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NEW CONCEPT OF CONNECTION IN BIDIRECTED GRAPHS

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Abstract. In bidirected graph an edge has a direction at each end. We introduce a new definition of connection in a bidirected graph. We prove some properties of this definition and we establish a relationship to connection and imbalance in the corresponding signed graph. The main result gives a sufficient condition for a signed graph to have a Biconnected biorientation.

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1. Introduction

In bidirected graph an edge has a direction at each end, so bidirected graphs generalize undirected and directed graphs. Harary [3] defined in 1954 the notion of the signed graphs. For any bidirected graph, we can associate a signed graph and conversely, any signed graph can be associated to a bidirected graph. The aim of this paper, is to introduce a new concept of connection to bidirected graphs, which is called Biconnectivity. Some properties will be established for this concept in bidirected graphs and, a relationship to sign-connectivity in signed graphs. Before that we give a notion of a bidirected path and a bidirected circuit in bidirected graphs. We consider finite undirected graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The elementary chains and cycles which are used in this paper are the usual elementary chains and cycles as defined in undirected graphs. We allow graphs to have loops and multiple edges.

2. Bidirected graphs

Let G be an undirected graph. The set of the incidences of G is a set $\Phi(G)$ defined as follows:

$$\boldsymbol{\Phi(G)} = \{(\boldsymbol{e},\, \boldsymbol{x}) \in \boldsymbol{E} \times \boldsymbol{V} /\ \boldsymbol{e} \text{ is incident to } \boldsymbol{x}\ \}$$

Definition 2.1 ([3])**.** A biorientation of G is a signature of its incidences

$$\boldsymbol{\tau : \Phi(G) \to \{-1, +1\}}$$

By convention $\tau(e,x) = 0$ if (e,x) is not an incidence of G, (what makes it possible to extend τ with any $(E \times V)$, which we will do henceforth).

A bidirected graph is a graph provided with a biorientation, it is denoted G_τ.

Keywords. Matroid, signed graphs, bidirected graphs.

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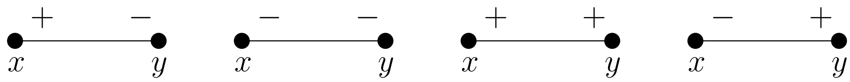

FIGURE 1. The four possible biorientations of an edge $\{x, y\}$ of G_τ.

Definition 2.2 ([2]).
A signed graph is a triple $(V, E; \sigma)$ where $V(G)$ and $E(G)$ are the vertex set and the edge set of the undirected graph G respectively and σ is a signature of the edges set E.

$$\begin{gathered} \sigma : E \to \{-1, +1\} \\ e \mapsto \sigma(e) \end{gathered}$$

A signed graph is denoted G_σ.

Definition 2.3. For any biorientation τ of a graph G_τ, we define a signature σ of E, for any edge e of E of ends x and y, by:

$$\sigma(e) = -\tau(e, x).\tau(e, y)$$

A bidirected graph determines a unique signed graph, but a signed graph does not determines a unique bidirected graph.

Definition 2.4. An elementary cycle of a signed graph is balanced, if the product of signs of its edges is positive, or the number of its negative edges is even. In the opposite case, it is unbalanced.

Definition 2.5. A signed graph is balanced if it is cycle free, or all its elementary cycles are balanced.

Definition 2.6. [7]
Given a signed graph G_σ, we define $M(G_\sigma)$ to be the matroid associated to G_σ. A subset F of the edge set E is a circuit of $M(G_\sigma)$ if, and only if, either

(i) F is a balanced cycle (Type (i)), or
(ii) F is the union of two unbalanced elementary cycles, having exactly one common vertex (Type (ii)), or
(iii) F is the union of two vertex-disjoint unbalanced elementary cycles and an elementary chain which is internally disjoint from both elementary cycles (Type (iii)).

(we represent in Fig. 2, a balanced (resp. unbalanced) elementary cycle by a quadrilateral (resp. triangle)).

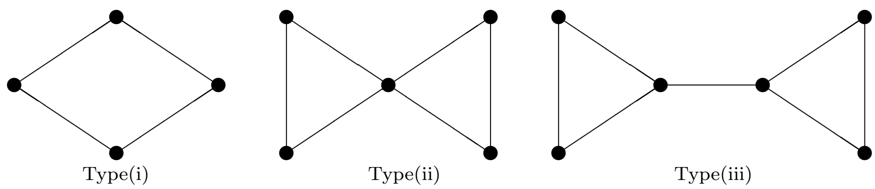

FIGURE 2.

