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MODELING FUZZY DATA ENVELOPMENT ANALYSIS UNDER ROBUST INPUT AND OUTPUT DATA

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Abstract. This paper offers a fuzzy optimization framework for data envelopment analysis (DEA) to evaluate the relative efficiency of decision making units (DMUs) with parametric interval-valued fuzzy variable-based inputs and outputs. The parametric interval-valued fuzzy variable-based inputs and outputs is employed to capture the uncertainty of data on the basis of professional judgements or empirical estimations. The DEA problem is formulated as fuzzy expectation model with credibility constraints. When the inputs and outputs are mutually independent parametric interval-valued triangular fuzzy variables, we investigate the parametric equivalent representations of expectation objective function and chance constraints. In order to find the optimal solution of our DEA model, a domain decomposition method is proposed. Finally, the numerical example on the sustainable supplier evaluation and selection problem is provided to demonstrate the efficiency of the proposed DEA model and domain decomposition method.

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1. Introduction

Data envelopment analysis, as an evaluation technology, was introduced by Charnes *et al.* [8] to measure the relative efficiency of a set of homogeneous decision-making units in multiple inputs and outputs systems. The original DEA model was called as DEA-CCR. Since its appearance, DEA has been developed quickly. In addition to the CCR model, many DEA theoretical models were established in the literature such as BCC model Banker *et al.* [4], FDH model Petersen [21], RAM model Cooper *et al.* [11], and SBM model Tone [25]. The advantages of DEA models include: (1) free of parameter estimation so that it minimizes the influence of subjectivity; (2) free of selection for the input-output weights to attempt reaching the efficient frontier and (3) simple operation. Therefore, DEA has been extensively used in the multifaceted applications. As shown in Liu *et al.* [16], the top-five industries addressed are banking, health care, agriculture and farm, transportation, and education. Furthermore, the applications that have the highest growth momentum recently are finance,

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as well as energy and environment. For example, Ignatius *et al.* [14] proposed a fuzzy DEA-based framework to assess the environmental evaluation at different levels of certainty, and further presented energy efficiency among 23 European Union member countries. More detailed discussion and in-depth review articles about the development of DEA technology and its application, the interested reader can refer to Cooper *et al.* [9], Cook and Seiford [10], Liu *et al.* [19], Azizi *et al.* [2], Liu *et al.* [17].

The traditional DEA methods require accurate measurement of both input and output parameters. In fact, the observed values of the input and output data present in real-world problems are often imprecise Amirteimoori and Emrouznejad [1]. To handle this situation, more and more researchers addressed the critical parameters by using fuzzy theory Zadeh [40], Liu and Liu [18] to construct fuzzy DEA models to express relative efficiencies of DMUs. For example, Sengupta [23] considered a fuzzy linear programming transformation to cope with DEA models with fuzzy input and output data. Triantis and Girod [24] suggested a mathematical programming approach to converting fuzzy input and output data into crisp data. Kao and Liu [15] employed the α-cut approach and presented a transformation of a fuzzy DEA model into a family of crisp DEA models, while Guo and Tanaka [22] advocated a method that changed a fuzzy DEA model to a bi-level linear programming model. Wen and Li [26] proposed a credibility DEA model and a ranking method in fuzzy environment. Wen *et al.* [27] discussed a technique for assessing the sensitivity of efficiency and inefficiency classification in DEA with fuzzy data. Wang and Chin [28] introduced a fuzzy expected value approach for data envelopment analysis in which fuzzy inputs and fuzzy outputs are first weighted, respectively, and used their expected values to measure the optimistic and pessimistic efficiencies of DMUs. Zerafat Angiz *et al.* [29] offered the definition of "local α-level" to develop a multi-objective linear DEA model to measure the efficiency of DMUs under uncertainty. Meng [30] proposed a satisficing DEA model with credibility criterion, and solved it by integrating approximation method, neural network and PSO algorithm. Based on hybrid simulated annealing algorithm, Feng *et al.* [31] studied the input-oriented and the output-oriented fuzzy DEA models. Murena *et al.* [20] provided a generalized fuzzy DEA model to improve numerous deficiencies of the fuzzy DEA. Dotoli *et al.* [12] offered a cross-efficiency fuzzy DEA technique and then defuzzified the results to provide a ranking of the DMUs. Ghasemi *et al.* [32] used the concept of expected value in generalized DEA model to realize the unification of fuzzy expected CCR, fuzzy expected BCC, and fuzzy expected FDH models as well as to handle both symmetrical and asymmetrical fuzzy numbers. Egilmez *et al.* [13] adopted a fuzzy DEA model coupled with an input-output-based life cycle assessment approach to perform a sustainability performance assessment of food manufacturing sectors.

The majority of existing literature devotes to measuring the efficiency of DMUs under the consideration of the fuzzy disturbance with fixed possibility distribution. However, some inputs and outputs in fuzzy DEA might be affected by various factors of uncertainty and information granularity. It is acknowledged that type-2 fuzzy sets Zadeh [38] demonstrated their advantages of modeling better this kind of uncertain data or imprecise information. Different from the set-based viewpoint, Liu and Liu [33] adopted a variable-based approach to depict type-2 fuzzy phenomenon and presented the fuzzy possibility theory which is a generalization of the possibility theory. Furthermore, four kinds of major reduction methods were defined in fuzzy possibility theory to reduce the uncertainty embedded in the secondary possibility distributions. Qin *et al.* [34, 35] gave the mean value reduction method by Choquet fuzzy integrals of regular fuzzy variables and the critical value reduction method by Sugeno fuzzy integrals of regular fuzzy variables. Wu and Liu [37] defined the equivalent value reduction method *via* classic Lebesgue-Stieltjes integrals of regular fuzzy variables. Bai and Liu [5,6] introduced the value-at-risk (VaR) reduction method based on the VaRs of regular fuzzy variables. For the recent development and other application of fuzzy possibility theory, we refer the reader to Bai [3], Bai and Liu [7] for detailed discussion. The theoretical development mentioned above paves a way for many scholars to study data envelopment analysis problem from a new perspective. Qin *et al.* [34, 35] built two classes of DEA models with type-2 fuzzy inputs and outputs in place of the risk-neutral and risk-averse criteria, respectively. Zhou *et al.* [39] developed a multi-objective DEA model in a setting of type-2 fuzzy modeling and employed it to evaluate and select the most appropriate sustainable suppliers.

In real-world DEA problem, the inputs or outputs are not crisp or the secondary possibility distribution functions for data are not clear, the parametric interval-valued fuzzy variables will be recommended. In fuzzy possibility theory, the parametric interval-valued fuzzy variables were investigated, and lambda selection variables were used to describe the corresponding interval-valued secondary possibility distributions Liu and Liu [36]. The notions about parametric interval-valued fuzzy variables and its lambda selection variables are helpful to deal with various decision-making problems, especially when the distribution of uncertain parameters are partially known. However, there is little research for modeling fuzzy DEA problem by virtue of interval type-2 fuzzy theory. This motivates us to consider how to evaluate the relative efficiency of DMUs in the case that the inputs and outputs are estimated as the parametric interval-valued fuzzy variables. In this research we are particularly interested in delivering a new type-2 fuzzy DEA model on the basis of parametric interval-valued fuzzy variables and its lambda selection variables to characterize the uncertain inputs and outputs. This paper intends to make the following contributions to the growing body of the DEA literature. First, an advanced tool – parametric interval-valued fuzzy variable is introduced into the DEA model. Therefore, imprecise information associated with the inputs and outputs of DEA can be well described. Second, a fuzzy expectation DEA model is articulated in a framework of type-2 interval-valued fuzzy variables. Third, the numeric equivalent transformation of the objective function and credibility conditions for the proposed model are discussed in case that the uncertain inputs and outputs are mutually independent interval-valued triangular fuzzy variables, which reduces computational complexity and makes our feasible domain decomposition procedure more understandable. Finally, we provide an application example about sustainable supplier evaluation and selection to demonstrate the effectiveness of the proposed optimization model.

The rest of this paper is organized as follows. Section 2 formulates a new class of fuzzy DEA model where the inputs and outputs are characterized by lambda selection variables of parametric interval-valued fuzzy variables. Section 3 focuses on the deterministic equivalent expressions to the expectation objective function and credibility constraint conditions. Taking the structural characteristics of the equivalent optimization model, a domain decomposition method is designed. To apply the proposed approach, we present an example of supplier evaluation and selection in Section 4. Finally, Section 5 gives the conclusions and future researches.

2. Formulation of fuzzy data envelopment analysis model

In this section, we review the conventional DEA model established by [8] and elaborate on the development of the parametric interval-valued fuzzy DEA model consequently.

