\theta); \quad P_i^*(0,\theta;z) = \sum_{n=1}^{\infty} z^n P_{n,i}^*(0;\theta);$$

$$Q_i^*(x,\theta;z) = \sum_{n=1}^{\infty} z^n Q_{n,i}^*(x;\theta); \quad Q_i^*(0,\theta;z) = \sum_{n=1}^{\infty} z^n Q_{n,i}^*(0;\theta);$$

$$U_N^*(\theta;z) = \sum_{n=0}^{N-1} U_n^*(\theta) z^n \quad \text{and} \quad U_n^*(\theta) = \int_0^\infty e^{-\theta t} dU_n(t);\ n = 0, 1, 2, \ldots, N-1.$$

Now performing Laplace transform with respect to equation (4.1), we get

$$\begin{aligned}(\lambda + \theta)U_n^*(\theta) - U_n(0) = \delta_{n,0} \sum_{i=1}^{2} [(1 - q_i) \int_0^\infty P_{n+1,i}^*(x;\theta)\mu_i(x)\mathrm{d}x + \int_0^\infty Q_{n+1,i}^*(x;\theta)\eta_i(x)\mathrm{d}x] \\ + (1 - \delta_{n,0})\lambda \sum_{k=1}^{n} c_k U_{n-k}^*(\theta); \quad n = 0, 1, 2, \ldots, N-1.\end{aligned} \tag{4.7}$$

Proceeding similarly with equations (4.2)–(4.6) for $i = 1, 2$ and for $n \geq 1$, we have

$$(\theta + \lambda + \alpha_i + \mu_i(x))P_{n,i}^*(x;\theta) + \frac{\partial}{\partial x} P_{n,i}^*(x;\theta) = \lambda \sum_{k=1}^{n} c_k P_{n-k,i}^*(x;\theta) \tag{4.8}$$

$$(\theta + \lambda + \alpha_i + \eta_i(x))Q_{n,i}^*(x;\theta) + \frac{\partial}{\partial x} Q_{n,i}^*(x;\theta) = \lambda \sum_{k=1}^{n} c_k Q_{n-k,i}^*(x;\theta) \tag{4.9}$$

$$\begin{aligned}P_{n,i}^*(0;\theta) = (1 - q_1)p_i \int_0^\infty P_{n+1,1}^*(x;\theta)\mu_1(x)\mathrm{d}x + (1 - q_2)p_i \int_0^\infty P_{n+1,2}^*(x;\theta)\mu_2(x)\mathrm{d}x \\ + p_i \int_0^\infty Q_{n+1,1}^*(x;\theta)\eta_1(x)\mathrm{d}x + p_i \int_0^\infty Q_{n+1,2}^*(x;\theta)\eta_2(x)\mathrm{d}x; \quad 1 \leq n \leq N-1\end{aligned} \tag{4.10}$$

$$P^*_{n,i}(0;\theta) = (1-q_1)p_i \int_0^\infty P^*_{n+1,1}(x;\theta)\mu_1(x)\mathrm{d}x + (1-q_2)p_i \int_0^\infty P^*_{n+1,2}(x;\theta)\mu_2(x)\mathrm{d}x$$
$$+p_i \int_0^\infty Q^*_{n+1,1}(x;\theta)\eta_1(x)\mathrm{d}x + p_i \int_0^\infty Q^*_{n+1,2}(x;\theta)\eta_2(x)\mathrm{d}x$$
$$+\lambda \sum_{k=0}^{N-1} c_{n-k} U^*_k(\theta); \quad n \geq N \tag{4.11}$$

$$Q^*_{n,i}(0;\theta) = q_i \int_0^\infty P^*_{n,i}(x;\theta)\mu_i(x)\mathrm{d}x. \tag{4.12}$$

Now multiplying equations (4.8) and (4.9) by $z^n$ and then taking summation over all possible values of $n \geq 1$, we get two set of differential equation of Lagrangian type whose solution is given by

$$P^*_i(x,\theta;z) = P^*_i(0,\theta;z)[1-S_i(x)]\exp\{-(\theta+\alpha_i+a(z))x\}; x>0 \quad \text{for } i=1,2 \tag{4.13}$$

$$Q^*_i(x,\theta;z) = Q^*_i(0,\theta;z)[1-B_i(x)]\exp\{-(\theta+\alpha_i+a(z))x\}; x>0 \quad \text{for } i=1,2, \tag{4.14}$$

where $a(z) = \lambda(1-c(z))$.