The matroid associated to a bidirected graph, is the matroid associated to its signed graph, this latest is given by Definition 2.3.

Definition 2.7 ([2]).
Let $G_\sigma = (V, E; \sigma)$ be a signed graph and let P be a walk connecting x and y in G_σ:

$$P : x, e_1, x_1, e_2, x_2, \ldots, y.$$

where $x, x_i, y \in V$ and $e_j \in E$.
We put $\sigma(P) = \prod_{e_i \in P} \sigma(e_i)$. We denote P^ε instead of P, if $\varepsilon = \sigma(P)$.
$P^\varepsilon(x, y)$ is called an ε-walk of sign ε ($\varepsilon = +1, -1$), connecting x and y.
An ε-walk P^ε is elementary if it is minimal for this property.
An ε-walk P^ε is a positive (resp. negative) walk P^+ (resp P^-) if, the product of the signs of the edges of P^ε is positive (resp. negative).

Example 2.8. $P^+(x, y) : x, x_1, x_2, x_3, x_2, x_4, y$ is a positive walk.
$P^-(x, y) : x, x_1, x_2, x_4, y$ is a negative walk.

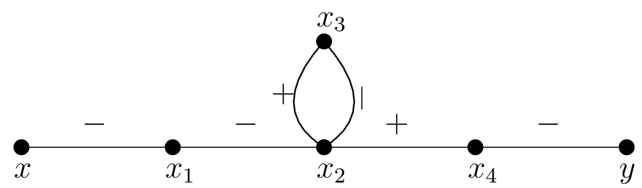

FIGURE 3.

We have : P^+ contains P^- (as sub walk), P^+ and P^- are elementary ε-walks because both are minimal ε-walks and their signs differ.

Definition 2.9. Let G_σ be a signed graph and let C be a closed walk defined as follows: $C : xe_1x_1 \ldots e_ix_{i+1}e_{i+1} \ldots x_{k-1}e_kx$.
where x, y, x_i are vertices of G_σ and $e_j \in G_\sigma$.
We say that C is an elementary ε-closed walk of G_σ, if the sequence:
$xe_1x_1 \ldots e_ix_{i+1}e_{i+1} \ldots x_{k-1}$ is an elementary ε-walk and C is one of the three circuits characterized in Definition 2.6. We have three types of ε-closed walk, type (i), (ii) and (iii) in signed graphs.

Definition 2.10. A connected signed graph G_σ is called sign-connected if, for every pair of vertices x and y, there exists both a positive walk and a negative walk, connecting them.

Example 2.11. The graph which is given in Figure 3 is sign-connected.

Proposition 2.12. *Let G_σ be a signed graph. If G_σ is an unbalanced elementary cycle, then G_σ is sign-connected.*

Proof. The proof is deduced from Definition 2.10. $\square$

Definition 2.13 ([2]). Let G_τ be a bidirected graph and W (resp. $\overline{W}$) be a function defined on $V(G_\tau)$ (resp. $E(G_\tau)$) as follows:

$$W : V \to \mathbb{Z} \qquad x \mapsto W(x) = \textstyle\sum_{e \in E} \tau(e, x)$$

$$\overline{W} : E \to \{-2, 0, 2\} \qquad e \mapsto \overline{W}(e) = \textstyle\sum_{x \in V} \tau(e, x)$$

Definition 2.14 ([2]).

Let $G_\tau = (V, E)$ be a bidirected graph, and let P be a walk connecting x and y in G_τ:

$$P : xe_1x_1 \dots e_ix_{i+1}e_{i+1} \dots x_{k-1}e_ky.$$

where $x, x_i, y \in V$ and $e_j \in E$.

Assume that: $\tau(e_1, x) = \alpha$ and $\tau(e_k, y) = \beta$ such that $\alpha, \beta \in \{-1, +1\}$.