DEA is a method for assessing the productive efficiency of DMUs which use the same type of resources (inputs) to produce the same kind of goods or services (outputs). The classical CCR model was given as

$$\begin{aligned} \max_{u,v} \ & \frac{v^T y_0}{u^T x_0} \\ \text{s.t.} \ & \frac{v^T y_i}{u^T x_i} \le 1, i = 1, 2, \ldots, n \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0, \end{aligned} \tag{2.1}$$

where n is the number of DMUs; DMU_i is the ith DMU, $i = 1, 2, \ldots, n$; DMU_0 is the target DMU; x_i represents the input column vector of DMU_i; x_0 represents the input column vector of DMU_0; y_i represents the output column vector of DMU_i; y_0 represents the output column vector of DMU_0; $u \in \Re^p$ is the weights of the input column vector, and $v \in \Re^q$ is the weights of the output column vector.

CCR model (2.1) is helpful to evaluate the efficiency of each DMU when the inputs and outputs data are real numbers. However, uncertain information and imprecise data are highly involved in many practical DEA models. As pointed out by Zhou *et al.* [39], it is more reasonable to treat some critical parameters as type-2 fuzzy variables because of the real difficulties of determining their numeric membership functions. With this concern in mind, type-2 interval-valued fuzzy variables are incorporated into model (2.1). Thus, CCR model (2.1) is developed as follows:

$$\begin{aligned}
\max_{u,v} \quad & \frac{v^T\tilde{\eta}_0}{u^T\tilde{\xi}_0} \\
\text{s.t.} \quad & \frac{v^T\tilde{\eta}_i}{u^T\tilde{\xi}_i} \leq 1, i = 1,\ldots,n \\
& u \geq 0, u \neq 0 \\
& v \geq 0, v \neq 0,
\end{aligned} \tag{2.2}$$

where $\tilde{\xi}_i$ represents the type-2 interval-valued fuzzy input column vector of DMU_i; $\tilde{\xi}_0$ represents the type-2 interval-valued fuzzy input column vector of DMU_0; $\tilde{\eta}_i$ represents the type-2 interval-valued fuzzy output column vector of DMU_i; $\tilde{\eta}_0$ represents the type-2 interval-valued fuzzy output column vector of DMU_0.

Since $\tilde{\xi}_i$ and $\tilde{\eta}_i$, $i = 1, 2, \ldots, n$, are type-2 interval-valued fuzzy variables, the objective function and constraint conditions don't have any specific mathematical meaning. In order to cope with this type-2 interval-valued fuzzy DEA model, we formulate two design steps:

Step 1. Transform the type-2 fuzzy DEA model into a fuzzy DEA model. This step is elaborated in Section 2.1, where the lambda selection method is employed.
Step 2. Take the expectation of fuzzy variable as the optimization objective and credibility of fuzzy event as the constraint condition to formulate a DEA model (Sect. 2.2).

2.1. Lambda selection method

Type-2 fuzzy variables were proposed by Liu and Liu [33] as a conceptional extension of type-2 fuzzy sets. Furthermore, Liu and Liu [36] defined the lower selection variable, upper selection variable and lambda selection variable for handling interval-valued fuzzy variable with variable lower and upper possibility distributions. In the following context, the parametric interval-valued triangular fuzzy variable and its λ selection variable will be used.

Definition 2.1 Liu and Liu [36]**.** Let $r_1 < r_2 < r_3$ be real numbers. Then a map $\tilde{\xi}$ is called a parametric interval-valued triangular fuzzy variable if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is the following subinterval

$$\left[\frac{x-r_1}{r_2-r_1} - \theta_l \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}, \frac{x-r_1}{r_2-r_1} + \theta_r \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}\right]$$

for any $x \in [r_1, r_2]$, and the next subinterval

$$\left[\frac{r_3-x}{r_3-r_2} - \theta_l \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{r_3-x}{r_3-r_2}\right\}, \frac{r_3-x}{r_3-r_2} + \theta_r \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}\right]$$

for any $x \in [r_2, r_3]$, where $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\xi}$ takes the value x. We denote the parametric interval-valued triangular fuzzy variable $\tilde{\xi}$ with the above distribution by $[\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r]$.

Definition 2.2. Liu and Liu [36] Assume that $\tilde{\xi}$ is a parametric interval-valued fuzzy variable with the secondary possibility distribution $\tilde{\mu}_{\xi}(x) = [\mu_{\xi^L}(x;\theta_l), \mu_{\xi^U}(x;\theta_r)]$, where $\mu_{\xi^L}(x;\theta_l)$ is the lower parametric possibility distribution of the lower selection ξ^L of $\tilde{\xi}$, and $\mu_{\xi^U}(x;\theta_r)$ is the upper parametric possibility distribution of the upper selection ξ^U of $\tilde{\xi}$. For any $\lambda \in [0, 1]$, a fuzzy variable ξ is called a λ selection of $\tilde{\xi}$ provided that ξ is characterized by the following parametric possibility distribution

$$\mu_{\xi}(x;\theta,\lambda) = (1-\lambda)\mu_{\xi^L}(x;\theta_l) + \lambda\mu_{\xi^U}(x;\theta_r), \theta = (\theta_l, \theta_r).$$

TABLE 1. List of notations for model (2.3).

Notations	Implication
Fuzzy variables:	
ξ_i	the λ selection variable of $\tilde{\xi}_i$, $i = 1, 2, \dots, n$;
ξ_0	the λ selection variable of $\tilde{\xi}_0$;
η_i	the λ selection variable of $\tilde{\eta}_i$, $i = 1, 2, \dots, n$;
η_0	the λ selection variable of $\tilde{\eta}_0$;
Decision variables:	
$u \in \Re^p$	the weights of the input column vector ξ_i;
$v \in \Re^q$	the weights of the output column vector η_i.

Theorem 2.3. *Let $\tilde{\xi} = [\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r]$ be a parametric interval-valued triangular fuzzy variable. If we denote $\theta = (\theta_l, \theta_r)$, then its λ selection fuzzy variable ξ has the following parametric possibility distribution*

$$\mu_\xi(x;\theta,\lambda) = \begin{cases} (1+\lambda\theta_r - (1-\lambda)\theta_l)\dfrac{x - r_1}{r_2 - r_1}, & \text{if } x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{(1-\lambda\theta_r + (1-\lambda)\theta_l)x + (\lambda\theta_r - (1-\lambda)\theta_l)r_2 - r_1}{r_2 - r_1}, & \text{if } x \in \left(\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{(-1+\lambda\theta_r - (1-\lambda)\theta_l)x - (\lambda\theta_r - (1-\lambda)\theta_l)r_2 + r_3}{r_3 - r_2}, & \text{if } x \in \left(r_2, \dfrac{r_2+r_3}{2}\right] \\ (1+\lambda\theta_r - (1-\lambda)\theta_l)\dfrac{r_3 - x}{r_3 - r_2}, & \text{if } x \in \left(\dfrac{r_2+r_3}{2}, r_3\right]. \end{cases}$$

On the basis of Theorem 2.3, we put forward a following model with λ selection fuzzy variable:

$$\begin{aligned} \max_{u,v} \quad & \frac{v^T\eta_0}{u^T\xi_0} \\ \text{s.t.} \quad & v^T\eta_i - u^T\xi_i \leq 0, i = 1, \dots, n \\ & u \geq 0, u \neq 0 \\ & v \geq 0, v \neq 0, \end{aligned} \tag{2.3}$$

where the notations are listed in Table 1.

2.2. Expectation DEA model with credibility constraints

Once lambda selection method has been completed, the parametric interval-valued fuzzy input $\tilde{\xi}_i$ and output $\tilde{\eta}_i$ can be represented by their λ selection variables ξ_i and η_i. Model (2.3) is not well-defined since the meanings of "max" in the objective as well as constraint conditions are not clear at all. In order to construct a specific model with mathematical meaning, a revision of the modeling process is necessary. On the one hand, the expected value operator of fuzzy variables, which was proposed by Liu and Liu [18], will be employed to formulate the objective function. On the other hand, we adopt the idea of chance constrained programming and use the credibility measure to address the constraint conditions. Thus, a fuzzy expectation DEA model with credibility constraints can be given as follows:

$$\begin{aligned} \max_{u,v} \quad & V = \mathrm{E}\left[\frac{v^T\eta_0}{u^T\xi_0}\right] \\ \text{s.t.} \quad & \mathrm{Cr}\{v^T\eta_i - u^T\xi_i \leq 0\} \geq \alpha_i, i = 1, \dots, n \\ & u \geq 0, u \neq 0 \\ & v \geq 0, v \neq 0. \end{aligned} \tag{2.4}$$

Model (2.4) wants to seek a optimal weight vector (u,v) with the maximum value of $\mathrm{E}[v^T\eta_0/u^T\xi_0]$, while the credibility of the fuzzy event $\{v^T\eta_i - u^T\xi_i \le 0\}$ is satisfied at least α_i.