Similarly equation (4.12) yields

$$Q^*_i(0,\theta;z) = q_i P^*_i(0,\theta;z) S^*_i(\theta+\alpha_i+a(z)) \quad \text{for } i=1,2. \tag{4.15}$$

Multiplying equations (4.10) and (4.11) by $z^n$ and then taking summation over all possible values of $n$ and utilizing equation (4.7), by noting $\sum_{n=N}^{\infty} z^n \sum_{k=0}^{N-1} c_{n-k} U^*_k(\theta) = \lambda U^*_0(\theta) - U^*_N(\theta;z)a(z)$, also utilizing equation (4.15), we get on simplification

$$[z-(1-q_1)p_1S^*_1(\theta+\alpha_1+a(z)) - p_1q_1S^*_1(\theta+\alpha_1+a(z))B^*_1(\theta+\alpha_1+a(z))]$$
$$P^*_1(0,\theta;z) + (a(z)+\theta)U^*_N(\theta;z)zp_1$$
$$= zp_1 + [(1-q_2)p_1S^*_2(\theta+\alpha_2+a(z)) + p_1q_2S^*_2(\theta+\alpha_2+a(z))B^*_2(\theta+\alpha_2+a(z))]P^*_2(0,\theta;z) \tag{4.16}$$

$$[z-(1-q_2)p_2S^*_2(\theta+\alpha_2+a(z)) - p_2q_2S^*_2(\theta+\alpha_2+a(z))B^*_2(\theta+\alpha_2+a(z))]$$
$$P^*_2(0,\theta;z) + (a(z)+\theta)U^*_N(\theta;z)zp_2$$
$$= zp_2 + [(1-q_1)p_2S^*_1(\theta+\alpha_1+a(z)) + p_2q_1S^*_1(\theta+\alpha_1+a(z))B^*_1(\theta+\alpha_1+a(z))]P^*_1(0,\theta;z). \tag{4.17}$$

Solving equations (4.16) and (4.17), we obtain

$$P^*_i(0,\theta;z) = \frac{[U^*_N(\theta;z)(\theta+a(z))-1]zp_i}{\begin{array}{l}\{(1-q_1)+q_1B^*_1(\theta+\alpha_1+a(z))\}p_1S^*_1(\theta+\alpha_1+a(z))\\ \quad +\{(1-q_2)+q_2B^*_2(\theta+\alpha_2+a(z))\}p_2S^*_2(\theta+\alpha_2+a(z))-z\end{array}} \quad \text{for } i=1,2. \tag{4.18}$$

Applying expression (4.18) in expression (4.15), we have

$$Q^*_i(0,\theta;z) = \frac{[U^*_N(\theta;z)(\theta+a(z))-1]zp_iq_iS^*_i(\theta+\alpha_i+a(z))}{\begin{array}{l}\{(1-q_1)+q_1B^*_1(\theta+\alpha_1+a(z))\}p_1S^*_1(\theta+\alpha_1+a(z))\\ \quad +\{(1-q_2)+q_2B^*_2(\theta+\alpha_2+a(z))\}p_2S^*_2(\theta+\alpha_2+a(z))-z\end{array}} \quad \text{for } i=1,2. \tag{4.19}$$

Now from equation (4.13), we obtain

$$P_i^*(\theta;z)=\int_0^\infty P_i^*(x,\theta;z)\mathrm{d}x$$
$$=\frac{[1-S_i^*(\theta+\alpha_i+a(z))][U_N^*(\theta;z)(\theta+a(z))-1]zp_i}{(\theta+\alpha_i+a(z))[\{(1-q_1)+q_1B_1^*(\theta+\alpha_1+a(z))\}p_1S_1^*(\theta+\alpha_1+a(z))+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2+a(z))\}p_2S_2^*(\theta+\alpha_2+a(z))-z]}\quad \text{for } i=1,2. \tag{4.20}$$

Similarly from equation (4.12) we get

$$Q_i^*(\theta;z)=\int_0^\infty Q_i^*(x,\theta;z)\mathrm{d}x$$
$$=\frac{[1-B_i^*(\theta+\alpha_i+a(z))][U_N^*(\theta;z)(\theta+a(z))-1]zp_iq_iS_i^*(\theta+\alpha_i+a(z))}{(\theta+\alpha_i+a(z))[\{(1-q_1)+q_1B_1^*(\theta+\alpha_1+a(z))\}p_1S_1^*(\theta+\alpha_1+a(z))+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2+a(z))\}p_2S_2^*(\theta+\alpha_2+a(z))-z]}\quad \text{for } i=1,2. \tag{4.21}$$