For every vertex $x_i \in V(P)$ we put $W_P(x_i) = \tau(e_i, x_i) + \tau(e_{i+1}, x_i)$, and we use the notation:

$$P_{(\alpha,\beta)}(x, y) : x^\alpha e_1x_1 \dots e_ix_{i+1}e_{i+1} \dots x_{k-1}e_ky^\beta.$$

The walk $P_{(\alpha,\beta)}(x, y)$ is called a bidirected path from x^α to y^β if:

(i) $k \geq 1$.
(ii) $\tau(e_1, x) = \alpha$, and $\tau(e_k, y) = \beta$.
(iii) $W_P(x_i) = 0, \forall i = 1, \dots, k-1$ if $k > 1$.
(iv) $P_{(\alpha,\beta)}(x, y)$ is minimal for the properties (i)−(iii).

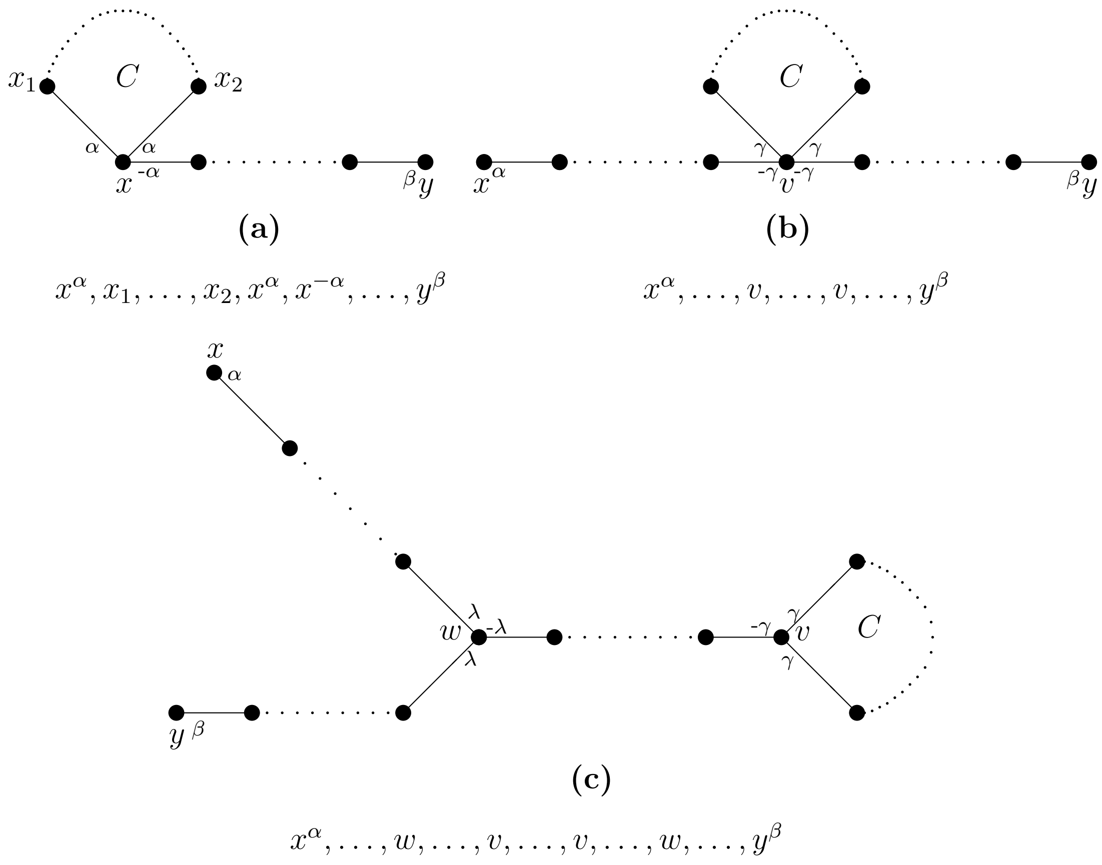

FIGURE 4. Examples of bidirected paths where C is a negative elementary cycle.

Remark 2.15. If $P_{(\alpha,\beta)}(x, y)$ is a bidirected path from x^α to y^β, then $P_{(\beta,\alpha)}(y, x)$ is also a bidirected path from y^β to x^α, where $P_{(\beta,\alpha)}(y, x) : y^\beta e_k x_{k-1} \dots e_{i+1} x_{i+1} e_i \dots x_1 e_1 x^\alpha$.