In the traditional CCR model (2.1), the value of v^Ty_0/u^Tx_0 is used to measure the relative efficiency of DMU_0. DMU_0 is efficient if and only if the optimal value is equal to 1 and there exists at least one optimal solution (u^*,v^*) with $u^*>0, v^*>0$.

Due to the existed uncertainty, we give the definition of mean efficiency value as follows.

Definition 2.4. In the expectation DEA model (2.4), the value $\mathrm{E}[v^T\eta_0/u^T\xi_0]$ is used to measure the mean efficiency value of DMU_0. DMU_0 is efficient if and only if the optimal value $V^* = \mathrm{E}[v^T\eta_0/u^T\xi_0]$ is the biggest one among the objective values of all DMUs.

Remark 2.5. In order to show that Definition 2.4 generalizes the efficiency value of traditional CCR model (2.1), we explain the reason as follows. Consider the case that $\alpha_i = 1$ $(i=1,2,\ldots,n)$. Namely, $v^T\eta_i - u^T\xi_i \le 0$ holds almost surely. In this case, $\mathrm{E}[v^T\eta_0/u^T\xi_0]=1$ is equivalent to $\mathrm{Cr}\{v^T\eta_0/u^T\xi_0 = 1\}=1$. That is to say, the input fuzzy variable $u^T\xi_0$ is equal to the output fuzzy variable $v^T\eta_0$ almost sure. As a matter of fact, it is very difficult to find such an efficient solution (u,v) for practical DEA model. When the input and output parameters are the crisp numbers, it is evident to the equation $\mathrm{Cr}\{v^T\eta_0/u^T\xi_0 = 1\}=1$ holds. So, if the input and output fuzzy variables reduce to the deterministic parameters, the mean efficiency value is that of traditional CCR model.

3. Model analysis and solution method

3.1. Equivalent transformation

To solve model (2.4), it is essential to devote research effort to compute the expectations of fuzzy variables in the objective and credibility of fuzzy event in the constraints. For convenience, some special cases are discussed, where the inputs and outputs are characterized by parametric interval-valued triangular fuzzy variables, and consequently the equivalent form to further simplify the objective and constraints in model (2.4) is deduced. The related theorems are proposed as well. For the sake of presentation, all technical details and proofs in this section are provided in Appendix A.

First of all, let us consider the analytical expression of objective function, *i.e.*, $\mathrm{E}\left[(v^T\eta_0)/(u^T\xi_0)\right]$.

Theorem 3.1. *Let $\xi_{j,0}$ and $\eta_{k,0}$ be the λ selection of the parametric interval-valued fuzzy inputs $\tilde{\xi}_{j,0} = [\xi_{j,0}^{r_1}, \xi_{j,0}^{r_2}, \xi_{j,0}^{r_3}; \theta_{l,j,0}, \theta_{r,j,0}]$ and outputs $\tilde{\eta}_{k,0} = [\eta_{k,0}^{r_1}, \eta_{k,0}^{r_2}, \eta_{k,0}^{r_3}; \overline{\theta}_{l,k,0}, \overline{\theta}_{r,k,0}]$, $j=1,2,\ldots,p$, $k=1,2,\ldots,q$. Suppose $\{\tilde{\xi}_{j,0}\}$ and $\{\tilde{\eta}_{k,0}\}$ are mutually independent, $\theta_{r,j,0}-\theta_{l,j,0} = \overline{\theta}_{r,k,0}-\overline{\theta}_{l,k,0} = \theta_{r,0}-\theta_{l,0}$, and $\lambda_{\xi_{j,0}} = \lambda_{\eta_{k,0}} = \lambda$. Then the expectation objective function of model (2.4) is equivalent to*

$$
\begin{aligned}
\mathrm{E}\left[\frac{v^T\eta_0}{u^T\xi_0}\right] &= \mathrm{E}\left[\frac{\sum_{k=1}^{q} v_k\eta_{k,0}}{\sum_{j=1}^{p} u_j\xi_{j,0}}\right] \\
&= \frac{\sum_{k=1}^{q}(\eta_{k,0}^{r_2}-\eta_{k,0}^{r_1})v_k}{2\sum_{j=1}^{p}(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_3})u_j} + \frac{\sum_{j=1}^{p}\xi_{j,0}^{r_2}u_j\sum_{k=1}^{q}\eta_{k,0}^{r_1}v_k - \sum_{j=1}^{p}\xi_{j,0}^{r_3}u_j\sum_{k=1}^{q}\eta_{k,0}^{r_2}v_k}{2\left(\sum_{j=1}^{p}(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_3})u_j\right)^2} \\
&\quad \times \left(2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln\sum_{j=1}^{p}(\xi_{j,0}^{r_2}+\xi_{j,0}^{r_3})u_j + (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2\sum_{j=1}^{p}\xi_{j,0}^{r_2}u_j\right.
\end{aligned}
$$

$$
\begin{aligned}
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2\sum_{j=1}^{p}\xi_{j,0}^{r_3}u_j\Bigg) \qquad (3.1)\\
&+\frac{\sum_{k=1}^{q}(\eta_{k,0}^{r_2}-\eta_{k,0}^{r_3})v_k}{2\sum_{j=1}^{p}(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_1})u_j}+\frac{\sum_{j=1}^{p}\xi_{j,0}^{r_2}u_j\sum_{k=1}^{q}\eta_{k,0}^{r_3}v_k-\sum_{j=1}^{p}\xi_{j,0}^{r_1}u_j\sum_{k=1}^{q}\eta_{k,0}^{r_2}v_k}{2\left(\sum_{j=1}^{p}(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_1})u_j\right)^2}\\
&\times\Bigg(2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln\sum_{j=1}^{p}(\xi_{j,0}^{r_1}+\xi_{j,0}^{r_2})u_j-(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2\sum_{j=1}^{p}\xi_{j,0}^{r_1}u_j\\
&+ (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2\sum_{j=1}^{p}\xi_{j,0}^{r_2}u_j\Bigg).
\end{aligned}
$$

In the following, let us deal with the analytical expressions of credibility constraint, *i.e.*,

$$\mathrm{Cr}\{v^T\eta_i-u^T\xi_i\le 0\}\ge\alpha_i, i=1,\dots,n.$$

Theorem 3.2. *Let $\xi_{j,i}$ and $\eta_{k,i}$ be the λ selection of the parametric interval-valued fuzzy inputs $\tilde{\xi}_{j,i}=[\xi_{j,i}^{r_1},\xi_{j,i}^{r_2},\xi_{j,i}^{r_3};\theta_{l,j,i},\theta_{r,j,i}]$ and outputs $\tilde{\eta}_{k,i}=[\eta_{k,i}^{r_1},\eta_{k,i}^{r_2},\eta_{k,i}^{r_3};\overline{\theta}_{l,k,i},\overline{\theta}_{r,k,i}]$, $j=1,2,\dots,p$, $k=1,2,\dots,q$. Suppose $\{\tilde{\xi}_{j,i}\}$ and $\{\tilde{\eta}_{k,i}\}$ are mutually independent, $\lambda_{\xi_{j,i}}=\lambda_{\eta_{k,i}}=\lambda$, and $\lambda\overline{\theta}_{r,1,i}-(1-\lambda)\overline{\theta}_{l,1,i}\le\lambda\overline{\theta}_{r,2,i}-(1-\lambda)\overline{\theta}_{l,2,i}\le\ldots\le\lambda\overline{\theta}_{r,q,i}-(1-\lambda)\overline{\theta}_{l,q,i}\le\lambda\theta_{r,1,i}-(1-\lambda)\theta_{l,1,i}\le\lambda\theta_{r,2,i}-(1-\lambda)\theta_{l,2,i}\le\ldots\le\lambda\theta_{r,p,i}-(1-\lambda)\theta_{l,p,i}$.*

(i) *If $\alpha_i\in(0,(1+\lambda\overline{\theta}_{r,1,i}-(1-\lambda)\overline{\theta}_{l,1,i})/4)$, then the credibility constraint of model (2.4) is equivalent to*

$$
\begin{aligned}
&\sum_{k=1}^{q}v_k\frac{(1-2\alpha_i+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_1}+2\alpha_i\eta_{k,i}^{r_2}}{1+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i}}\\
&-\sum_{j=1}^{p}u_j\frac{2\alpha_i\xi_{j,i}^{r_2}+(1-2\alpha_i+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i}}\le 0.
\end{aligned}
$$