Now considering the coefficient

$$f(z)=\{(1-q_1)+q_1B_1^*(\theta+\alpha_1+a(z))\}p_1S_1^*(\theta+\alpha_1+a(z))$$
$$+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2+a(z))\}p_2S_2^*(\theta+\alpha_2+a(z))-z$$

It can be shown that the function $f(z)$ is convex. Hence by Rouche's theorem $f(z)$ has exactly one root $k_0(\theta)$ inside the unit circle $|z|=1$ for Re(z). Therefore we have

$$U_N^*(\theta;z)=\sum_{n=0}^{N-1}\left[\frac{1}{\lambda+\theta-\lambda[k_0(\theta)]^{n+1}}\right]z^n$$

again $k_0(\theta)$ is the unique root of the equation

$$\mathrm{z}=\{(1-q_1)+q_1B_1^*(\theta+\alpha_1+a(z))\}p_1S_1^*(\theta+\alpha_1+a(z))$$
$$+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2+a(z))\}p_2S_2^*(\theta+\alpha_2+a(z)).$$

Hence from equations (4.20) and (4.21), we have

$$\mathrm{R}^*(\theta)=U_N^*(\theta)+\sum_{i=1}^{2}[P_i^*(\theta)+Q_i^*(\theta)]$$
$$=U_N^*(\theta;1)+\sum_{i=1}^{2}[P_i^*(\theta;1)+Q_i^*(\theta;1)], \tag{4.22}$$

where,

$$P_i^*(\theta;1)=\frac{[1-S_i^*(\theta+\alpha_i)][\theta U_N^*(\theta)-1]p_i}{(\theta+\alpha_i)[\{(1-q_1)+q_1B_1^*(\theta+\alpha_1)\}p_1S_1^*(\theta+\alpha_1)+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2)\}p_2S_2^*(\theta+\alpha_2)-1]}\quad \text{for } i=1,2$$

$$Q_i^*(\theta;1)=\frac{[1-B_i^*(\theta+\alpha_i)][\theta U_N^*(\theta)-1]p_iq_iS_i^*(\theta+\alpha_i)}{(\theta+\alpha_i)[\{(1-q_1)+q_1S_1^*(\theta+\alpha_1)\}p_1S_1^*(\theta+\alpha_1)+\{(1-q_2)+q_2S_2^*(\theta+\alpha_2)\}p_2S_2^*(\theta+\alpha_2)-1]}\quad \text{for } i=1,2.$$

The above results may be summarize in the following theorem.

**Theorem 4.1.** *The Laplace transform of the reliability function R(t) is given by*

$$R^*(\theta)=U_N^*(\theta)+\frac{\begin{array}{c}[p_1\{(1-S_1^*(\theta+\alpha_1))+(1-B_1^*(\theta+\alpha_1))q_1S_1^*(\theta+\alpha_1)\}(\theta+\alpha_2)\\ +p_2\{(1-S_2^*(\theta+\alpha_2))+(1-B_2^*(\theta+\alpha_2))q_2S_2^*(\theta+\alpha_2)\}(\theta+\alpha_1)][\theta U_N^*(\theta)-1]\end{array}}{\begin{array}{c}(\theta+\alpha_1)(\theta+\alpha_2)[\{(1-q_1)+q_1B_1^*(\theta+\alpha_1)\}\\ p_1S_1^*(\theta+\alpha_1)+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2)\}p_2S_2^*(\theta+\alpha_2)-1]\end{array}}$$

*where* $U_N^*(\theta)=\sum_{n=0}^{N-1}[1/(\lambda+\theta-\lambda[k_0(\theta)]^{n+1})]$ *and* $k_0(\theta)$ *is the unique root of the equation*

$$\begin{aligned}z=&\{(1-q_1)+q_1B_1^*(\theta+\alpha_1+a(z))\}p_1S_1^*(\theta+\alpha_1+a(z))\\&+\{(1-q_2)+q_2B_2^*(\theta+\alpha_2+a(z))\}p_2S_2^*(\theta+\alpha_2+a(z))\end{aligned}$$

inside $|z|=1, Re(\theta)>0$ and $a(z)=\lambda(1-c(z))$.