Definition 2.16 ([2])**.** Let G_τ be a bidirected graph and let C be a closed bidirected path defined as follows:

$$C : x^\alpha \; e_1 \; x_1 \; \dots \; e_i \; x_{i+1} \; e_{i+1} \; \dots \; x_{k-1} \; e_k \; x^\beta.$$

The closed bidirected path C is called a bidirected circuit of G_τ if: $\alpha = -\beta$; $\forall \alpha, \beta \in \{-1, +1\}$.

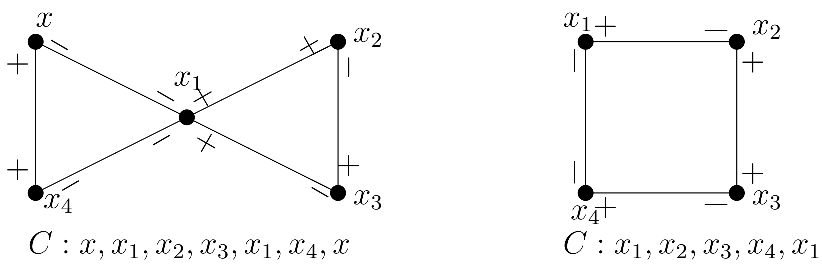

Figure 5. Examples of bidirected circuits.

Proposition 2.17. *The sign $\sigma(P_{(\alpha,\beta)}(x, y))$ of the bidirected path $P_{(\alpha,\beta)}(x, y)$ is given by the following formula: $\sigma(P_{(\alpha,\beta)}(x, y)) = -\alpha\beta$.*

Proof. Let $P_{(\alpha,\beta)}(x, y)$ be a bidirected path from x^α to y^β, which is represented by the following sequence: $P_{(\alpha,\beta)}(x, y) : x^\alpha e_1 x_1 \dots e_i x_{i+1} e_{i+1} \dots x_{k-1} e_k y^\beta$. We have: $\sigma(P_{(\alpha,\beta)}(x, y)) = \prod_{e \in P_{(\alpha,\beta)}(x,y)} \sigma(e)$

$= [-\tau(e_1, x)\tau(e_1, x_1)] \dots [-\tau(e_{k-2}, x_{k-2})\tau(e_{k-1}, x_{k-1})][-\tau(e_k, x_{k-1})\tau(e_k, y)]$

$= -\tau(e_1, x)[-\tau(e_1, x_1)\tau(e_2, x_1)] \dots [-\tau(e_{k-1}, x_{k-1})\tau(e_k, x_{k-1})]\tau(e_k, y)$.

According to the definition of the bidirected path we have $W_P(x_i) = \tau(e_i, x_i) + \tau(e_{i+1}, x_i) = 0$. This implies that, $\tau(e_i, x_i)\tau(e_{i+1}, x_i) = -1 \; \forall \; i = 1, \dots, k-1$ if $k > 1$. Thus $\sigma(P_{(\alpha,\beta)}) = -\tau(e_1, x)\tau(e_k, y) = -\alpha\beta$. ☐

Proposition 2.18. *The weight $\overline{W}(P_{(\alpha,\beta)}(x, y))$ of the bidirected path $P_{(\alpha,\beta)}(x, y)$ is given by the following formula: $\overline{W}(P_{(\alpha,\beta)}(x, y)) = \alpha + \beta$.*

Proof. Let $P_{(\alpha,\beta)}(x, y)$ be a bidirected path from x^α to y^β, which is represented by the following sequence : $P_{(\alpha,\beta)}(x, y) : x^\alpha e_1 x_1 \dots e_i x_{i+1} e_{i+1} \dots x_{k-1} e_k y^\beta$.

We have: $\overline{W}(P_{(\alpha,\beta)}(x, y)) = \sum_{e \in P_{(\alpha,\beta)}(x,y)} \overline{W}(e)$

$= \tau(e_1, x) + W_P(x_{1)}) + W_P(x_2) + \dots\dots + W_P(x_{k-1}) + \tau(e_k, y)$.