(ii) *If there exists a k_0, $1\le k_0<q$ such that $\alpha_i\in[(1+\lambda\overline{\theta}_{r,k_0,i}-(1-\lambda)\overline{\theta}_{l,k_0,i})/4,(1+\lambda\overline{\theta}_{r,k_0+1,i}-(1-\lambda)\overline{\theta}_{l,k_0+1,i})/4)$, then the credibility constraint of model (2.4) is equivalent to*

$$
\begin{aligned}
&\sum_{k=1}^{k_0}v_k\frac{(1-2\alpha_i)\eta_{k,i}^{r_1}+(2\alpha_i-\lambda\overline{\theta}_{r,k,i}+(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1-\lambda\overline{\theta}_{r,k,i}+(1-\lambda)\overline{\theta}_{l,k,i}}\\
&\sum_{k=k_0+1}^{q}v_k\frac{(1-2\alpha_i+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_1}+2\alpha_i\eta_{k,i}^{r_2}}{1+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i}}\\
&-\sum_{j=1}^{p}u_j\frac{2\alpha_i\xi_{j,i}^{r_2}+(1-2\alpha_i+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i}}\le 0.
\end{aligned}
$$

(iii) *If there exists a j_0, $1\le j_0<p$ such that $\alpha_i\in[(1+\lambda\theta_{r,j_0,i}-(1-\lambda)\theta_{l,j_0,i})/4,(1+\lambda\theta_{r,j_0+1,i}-(1-\lambda)\theta_{l,j_0+1,i})/4)$, then the credibility constraint of model (2.4) is equivalent to*

$$\sum_{k=1}^{q}v_k\frac{(1-2\alpha_i)\eta_{k,i}^{r_1}+(2\alpha_i-\lambda\overline{\theta}_{r,k,i}+(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1-\lambda\overline{\theta}_{r,k,i}+(1-\lambda)\overline{\theta}_{l,k,i}}$$

$$-\sum_{j=1}^{j_0} u_j \frac{(2\alpha_i - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2} + (1-2\alpha_i)\xi_{j,i}^{r_3}}{1 - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i}}$$
$$- \sum_{j=j_0+1}^{p} u_j \frac{2\alpha_i\xi_{j,i}^{r_2} + (1-2\alpha_i + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \leq 0.$$

(iv) *If* $\alpha_i \in [(1 + \lambda\theta_{r,p,i} - (1-\lambda)\theta_{l,p,i})/4, 0.5)$, *then the credibility constraint of model* (2.4) *is equivalent to*

$$\sum_{k=1}^{q} v_k \frac{(1-2\alpha_i)\eta_{k,i}^{r_1} + (2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2} + (1-2\alpha_i)\xi_{j,i}^{r_3}}{1 - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i}} \leq 0.$$

(v) *If* $\alpha_i \in [0.5, (3 - \lambda\theta_{r,p,i} + (1-\lambda)\theta_{l,p,i})/4)$, *then the credibility constraint of model* (2.4) *is equivalent to*

$$\sum_{k=1}^{q} v_k \frac{(2 - 2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2} + (2\alpha_i - 1)\eta_{k,i}^{r_3}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - 1)\xi_{j,i}^{r_1} + (2 - 2\alpha_i - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2}}{1 - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i}} \leq 0.$$

(vi) *If there exists a* j_0, $1 \leq j_0 < p$ *such that* $\alpha_i \in [(3-\lambda\theta_{r,j_0+1,i}+(1-\lambda)\theta_{l,j_0+1,i})/4, (3-\lambda\theta_{r,j_0,i}+(1-\lambda)\theta_{l,j_0,i})/4)$, *then the credibility constraint of model* (2.4) *is equivalent to*

$$\sum_{k=1}^{q} v_k \frac{(2 - 2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2} + (2\alpha_i - 1)\eta_{k,i}^{r_3}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{j_0} u_j \frac{(2\alpha_i - 1)\xi_{j,i}^{r_1} + (2 - 2\alpha_i - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2}}{1 - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i}}$$
$$-\sum_{j=j_0+1}^{p} u_j \frac{(2\alpha_i - 1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_1} + (2-2\alpha_i)\xi_{j,i}^{r_2}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \leq 0.$$

(vii) *If there exists a* k_0, $1 \leq k_0 < q$ *such that* $\alpha_i \in [(3 - \lambda\overline{\theta}_{r,k_0+1,i} + (1-\lambda)\overline{\theta}_{l,k_0+1,i})/4, (3 - \lambda\overline{\theta}_{r,k_0,i} + (1-\lambda)\overline{\theta}_{l,k_0,i})/4)$, *then the credibility constraint of model* (2.4) *is equivalent to*

$$\sum_{k=1}^{k_0} v_k \frac{(2 - 2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2} + (2\alpha_i - 1)\eta_{k,i}^{r_3}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$+ \sum_{k=k_0+1}^{q} v_k \frac{(2-2\alpha_i)\eta_{k,i}^{r_2} + (2\alpha_i - 1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_3}}{1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - 1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_1} + (2-2\alpha_i)\xi_{j,i}^{r_2}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \leq 0.$$

(viii) *If* $\alpha_i \in [(3 - \lambda\overline{\theta}_{r,1,i} + (1-\lambda)\overline{\theta}_{l,1,i})/4, 1]$, *then the credibility constraint of model* (2.4) *is equivalent to*

$$\sum_{k=1}^{q} v_k \frac{(2-2\alpha_i)\eta_{k,i}^{r_2} + (2\alpha_i - 1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_3}}{1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i}}$$

$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - 1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_1} + (2-2\alpha_i)\xi_{j,i}^{r_2}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \le 0.$$

3.2. Domain decomposition method

In what follows, assume that $\{\tilde{\xi}_{j,i}\}$ and $\{\tilde{\eta}_{k,i}\}$ are mutually independent, $\lambda_{\xi_{j,i}} = \lambda_{\eta_{k,i}} = \lambda$, and $\lambda\overline{\theta}_{r,1,i} - (1-\lambda)\overline{\theta}_{l,1,i} \le \lambda\overline{\theta}_{r,2,i} - (1-\lambda)\overline{\theta}_{l,2,i} \le \ldots \le \lambda\overline{\theta}_{r,q,i} - (1-\lambda)\overline{\theta}_{l,q,i} \le \lambda\theta_{r,1,i} - (1-\lambda)\theta_{l,1,i} \le \lambda\theta_{r,2,i} - (1-\lambda)\theta_{l,2,i} \le \ldots \le \lambda\theta_{r,p,i} - (1-\lambda)\theta_{l,p,i}$. Let $A = \{(j,k) \mid 0.5 \le \alpha_i < (3 - \lambda\theta_{r,p,i} + (1-\lambda)\theta_{l,p,i})/4\}$, $B = \{(j,k) \mid \exists j_0, 1 \le j_0 < p, (3 - \lambda\theta_{r,j_0+1,i} + (1-\lambda)\theta_{l,j_0+1,i})/4 \le \alpha_i < (3 - \lambda\theta_{r,j_0,i} + (1-\lambda)\theta_{l,j_0,i})/4\}$, $C = \{(j,k) \mid \exists k_0, 1 \le k_0 < q, (3 - \lambda\overline{\theta}_{r,k_0+1,i} + (1-\lambda)\overline{\theta}_{l,k_0+1,i})/4 \le \alpha_i < (3 - \lambda\overline{\theta}_{r,k_0,i} + (1-\lambda)\overline{\theta}_{l,k_0,i})/4\}$, and $D = \{(j,k) \mid (3 - \lambda\overline{\theta}_{r,1,i} + (1-\lambda)\overline{\theta}_{l,1,i})/4 \le \alpha_i < 1\}$. According to Theorem 3.2, if $\alpha_i > 0.5$, then $\mathrm{Cr}\{v^T\eta_i - u^T\xi_i \le 0\} \ge \alpha_i$ is equivalent to

$$\sum_{k=1}^{q} v_k \frac{(2-2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2} + (2\alpha_i - 1)\eta_{k,i}^{r_3}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - 1)\xi_{j,i}^{r_1} + (2-2\alpha_i - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2}}{1 - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i}} \le 0, \quad \text{for } (j,k) \in A,$$

or

$$\sum_{k=1}^{q} v_k \frac{(2-2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2} + (2\alpha_i - 1)\eta_{k,i}^{r_3}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{j_0} u_j \frac{(2\alpha_i - 1)\xi_{j,i}^{r_1} + (2-2\alpha_i - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2}}{1 - \lambda\theta_{r,j,i} + (1-\lambda)\theta_{l,j,i}}$$
$$-\sum_{j=j_0+1}^{p} u_j \frac{(2\alpha_i - 1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_1} + (2-2\alpha_i)\xi_{j,i}^{r_2}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \le 0, \quad \text{for } (j,k) \in B,$$

or

$$\sum_{k=1}^{k_0} v_k \frac{(2-2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2} + (2\alpha_i - 1)\eta_{k,i}^{r_3}}{1 - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$+\sum_{k=k_0+1}^{q} v_k \frac{(2-2\alpha_i)\eta_{k,i}^{r_2} + (2\alpha_i - 1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_3}}{1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - 1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_1} + (2-2\alpha_i)\xi_{j,i}^{r_2}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \le 0, \quad \text{for } (j,k) \in C,$$

or

$$\sum_{k=1}^{q} v_k \frac{(2-2\alpha_i)\eta_{k,i}^{r_2} + (2\alpha_i - 1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_3}}{1 + \lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{(2\alpha_i - 1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_1} + (2-2\alpha_i)\xi_{j,i}^{r_2}}{1 + \lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i}} \le 0, \quad \text{for } (j,k) \in D.$$