***Performance measure:*** The mean time of the first failure (MTFF) of the server is given by

$$MTFF=\int_0^\infty R(t)\mathrm{d}t=R^*(\theta)|_{\theta=0}. \tag{4.23}$$

By Tauberian theorem of Laplace Transform, we have

$$\lim_{\theta\to\ 0}\theta U_N^*(\theta)=\lim_{t\to\infty}U_N(t)=U_N.$$

Now, let us define

$$U_n=U_N\psi_n;n=0,1,2,\ldots,(N-1)\text{ and }\psi_0=1. \tag{4.24}$$

where $U_n=\lim_{t\to\infty}U_n(t)$, $\psi_n$= Pr {A batches of '$n$' units arrive in the system during an idle period} and $U_N$ is a normalizing constant.

Now considering the normalizing condition under steady state condition

$$\sum_{n=0}^{N-1}U_n+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\left[\int_0^\infty P_{n,i}(x)\mathrm{d}x+\int_0^\infty Q_{n,i}(x)\mathrm{d}x\right]=1,$$

we have

$$U_N=\frac{1-\rho_s}{\sum\limits_{n=0}^{N-1}\psi_n}, \tag{4.25}$$

where $\rho_s=p_1(\rho_{s_1}+q_1\rho_{b_1})+p_2(\rho_{s_2}+q_2\rho_{b_2})$ such that $\rho_{s_i}=\lambda c_{[1]}s_i^{(1)}$ and $\rho_{b_i}=\lambda c_{[1]}b_i^{(1)}$ for $i=1,2$.

Hence by substituting $U_N$ in (4.23), we finally obtain

$$MTFF = U_N^*(0) + \frac{\left[1-\rho_s-\sum_{n=0}^{N-1}\psi_n\right]\begin{bmatrix} p_1\{(1-S_1^*(\alpha_1))+(1-B_1^*(\alpha_1))q_1S_1^*(\alpha_1)\}(\alpha_2) \\ +\,p_2\{(1-S_2^*(\alpha_2))+(1-B_2^*(\alpha_1))q_2S_2^*(\alpha_1)\}(\alpha_1)\end{bmatrix}}{\left(\sum_{n=0}^{N-1}\psi_n\right)(\alpha_1)(\alpha_2)\left[[\{(1-q_1)+q_1B_1^*(\alpha_1)\}p_1S_1^*(\alpha_1)+\{(1-q_2)+q_2B_2^*(\alpha_2)\}p_2S_2^*(\alpha_2)]-1\right]}.$$

## 5. Numerical illustration

In this section, we illustrate the effect of system parameters on the reliability measures MTFF. For illustrative purpose, we assume that the arrival batch size follows geometric distribution with parameter $\varepsilon$ for which the PGF is given by

$$c(z)=\frac{\varepsilon}{1-(1-\varepsilon)z}.$$

The service time and repeated service time random variables is assumed to follow exponential distributions with service rates $\beta_i$ and $\gamma_i$ respectively for $i=1,2$ (denoting type 1 and type 2 service) for which LST is given by

$$S_i^*(\theta)=\frac{\beta_i}{\theta+\beta_i}\quad \text{for } i=1,2 \quad\text{and}\quad B_i^*(\theta)=\frac{\gamma_i}{\theta+\gamma_i}\quad \text{for } i=1,2.$$

Now, the MTFF obtained in the previous section can be written as follows:

$$MTFF=\sum_{n=0}^{N-1}\left[\frac{1}{\lambda-\lambda[k_0(0)]^{n+1}}\right] + \frac{\left(\left[1-\rho_s-\sum_{n=0}^{N-1}\psi_n\right]\begin{bmatrix} p_1\left\{\left(1-\frac{\beta_1}{\alpha_1+\beta_1}\right)+\left(1-\frac{\gamma_1}{\alpha_1+\gamma_1}\right)\left(\frac{q_1\beta_1}{\alpha_1+\beta_1}\right)\right\}(\alpha_2) \\ +\,p_2\left\{\left(1-\frac{\beta_2}{\alpha_2+\beta_2}\right)+\left(1-\frac{\gamma_2}{\alpha_2+\gamma_2}\right)\left(\frac{q_2\beta_2}{\alpha_2+\beta_2})\right)\right\}(\alpha_1)\end{bmatrix}\right)}{\left(\sum_{n=0}^{N-1}\psi_n\right)(\alpha_1)(\alpha_2)\left[\left[\left\{(1-q_1)+\frac{q_1\gamma_1}{\alpha_1+\gamma_1}\right\}\left(\frac{p_1\beta_1}{\alpha_1+\beta_1}\right)+\left\{(1-q_2)+\frac{q_2\gamma_2}{\alpha_2+\gamma_2}\right\}\left(\frac{p_2\beta_2}{\alpha_2+\beta_2}\right)\right]-1\right]} \tag{5.1}$$