According to the definition of the bidirected path we have $W_P(x_i) = 0$
$\forall \; i = 1, \dots, k-1$ if $k > 1$. Thus $\overline{W}(P_{(\alpha,\beta(}(x, y)) = \tau(e_1, x) + \tau(e_k, y) = \alpha + \beta$. ☐

3. Biconnectivity in bidirected graphs

Let G_τ be a bidirected graph. Let us consider the relation R in $V(G_\tau)$ defined by:

$$xRy \Leftrightarrow \begin{cases} x = y \\ \text{or} \\ \text{There exists a bidirected path } P_{(\alpha,\beta)}(x, y) \text{ from } x^\alpha \text{ to } y^\beta;\ \forall \alpha, \beta \in \{-1, +1\}. \end{cases}$$

Proposition 3.1. *R is an equivalence relation.*

Proof.

(1) *R is reflexive.* $\forall x \in V$, we have $x = x$. So that xRx.
(2) *R is symmetric.* $xRy \Leftrightarrow$ there exists a bidirected path $P_{(\alpha,\beta)}(x,y)$ from x^α to y^β; $\forall \alpha, \beta \in \{-1,+1\}$. Then it follows from the remark given in Definition 2.14 that there exists a bidirected path $P_{(\beta,\alpha)}(y,x)$ from y^β to x^α for all $\alpha, \beta \in \{-1,+1\}$. So that yRx.
(3) *R is transitive.* $\forall x, y$ and $z \in V$, there exists a bidirected path $P_{(\alpha,\beta)}(x,y)$ from x^α to y^β for all $\alpha, \beta \in \{-1,+1\}$ and there exists also a bidirected path $P_{(\acute{\alpha},\acute{\beta})}(y,z)$ from $y^{\acute{\alpha}}$ to $z^{\acute{\beta}}$ for all $\acute{\alpha}, \acute{\beta} \in \{-1,+1\}$. Choose $\acute{\alpha} \in \{-1,+1\}$ so that $\acute{\alpha} + \beta = 0$. Then there exists a bidirected path $P_{(\alpha,\acute{\beta})}(x,z)$ from x^α to $z^{\acute{\beta}}$ for all $\alpha, \acute{\beta} \in \{-1,+1\}$. So that xRz. □

Definition 3.2. A bidirected graph G_τ is called Biconnected, if for every pair of vertices x and y, there exists a bidirected path $P_{(\alpha,\beta)}(x,y)$ from x^α to y^β; for all $\alpha, \beta \in \{-1,+1\}$.

The classes of equivalence of the equivalence relation R form a partition $(V_1, \ldots, V_q)$ of $V(G_\tau)$. The subgraphs of G_τ generated by the subsets
V_i $(i = 1, \ldots, q)$ are called the Biconnected components of G_τ.

Proposition 3.3. *Let G_τ be a bidirected graph such that $|V(G_\tau)| \geq 2$. If G_τ is biconnected, then it is unbalanced.*

Proof. Since G_τ is a biconnected graph, then there exists two vertices, and they are connected by a positive bidirected path and a negative bidirected path. If one of the bidirected paths contains a cycle, it contains a negative cycle and G_τ is unbalanced. If both bidirected paths do not contain cycles, they are elementary chains and, by Harary's signed paths Theorem [3], G_τ is unbalanced. □

Proposition 3.4. *Let G_τ be a bidirected graph. If G_τ is biconnected, then it is pendant-vertex-free.*

Proof. If G_τ admits a pendant vertex, then this vertex is an extremity of at most one bidirected path. This contradicts the definition of the biconnectivity of bidirected graphs. □

Proposition 3.5. *Let G_τ be a bidirected graph. The matroid circuits of the type* (ii) *are biconnected.*

Proof. Let $C = C_1 \cup C_2$ be a bidirected circuit of type (ii) and let x, y and v be a vertices of C such that $C_1 \cap C_2 = \{v\}$.

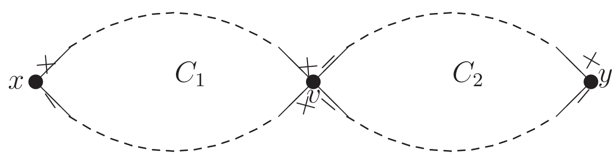

FIGURE 6.