For simplicity, we denote the functions on the left hand sides of these inequalities as $G_1(u, v; \theta, \alpha, \lambda)$, $G_2(u, v; \theta, \alpha, \lambda)$, $G_3(u, v; \theta, \alpha, \lambda)$ and $G_4(u, v; \theta, \alpha, \lambda)$, respectively, where $\theta = (\theta_l, \theta_r)$.

Based on Theorem 3.1, we can observe that if the inputs and outputs are described by mutually independent parametric interval-valued triangular fuzzy variables, $\theta_{r,j,0} - \theta_{l,j,0} = \overline{\theta}_{r,k,0} - \overline{\theta}_{l,k,0} = \theta_{r,0} - \theta_{l,0}$, and $\lambda_{\xi_{j,0}} = \lambda_{\eta_{k,0}} = \lambda$, then the model (2.4) is equivalent to the following four parametric programming sub-models

$$\begin{aligned} \max_{u,v} \quad & f_0(u,v;\theta,\lambda) \\ \text{s.t.} \quad & G_1(u,v;\theta,\alpha,\lambda) \le 0, \quad i=1,\ldots,n,(j,k)\in A \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0; \end{aligned} \tag{3.2}$$

$$\begin{aligned} \max_{u,v} \quad & f_0(u,v;\theta,\lambda) \\ \text{s.t.} \quad & G_2(u,v;\theta,\alpha,\lambda) \le 0, \quad i=1,\ldots,n,(j,k)\in B \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0; \end{aligned} \tag{3.3}$$

$$\begin{aligned} \max_{u,v} \quad & f_0(u,v;\theta,\lambda) \\ \text{s.t.} \quad & G_3(u,v;\theta,\alpha,\lambda) \le 0, \quad i=1,\ldots,n,(j,k)\in C \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0; \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \max_{u,v} \quad & f_0(u,v;\theta,\lambda) \\ \text{s.t.} \quad & G_4(u,v;\theta,\alpha,\lambda) \le 0, \quad i=1,\ldots,n,(j,k)\in D \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0, \end{aligned} \tag{3.5}$$

where $f_0(u,v;\theta,\lambda)$ is determined by equation (3.1).

On the basis of Theorems 3.1 and 3.2, we have decomposed the original model (2.4) into four sub-models (3.2)–(3.5). As a result, the feasible region of model (2.4) is decomposed into four disjoint subregions according to the values of parameter α_i. The four subregions are just the feasible regions of sub-models (3.2)–(3.5). From this observation, we know that the global optimal solution of model (2.4) can be obtained by solving sub-models (3.2)–(3.5). The four sub-models (3.2)–(3.5) are the nonlinear parametric programming that can be solved by using conventional optimization algorithms when the parameters vary in their domains. As an example, for any given values of the parameters α,λ and θ, we can make use of LINGO software to solve it. Since we do not know in advance which subregion the global optimal solution locates in, we have to solve all sub-models to find four local optimal solutions of model (2.4). By comparing the objective values of the obtained local optimal solutions, we can find the global optimal solution. This solution procedure is called as *the domain decomposition method*.

Given the values of model parameters θ,α and λ, the solution process described in the above statement can be summarized as follows.

Step 1. Solve sub-models (3.2)–(3.5) by LINGO software, and denote the obtained local optimal solutions as $(u,v)_I$, $I=1,2,3,4$.

Step 2. Compute objective value $V_I(u,v;\theta,\alpha,\lambda)$ at local optimal solution $(u,v)_I$ for $I=1,2,3,4$, and find the global maximum expected value by the following formula

$$V(u,v;\theta,\alpha,\lambda) = \max_{1\le I\le 4} V_I(u,v;\theta,\alpha,\lambda).$$

Step 3. Return (u,v) as the global optimal solution of model (2.4) with the maximum value $V(u,v;\theta,\alpha,\lambda)$.

TABLE 2. The parametric interval-valued triangular fuzzy inputs for five suppliers.

Supplier_i	TC	FC	EC	SC
$i=1$	$[\widetilde{2.6}, \widetilde{3.0}, \widetilde{3.3}; \theta_l, \theta_r]$	$[\widetilde{3.7}, \widetilde{3.9}, \widetilde{4.0}; \theta_l, \theta_r]$	$[\widetilde{3.8}, \widetilde{4.0}, \widetilde{4.2}; \theta_l, \theta_r]$	$[\widetilde{3.3}, \widetilde{3.7}, \widetilde{4.3}; \theta_l, \theta_r]$
$i=2$	$[\widetilde{1.1}, \widetilde{1.5}, \widetilde{1.7}; \theta_l, \theta_r]$	$[\widetilde{1.8}, \widetilde{2.0}, \widetilde{2.1}; \theta_l, \theta_r]$	$[\widetilde{2.0}, \widetilde{2.3}, \widetilde{2.5}; \theta_l, \theta_r]$	$[\widetilde{1.8}, \widetilde{2.0}, \widetilde{2.1}; \theta_l, \theta_r]$
$i=3$	$[\widetilde{2.3}, \widetilde{2.4}, \widetilde{2.5}; \theta_l, \theta_r]$	$[\widetilde{3.0}, \widetilde{3.2}, \widetilde{3.5}; \theta_l, \theta_r]$	$[\widetilde{2.4}, \widetilde{2.7}, \widetilde{2.9}; \theta_l, \theta_r]$	$[\widetilde{2.7}, \widetilde{2.8}, \widetilde{3.0}; \theta_l, \theta_r]$
$i=4$	$[\widetilde{1.6}, \widetilde{1.8}, \widetilde{1.9}; \theta_l, \theta_r]$	$[\widetilde{2.2}, \widetilde{2.3}, \widetilde{2.5}; \theta_l, \theta_r]$	$[\widetilde{2.7}, \widetilde{2.8}, \widetilde{3.0}; \theta_l, \theta_r]$	$[\widetilde{3.3}, \widetilde{3.4}, \widetilde{3.6}; \theta_l, \theta_r]$
$i=5$	$[\widetilde{3.4}, \widetilde{3.5}, \widetilde{3.7}; \theta_l, \theta_r]$	$[\widetilde{2.2}, \widetilde{2.5}, \widetilde{2.8}; \theta_l, \theta_r]$	$[\widetilde{3.8}, \widetilde{4.0}, \widetilde{4.2}; \theta_l, \theta_r]$	$[\widetilde{3.9}, \widetilde{4.1}, \widetilde{4.3}; \theta_l, \theta_r]$

TABLE 3. The parametric interval-valued triangular fuzzy outputs for five suppliers.