where $k_0(0)$ is the unique root of the equation

$$z = \left\{(1-q_1) + \frac{q_1\gamma_1}{\alpha_1+\gamma_1+\lambda\left(1-\frac{\varepsilon}{1-(1-\varepsilon)z}\right)}\right\}\left(\frac{p_1\beta_1}{\alpha_1+\beta_1+\lambda\left(1-\frac{\varepsilon}{1-(1-\varepsilon)z}\right)}\right) + \left\{(1-q_2) + \frac{q_2\gamma_2}{\alpha_2+\gamma_2+\lambda\left(1-\frac{\varepsilon}{1-(1-\varepsilon)z}\right)}\right\}\left(\frac{p_2\beta_2}{\alpha_2+\beta_2+\lambda\left(1-\frac{\varepsilon}{1-(1-\varepsilon)z}\right)}\right) \tag{5.2}$$

inside $|z| = 1$ which we solve for $z$ numerically.

Further, for the sake of computational convenience, the settings of system's parameter are as follows:

- The arrival rate of batches $\lambda = 0.6$
- $\varepsilon = 0.5$, so that $c_{[1]} = 1/(\varepsilon) = 0.5$. It is important to note that, in this case, it is straightforward to show that $\psi_n = \varepsilon$ for all $n = 1, 2, \ldots, N-1$ and thus $\sum_{n=0}^{N-1} \psi_n = 1 + N\varepsilon$.
- $\beta_1 = 2$ and $\beta_2 = 1.5$ for type 1 and type 2 services respectively
- $\gamma_1 = 3.5$ and $\gamma_2 = 2.5$ for type 1 and type 2 services respectively
- $p_1 = p_2 = 0.5, q_1 = 0.15$ and $q_2 = 0.2$

The result of MTFF are shown for the following two cases:

Case 1: We choose $\alpha_2 = 0.4$ and vary the value of $\alpha_1$ from 0 to 1 for different values of $N$.
Case 2: We choose $\alpha_1 = 0.4$ and vary the value of $\alpha_2$ from 0 to 1 for different values of $N$.

Now for the above values of the system parameters the equation (5.2) reduces for case 1 and 2 as follows:

$$\begin{aligned}&(0.25\alpha_1^2 + 1.675\alpha_1 + 2.665)z^5 - (1.96785\alpha_1^2 + 13.0971\alpha_1 + 20.9749)z^4\\ &+(5.94857\alpha_1^2 + 39.7066\alpha_1 + 65.081)z^3 - (8.44227\alpha_1^2 + 57.5321\alpha_1 + 99.1455)z^2\\ &+(5.35771\alpha_1^2 + 38.7826\alpha_1 + 73.7565)z - (1.04914\alpha_1^2 + 9.13531\alpha_1 + 21.2542) = 0\end{aligned}$$