We distinguish the following cases:

Case 1. Let x be on $C_1 \backslash v$ and let y on $C_2 \backslash v$ (see Fig. 6). Since for every vertex in C we have $W(x) = W(v) = W(y) = 0$, then x is in an edge $e_1 \in C_1$ such that $\tau(x, e_1) = +1$ and also in an edge $e_2 \in C_1$ such that $\tau(x, e_2) = -1$. Let P_i be the bidirected path from x to v in C_1 that contains e_i, $i = 1, 2$. Similarly, y is in an edge $f_1 \in C_2$ such that $\tau(y, f_1) = +1$ and also in an edge $f_2 \in C_2$ such that $\tau(y, f_2) = -1$.

Let Q_j be the bidirected path from y to v in C_2 that contains f_j, $j = 1, 2$. Now the concatenations P_iQ_j, $i, j = 1, 2$ are four bidirected paths $P_{(\alpha,\beta)}(x, y)$ that prove x and y are biconnected.

Case 2. Let x, y be on $C_1\backslash v$. Then $C_1\backslash v$ contains an (α, β) bidirected path from x to y, for some α, $\beta \in \{+1, -1\}$. Deleting the edges of P, the remainder of C is a $(-\alpha, -\beta)$ bidirected path Q from x to y that contains C_2. Now you need an $(\alpha, -\beta)$ bidirected. The former is obtained by taking P followed by the bidirected path P_y in C_1 from y to v, then C_2, then P_y in the opposite direction, ending at y. The latter is obtained by taking the path P_x from x to v in C_1, then C_2 then retracting P_x from v to x and continuing along P to y.

Case 3. Let $y = v$ and let x be on $C_1\backslash v$, as in Case 1.The bidirected paths P_1 and P_2 are respectively, a $(+1, \beta)$ bidirected path and a $(-1, \beta)$ bidirected path from x to y for some $\beta \in \{+1, -1\}$. The bidirecred paths P_1C_2 and P_2C_2 are, respectively, a $(+1, -\beta)$ bidirected path and a $(-1, -\beta)$ bidirected path from x to y. Therefore, x and y are biconnected. □

Proposition 3.6. *Let G_τ be a bidirected graph. The matroid circuits of the type* (iii) *are biconnected.*

Proof. The proof is similar to that given in Proposition 3.5. □

Theorem 3.7. *Let G_τ be a bidirected graph, such that $|E(G_\tau)| \geq 1$. If G_τ is biconnected, then every edge of E belongs to a bidirected circuit.*

Proof. Let $\{x^\alpha, y^\beta\}$ be an edge of G_τ such that $\alpha, \beta \in \{-1, +1\}$, since G_τ is biconnected then there exists a bidirected path from $x^{-\alpha}$ to $y^{-\beta}$ connecting the two vertices x and y. Thus this bidirected path together with the edge $\{x^\alpha, y^\beta\}$ makes a bidirected circuit. □

Theorem 3.8. *Let G_σ be a connected signed graph. It is possible to give a biorientation to G_σ in order to obtain a biconnected bidirected graph G_τ, if every pair of vertices belongs to matroid circuit of type* (ii) *or* (iii).

Proof. Suppose that G_σ admits an ε-closed walk of the type (ii) or (iii), it is sign-connected. Then for the proof proceed as follows:

Let W be an ε-closed walk of G_σ. Let V_1 be the set of the vertices of W. We give a biorientation compatible with the edge signs as in Definition 2.3 to the edges of this ε-closed walk such that a bidirected circuit is obtained, and for each other edge having its two ends on V_1 we give an unspecified biorientation. The subgraph generated by V_1 is biconnected according to Propositions 3.5 and 3.6.

If $V_1 = V$ the result is obtained. If not $(V - V_1 \neq \varnothing)$, then there exists a vertex $x \notin V_1$ which is adjacent to a vertex $y \in V_1$ (since G_σ is connected). It follows from the hypothesis that this edge is on an ε-closed walk W' of G_σ, which contains an ε'-walk from y to an other vertex of V_1, then we give a biorientation compatible with the edge signs as in Definition 2.3 to this ε-closed walk W' in order to obtain a bidirected circuit.

Let V_2 be the union of V_1 and all vertices of W'. We give an arbitrary biorientation of the edges which have the two ends on V_2 and have not been bioriented. The subgraph generated by V_2 is biconnected according to Propositions 3.5 and 3.6. If $V_2 = V$ the result is obtained. Otherwise we repeat the same operation. □

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