Supplier_i	NOT	NB	PQ	SR
$i=1$	$[\widetilde{5.4}, \widetilde{5.5}, \widetilde{5.7}; \theta_l, \theta_r]$	$[\widetilde{5.2}, \widetilde{5.3}, \widetilde{5.5}; \theta_l, \theta_r]$	$[\widetilde{6.0}, \widetilde{6.2}, \widetilde{6.4}; \theta_l, \theta_r]$	$[\widetilde{4.8}, \widetilde{5.0}, \widetilde{5.5}; \theta_l, \theta_r]$
$i=2$	$[\widetilde{4.0}, \widetilde{4.2}, \widetilde{4.3}; \theta_l, \theta_r]$	$[\widetilde{4.2}, \widetilde{4.4}, \widetilde{4.7}; \theta_l, \theta_r]$	$[\widetilde{5.0}, \widetilde{5.1}, \widetilde{5.3}; \theta_l, \theta_r]$	$[\widetilde{3.1}, \widetilde{3.3}, \widetilde{3.4}; \theta_l, \theta_r]$
$i=3$	$[\widetilde{4.2}, \widetilde{4.5}, \widetilde{4.7}; \theta_l, \theta_r]$	$[\widetilde{4.0}, \widetilde{4.1}, \widetilde{4.3}; \theta_l, \theta_r]$	$[\widetilde{5.3}, \widetilde{5.5}, \widetilde{5.6}; \theta_l, \theta_r]$	$[\widetilde{3.0}, \widetilde{3.3}, \widetilde{3.5}; \theta_l, \theta_r]$
$i=4$	$[\widetilde{4.1}, \widetilde{4.3}, \widetilde{4.4}; \theta_l, \theta_r]$	$[\widetilde{3.8}, \widetilde{4.0}, \widetilde{4.2}; \theta_l, \theta_r]$	$[\widetilde{5.2}, \widetilde{5.3}, \widetilde{5.5}; \theta_l, \theta_r]$	$[\widetilde{3.5}, \widetilde{3.7}, \widetilde{3.9}; \theta_l, \theta_r]$
$i=5$	$[\widetilde{5.5}, \widetilde{5.8}, \widetilde{6.0}; \theta_l, \theta_r]$	$[\widetilde{5.1}, \widetilde{5.5}, \widetilde{5.8}; \theta_l, \theta_r]$	$[\widetilde{6.6}, \widetilde{6.7}, \widetilde{6.9}; \theta_l, \theta_r]$	$[\widetilde{5.0}, \widetilde{5.4}, \widetilde{5.7}; \theta_l, \theta_r]$

4. One numerical example

In this section, we consider a sustainable supplier evaluation and selection example to demonstrate the performance of fuzzy DEA with robust inputs and outputs.

4.1. Problem description

Sustainable supplier evaluation and selection plays an important role in establishing an effective supply chain management, and has received more and more consideration from corporate and academic over the past decade. In order to evaluate the supplier's sustainability, it is necessary to choose some representative selection criteria. Based on the existing research, the typical used selection criteria include the economic, environmental and social criteria, which are usually the cardinal inputs. Specifically, the economic criteria are composed of technology capability (TC) and financial capability (FC). The environmental criteria mainly takes environmental cost (EC) into consideration. The social criteria is the cost of work safety and labor health (SC). Generally, there are four output criteria involved in supplier evaluation and selection. They contain the number of shipments to arrive on time (NOT), the number of bills received from suppliers without errors (NB), product quality (PQ) and supplier reputation (SR), which reflect the suppliers' ability to some extent. If the supplier's ability is stronger, the more we are willing to choose it. With these inputs and outputs, the decision makers can quantify the performance when selecting sustainable suppliers. Nevertheless, because of the uncertainty of decision making process, we explicitly introduce the parametric interval-valued fuzzy variables to capture the inaccuracy of inputs and outputs on the basis of empirical estimates. Tables 2 and 3 provide the inputs and outputs for five suppliers. Furthermore, suppose that $\alpha_1 = \ldots = \alpha_5 = \alpha$ in this system. Now there is an enterprise with the need of choosing the optimal supplier. The enterprise wishes to evaluate 5 candidates involved in the assessment and choose the most sustainable raw materials suppliers.

4.2. Computational results

When model parameters are set to $(\theta_l, \theta_r) = (0.5, 0.5)$, $\lambda = 0.78$ and $\alpha = 0.95$, we can find that $A = B = C = \varnothing$ and all the (j,k)s belong to set D. According to the domain decomposition method, DEA model (2.4)

TABLE 4. Evaluating results of each supplier with $(\theta_l, \theta_r) = (0.5, 0.5)$ and $\lambda = 0.78$ under $\alpha = 0.95$.

Supplier$_i$	Optimal solution $(u; v)$	MEV
$i = 1$	(0.0000,0.0000,1.5781,0.0000;0.0000,0.0000,0.0000,0.9413)	0.7601990
$i = 2$	(0.0000,7335.7,0.0000,2593.5;4200.1,0.0000,0.0000,0.0000)	0.8981914
$i = 3$	(0.0000,0.0000,4.8972,0.0000;2.3086,0.0000,0.0000,0.0000)	0.7935827
$i = 4$	(3.0490,5.6678,0.0000,0.0000;0.0000,0.0000,0.0000,4.0504)	0.8086242
$i = 5$	(0.0000,3.6838,6.9882,0.0000;0.0000,0.0000,0.0000,6.1402)	0.8907104

of the supplier selection system can be built as follows

$$\begin{array}{ll} \max\limits_{u,v} & f_0(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \\ \text{s.t.} & g_i(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \leq 0, i = 1, \ldots, 5 \\ & u \geq 0, u \neq 0 \\ & v \geq 0, v \neq 0, \end{array} \tag{4.1}$$

where the objective function for each supplier, $i = 1, \ldots, 5$, is modeled as the target supplier, denoted as supplier$_0$. The analytical expressions for the objective functions and constraint conditions in model (4.1) are found in Appendix B.

With the calculation of LINGO software, the evaluating results of each supplier are reported in Table 4. The mean efficiency value of every supplier is listed in the column labeled "MEV". The results show supplier$_2$ has the biggest mean efficiency value 0.8981914, followed by supplier$_5$ and supplier$_4$, which shows that supplier$_2$ is the most efficient supplier.

In order to identify parameters' influence on evaluating results, the optimal mean efficiency values are calculated by adjusting independently the values of parameters (θ_l, θ_r) and λ.

Case I. The influence of parameter θ

When fixing the parameters α and λ and changing the value of parameter (θ_l, θ_r), the computational results are displayed in Table 5. From the evaluating results in Table 5, it is obvious that for each supplier the efficiency values change as θ_l and θ_r vary between 0 and 1. Moreover, the efficiency values increase while the differences of $\theta_r - \theta_l$ decrease.

Case II. The influence of parameter λ

When fixing the parameters α and (θ_l, θ_r) and changing the value of parameter λ, the corresponding computational results are displayed in Table 6. For each supplier, the mean efficiency values gradually decrease with respect to the values of λ in the interval [0,1]. The computational results demonstrate the advantages of parametric interval-valued fuzzy variables and lambda selection method in assessing the supplier performance.

4.3. Comparative studies

In order to demonstrate the advantage of using parametric interval-valued fuzzy inputs and outputs, we compare it with both the deterministic DEA method and type-1 fuzzy DEA method. In the deterministic DEA model, the inputs and outputs are the crisp numbers. In the fuzzy DEA model, the inputs and outputs are triangular fuzzy variables with fixed possibility distributions.

Case I. Optimal supplier selection decision under deterministic data

We first solve DEA model with deterministic input and output data. For the sake of comparison, the deterministic input and output data in Table 7 are the mean values of triangular fuzzy variables collected in Tables 9 and 10.

TABLE 5. Evaluating results of each supplier with different (θ_l, θ_r) under $\alpha = 0.95$ and $\lambda = 0.5$.

(θ_l, θ_r)	Supplier$_i$	Optimal solution $(u; v)$	MEV
$(0.2, 0.8)$	$i = 1$	(0.0000,0.0000,1.5787,0.0000;0.0000,0.0000,0.0000,0.9415)	0.7601711
	$i = 2$	(0,0.324E+15,0,0.4819E+14;0.7315E+14,0,0.6844E+14,776.2209)	0.8974317
	$i = 3$	(0.0000,0.0000,4.9040,0.0000;2.3114,0.0000,0.0000,0.0000)	0.7935049
	$i = 4$	(3.0454,5.6630,0.0000,0.0000;0.0000,0.0000,0.0000,4.0457)	0.8084473
	$i = 5$	(0.0000,3.6996,7.0142,0.0000;0.0000,0.0000,0.0000,6.1629)	0.8905694
$(0.4, 0.7)$	$i = 1$	(0.0000,0.0000,1.5746,0.0000;0.0000,0.0000,0.0000,0.9408)	0.7605170
	$i = 2$	(0.0000,6812.211,0.0000,2514.977;3950.116,0.0000,0.0000,0.0000)	0.8986081
	$i = 3$	(0.0000,0.0000,4.8551,0.0000;2.2923,0.0000,0.0000,0.0000)	0.7942259
	$i = 4$	(4636.6,8595.9,0.0000,0.0000;0.0000,0.0000,0.0000,6158.0)	0.8099296
	$i = 5$	(0.0000,3.5575,6.7770,0.0000;0.0000,0.0000,0.0000,5.9553)	0.8917459
$(0.6, 0.5)$	$i = 1$	(0.0000,0.0000,1.5697,0.0000;0.0000,0.0000,0.0000,0.9408)	0.7616389
	$i = 2$	(0.0000,24506.19,0.0000,6496.42;13161.84,0.0000,0.0000,0.0000)	0.8998124
	$i = 3$	(0.0000,0.0000,4.8019,0.0000;2.2743,0.0000,0.0000,0.0000)	0.7958554
	$i = 4$	(4633.002,8541.693,0.0000,0.0000;0.0000,0.0000,0.0000,6150.291)	0.8126613
	$i = 5$	(0.0000,3.2699,6.2839,0.0000;0.0000,0.0000,0.0000,5.5233)	0.8938884
$(0.8, 0.2)$	$i = 1$	(0.0000,0.0000,1.5586,0.0000;0.0000,0.0000,0.0000,0.9404)	0.7650168
	$i = 2$	(0,0.1615E+15,0,0.3566E+14;0.8411E+14,0,1485.97,972.5537)	0.9030760
	$i = 3$	(0.0000,0.0000,4.7674,0.0000;2.2724,0.0000,0.0000,0.0000)	0.7998895
	$i = 4$	(4607.841,8398.31,0.0000,0.0000;0.0000,0.0000,0.0000,6110.628)	0.8183420
	$i = 5$	(0.0000,2.7143,5.3116,0.0000;0.0000,0.0000,0.0000,4.6709)	0.8982800