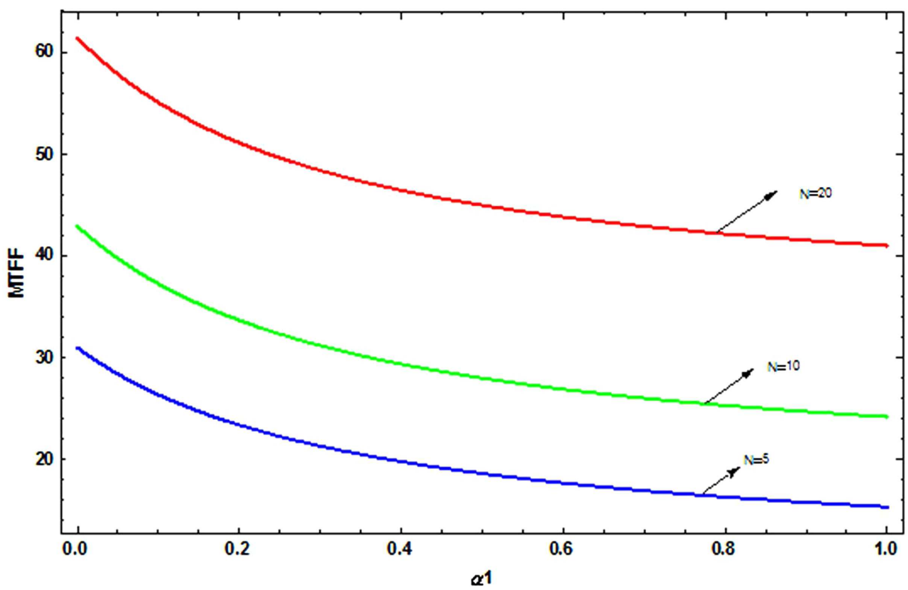


FIGURE 1. Effect of $\alpha_1$ on MTFF at different threshold level (color online).

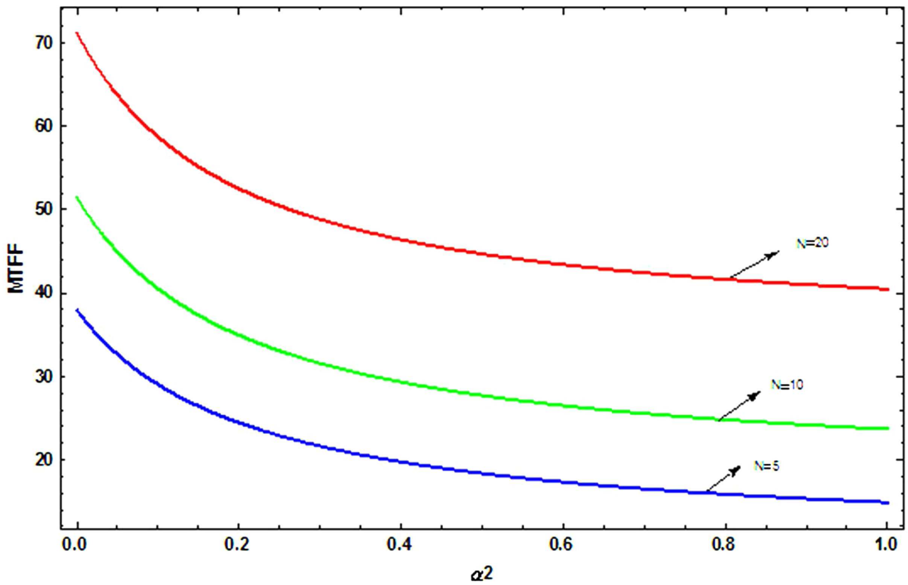


FIGURE 2. Effect of $\alpha_2$ on MTFF at different threshold level (color online).

and

$$(0.25\alpha_2^2 + 1.3\alpha_2 + 1.6275)z^5 - (1.99722\alpha_2^2 + 10.2356\alpha_2 + 12.7807)z^4$$
$$+(6.14167\alpha_2^2 + 31.29\alpha_2 + 39.5947)z^3 - (8.90333\alpha_2^2 + 45.8247\alpha_2 + 60.2978)z^2$$
$$+(5.82444\alpha_2^2 + 31.3724\alpha_2 + 44.9346)z - (1.21333\alpha_2^2 + 7.59733\alpha_2 + 13.02) = 0.$$

The solutions of these two equations can easily be obtained using MATLAB/ MATHEMATICA software which yields five roots for each of the equation. Here we consider only that root for each of the equation which lie inside the unit circle $|z| = 1$.

The figures below show the effect of breakdown rates on the MTFF. From Figures 1 and 2 we observe that the value of the MTFF decreases as breakdown rates increases at all threshold level. The figures also reveal the effect of threshold level on MTFF. The figures show that MTFF increases as threshold value increases.

## 6. Conclusion

In this paper, we have studied reliability analysis of a batch arrival $N$ policy unreliable queue with two types of general heterogeneous service and optional repeated service where the server does not start service until the queue size becomes "$N$" as model building of a production system in which production does not starts until "$N$" specified raw materials accumulated in the system .We have obtained LST of reliability function and MTFF of the model under the study. Further, we have performed some numerical experiments to investigate parameter effects on MTFF. The result can be further generalized by introducing the concept of a setup period. As possible extension of our model we mentioned possibility of assuming batch arrival with vacation under different vacation policies as auxiliary tools leading to the development of more versatile queuing model.

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