Based on Charnes−Cooper transformation,

$$t = \frac{1}{u^T x_0}, \omega = tu, \mu = tv$$

the CCR model (2.1) is converted to the equivalent linear programming. In this situation, we obtain the supplier$_2$ and supplier$_5$ are the most efficient suppliers, and the optimal solution is provided in Table 8.

In the first comparative study, the enterprise decision maker prefers to solving deterministic optimization model, then he may replace all uncertain parameters in the data envelope analysis problem with their mean values. This method isn't acceptable when the optimal solution is sensitive to the distributions of uncertain parameters. Therefore, the decision maker cannot ignore the uncertain factors to model the DEA problem.

Case II. Optimal supplier selection decision under fixed possibility distributions

We next solve our DEA model when the input and output data are fuzzy parameters with fixed possibility distributions, which are collected in Tables 9 and 10.

In the case of $\theta_l = \theta_r = 0$, the parametric interval-valued fuzzy inputs and outputs degenerate to their corresponding type-1 fuzzy variable, and model (4.1) is transformed into credibility constrained fuzzy supplier selection model. If model parameter α is set to 0.95, the evaluating results of each supplier are reported in Table 11. As Table 11 shown, the supplier$_2$ is the most efficient supplier with biggest efficiency value 0.8994293.

For the sake of comparison, the optimal solution of each supplier is called the nominal solution for fuzzy supplier selection problem under fixed possibility distributions. That is to say, the nominal solution of each supplier is given in Table 11. In comparison with computational results in Table 4, there exist many changes with respect to the optimal solution. It is shown that the nominal solutions are not optimal under $(\theta_l, \theta_r) = (0.5, 0.5), \lambda = 0.78$

TABLE 6. Evaluating results of each supplier with different λ under $\alpha = 0.95$ and $(\theta_l, \theta_r) = (0.5, 0.5)$.

λ	Supplier$_i$	Optimal solution $(u; v)$	MEV
0	$i = 1$	(0.0000,0.0000,20.361,0.0000;0.0000,0.0000,0.0000,12.4095)	0.7713665
	$i = 2$	(0.0000,9176.6,0.0000,3197.4;5319.6,0.0000,0.0000,0.0000)	0.9089306
	$i = 3$	(0.0000,0.0000,4.7974,0.0000;2.3090,0.0000,0.0000,0.0000)	0.8068014
	$i = 4$	(4553.8,8154.5,0.0000,0.0000;0.0000,0.0000,0.0000,6028.9)	0.8270942
	$i = 5$	(0.0000,2.0888,4.2039,0.0000;0.0000,0.0000,0.0000,3.6992)	0.9049476
0.2	$i = 1$	(0.0000,0.0000,1.5586,0.0000;0.0000,0.0000,0.0000,0.9404)	0.7650168
	$i = 2$	(0,0.1615E+15,0,0.3566E+14;0.8411E+14,0,1485.970,972.5537)	0.9030760
	$i = 3$	(0.0000,0.0000,4.7674,0.0000;2.2724,0.0000,0.0000,0.0000)	0.7998895
	$i = 4$	(4607.841,8398.310,0.0000,0.0000;0.0000,0.0000,0.0000,6110.628)	0.8183420
	$i = 5$	(0.0000,2.7143,5.3116,0.0000;0.0000,0.0000,0.0000,4.6709)	0.8982800
0.6	$i = 1$	(0.0000,0.0000,1.5736,0.0000;0.0000,0.0000,0.0000,0.9408)	0.7607138
	$i = 2$	(0.0000,197243.2,0.0000,44840.28;102577.7,0.0000,0.0000,0.0000)	0.8988347
	$i = 3$	(0.0000,0.0000,4.8403,0.0000;2.2868,0.0000,0.0000,0.0000)	0.7945487
	$i = 4$	(4636.563,8585.586,0.0000,0.0000;0.0000,0.0000,0.0000,6157.339)	0.8105168
	$i = 5$	(0.0000,3.4970,6.6743,0.0000;0.0000,0.0000,0.0000,5.8653)	0.8922088
1.0	$i = 1$	(0.0000,0.0000,1032.714,0.0000;0.0000,0.0000,0.0000,614.7593)	0.7601277
	$i = 2$	(0.0000,3948.027,0.0000,1770.002;2415.068,0.0000,0.0000,0.0000)	0.8979021
	$i = 3$	(0.0000,0.0000,4.9783,0.0000;2.3423,0.0000,0.0000,0.0000)	0.7929664
	$i = 4$	(3.0121,5.6186,0.0000,0.0000;0.0000,0.0000,0.0000,4.0025)	0.8069476
	$i = 5$	(0.0000,3.8138,7.1956,0.0000;0.0000,0.0000,0.0000,6.3215)	0.8893641

TABLE 7. The deterministic inputs and outputs for five suppliers.

Supplier$_i$	TC	FC	EC	SC	NOT	NB	PQ	SR
$i = 1$	2.975	3.875	4.000	3.750	5.525	5.325	6.200	5.075
$i = 2$	1.450	1.975	2.275	1.975	4.175	4.425	5.125	3.275
$i = 3$	2.400	3.225	2.675	2.825	4.475	4.125	5.475	3.275
$i = 4$	1.775	2.325	2.825	3.425	4.275	4.000	5.325	3.700
$i = 5$	3.525	2.500	4.000	4.100	5.775	5.475	6.725	5.375

TABLE 8. Evaluating results of each supplier with deterministic data.

Supplier$_i$	Optimal solution $(\omega; \mu)$	Efficiency value
$i = 1$	(0.0000,0.0000,0.2500,0.0000;0.0000,0.0000,0.0000,0.1737)	0.8813454
$i = 2$	(0.0000,0.0000,0.0000,0.5063;0.0000,0.0000,0.1951,0.0000)	1.0000000
$i = 3$	(0.0000,0.0000,0.3738,0.0000;0.2037,0.0000,0.0000,0.0000)	0.9115787
$i = 4$	(0.1863,0.2878,0.0000,0.0000;0.0000,0.0000,0.0000,0.2561)	0.9475302
$i = 5$	(0.0000,0.3638,0.0000,0.0221;0.0000,0.0000,0.1487,0.0000)	1.0000000

Table 9. The triangular fuzzy inputs for five suppliers.

Supplier$_i$	TC	FC	EC	SC
$i=1$	$(2.6, 3.0, 3.3)$	$(3.7, 3.9, 4.0)$	$(3.8, 4.0, 4.2)$	$(3.3, 3.7, 4.3)$
$i=2$	$(1.1, 1.5, 1.7)$	$(1.8, 2.0, 2.1)$	$(2.0, 2.3, 2.5)$	$(1.8, 2.0, 2.1)$
$i=3$	$(2.3, 2.4, 2.5)$	$(3.0, 3.2, 3.5)$	$(2.4, 2.7, 2.9)$	$(2.7, 2.8, 3.0)$
$i=4$	$(1.6, 1.8, 1.9)$	$(2.2, 2.3, 2.5)$	$(2.7, 2.8, 3.0)$	$(3.3, 3.4, 3.6)$
$i=5$	$(3.4, 3.5, 3.7)$	$(2.2, 2.5, 2.8)$	$(3.8, 4.0, 4.2)$	$(3.9, 4.1, 4.3)$

Table 10. The triangular fuzzy outputs for five suppliers.

Supplier$_i$	NOT	NB	PQ	SR
$i=1$	$(5.4, 5.5, 5.7)$	$(5.2, 5.3, 5.5)$	$(6.0, 6.2, 6.4)$	$(4.8, 5.0, 5.5)$
$i=2$	$(4.0, 4.2, 4.3)$	$(4.2, 4.4, 4.7)$	$(5.0, 5.1, 5.3)$	$(3.1, 3.3, 3.4)$
$i=3$	$(4.2, 4.5, 4.7)$	$(4.0, 4.1, 4.3)$	$(5.3, 5.5, 5.6)$	$(3.0, 3.3, 3.5)$
$i=4$	$(4.1, 4.3, 4.4)$	$(3.8, 4.0, 4.2)$	$(5.2, 5.3, 5.5)$	$(3.5, 3.7, 3.9)$
$i=5$	$(5.5, 5.8, 6.0)$	$(5.1, 5.5, 5.8)$	$(6.6, 6.7, 6.9)$	$(5.0, 5.4, 5.7)$

Table 11. Evaluating results of each supplier with triangular fuzzy data under $\alpha = 0.95$.

Supplier$_i$	Optimal solution $(u; v)$	MEV
$i=1$	(0.0000,0.0000,1.5711,0.0000;0.0000,0.0000,0.0000,0.9408)	0.7612663
$i=2$	(0,0.5021E+15,0,0.1114E+15;0.2603E+15,0,912.7652,484.8061)	0.8994293
$i=3$	(0.0000,0.0000,4.8136,0.0000;2.2778,0.0000,0.0000,0.0000)	0.7953549
$i=4$	(4634.868,8558.811,0.0000,0.0000;0.0000,0.0000,0.0000,6153.624)	0.8118730
$i=5$	(0.0000,3.3537,6.4285,0.0000;0.0000,0.0000,0.0000,5.6501)	0.8932728

and $\alpha = 0.95$. To analyze further the influences on the nominal optimal solution under different values of theta and lambda, we compare the nominal solution with the computational results in Tables 5 and 6. From Tables 5 and 6, we observe that the input weights and output weights vary greatly for each supplier when parameters θ_l, θ_r and λ take different values. Not only that, but the corresponding efficiency value also has the positive or negative deviations comparing with that of the nominal solution. This implies that the nominal solution no longer is optimal.

In the second comparative study, an accurate possibilistic description of the fuzzy inputs and outputs is assumed available in the form of fixed possibility distributions. If the possibility distributions cannot be determined accurately in the modeling process, then the optimal supplier evaluation may be invalid, necessitating other techniques. Thus, the proposed fuzzy DEA under robust input and output data paves an effective way for enterprise decision maker.

5. Conclusions and future studies

In this study we considered the data envelopment analysis problem in a fuzzy decision system. The innovation contents include the following three aspects:

- A fuzzy expectation data envelopment analysis model with credibility constraints is built, in which uncertain inputs and outputs are characterized by parametric interval-valued triangular fuzzy variables. The variable

possibility distributions are obtained by using the lambda selection method to fuzzy inputs and outputs, and controlled by two types of parameters.
- Some basic properties of presented model are discussed (Thms. 3.1 and 3.2). Theoretically, the analytical expressions of the expected value objective and credibility constraints are obtained, which transform the original fuzzy model into its equivalent parametric nonlinear programming.
- A practice-oriented example of supplier evaluation and selection is performed to verify the effectiveness of the proposed fuzzy DEA model for illustration purpose. The computational results and comparative study verify that our solution method works well and provides acceptable solution, which assists decision makers to make proper decisions.

Future research might address the following topics. First, as far as the expectation objective function incurred from DEA is concerned, our current model assumes a risk-neural decision maker. An extension of the model to the case of risk-averse decision maker is possible. Second, the current model considers the efficiency as a single objective function, a multi-objective optimization model that accounts for both efficiency and effectiveness objectives may be helpful in practice. Third, the extension of the parametric interval-valued fuzzy variable and lambda selection methodology lies in other optimization problems such as transportation problem, emergency supplies prepositioning, facility location, and assignment problem.

Appendix A. Proofs of main results

Proof of Theorem 2. Since $\xi_{j,0}$ and $\eta_{k,0}$ are the λ selection of the parametric interval-valued fuzzy variables $\tilde{\xi}_{j,0} = [\xi^{r_1}_{j,0}, \xi^{r_2}_{j,0}, \xi^{r_3}_{j,0}; \theta_{l,j,0}, \theta_{r,j,0}]$ and $\tilde{\eta}_{k,0} = [\eta^{r_1}_{k,0}, \eta^{r_2}_{k,0}, \eta^{r_3}_{k,0}; \overline{\theta}_{l,k,0}, \overline{\theta}_{r,k,0}]$, based on Theorem 2.3, their parametric possibility distributions are

$$\mu_{\xi_{j,0}}(x;\theta,\lambda) = \begin{cases} (1+\lambda\theta_{r,j,0}-(1-\lambda)\theta_{l,j,0})\frac{x-\xi^{r_1}_{j,0}}{\xi^{r_2}_{j,0}-\xi^{r_1}_{j,0}}, & \text{if } x\in\left[\xi^{r_1}_{j,0}, \frac{\xi^{r_1}_{j,0}+\xi^{r_2}_{j,0}}{2}\right] \\ \frac{(1-\lambda\theta_{r,j,0}+(1-\lambda)\theta_{l,j,0})x+(\lambda\theta_{r,j,0}-(1-\lambda)\theta_{l,j,0})\xi^{r_2}_{j,0}-\xi^{r_1}_{j,0}}{\xi^{r_2}_{j,0}-\xi^{r_1}_{j,0}}, & \text{if } x\in\left(\frac{\xi^{r_1}_{j,0}+\xi^{r_2}_{j,0}}{2}, \xi^{r_2}_{j,0}\right] \\ \frac{(-1+\lambda\theta_{r,j,0}-(1-\lambda)\theta_{l,j,0})x-(\lambda\theta_{r,j,0}-(1-\lambda)\theta_{l,j,0})\xi^{r_2}_{j,0}+\xi^{r_3}_{j,0}}{\xi^{r_3}_{j,0}-\xi^{r_2}_{j,0}}, & \text{if } x\in\left(\xi^{r_2}_{j,0}, \frac{\xi^{r_2}_{j,0}+\xi^{r_3}_{j,0}}{2}\right] \\ (1+\lambda\theta_{r,j,0}-(1-\lambda)\theta_{l,j,0})\frac{\xi^{r_3}_{j,0}-x}{\xi^{r_3}_{j,0}-\xi^{r_2}_{j,0}}, & \text{if } x\in\left(\frac{\xi^{r_2}_{j,0}+\xi^{r_3}_{j,0}}{2}, \xi^{r_3}_{j,0}\right], \end{cases}$$

for $j = 1,2,\ldots,p$, and

$$\mu_{\eta_{k,0}}(x;\theta,\lambda) = \begin{cases} (1+\lambda\overline{\theta}_{r,k,0}-(1-\lambda)\overline{\theta}_{l,k,0})\frac{x-\eta^{r_1}_{k,0}}{\eta^{r_2}_{k,0}-\eta^{r_1}_{k,0}}, & \text{if } x\in\left[\eta^{r_1}_{k,0}, \frac{\eta^{r_1}_{k,0}+\eta^{r_2}_{k,0}}{2}\right] \\ \frac{(1-\lambda\overline{\theta}_{r,k,0}+(1-\lambda)\overline{\theta}_{l,k,0})x+(\lambda\overline{\theta}_{r,k,0}-(1-\lambda)\overline{\theta}_{l,k,0})\eta^{r_2}_{k,0}-\eta^{r_1}_{k,0}}{\eta^{r_2}_{k,0}-\eta^{r_1}_{k,0}}, & \text{if } x\in\left(\frac{\eta^{r_1}_{k,0}+\eta^{r_2}_{k,0}}{2}, \eta^{r_2}_{k,0}\right] \\ \frac{(-1+\lambda\overline{\theta}_{r,k,0}-(1-\lambda)\overline{\theta}_{l,k,0})x-(\lambda\overline{\theta}_{r,k,0}-(1-\lambda)\overline{\theta}_{l,k,0})\eta^{r_2}_{k,0}+\eta^{r_3}_{k,0}}{\eta^{r_3}_{k,0}-\eta^{r_2}_{k,0}}, & \text{if } x\in\left(\eta^{r_2}_{k,0}, \frac{\eta^{r_2}_{k,0}+\eta^{r_3}_{k,0}}{2}\right] \\ (1+\lambda\overline{\theta}_{r,k,0}-(1-\lambda)\overline{\theta}_{l,k,0})\frac{\eta^{r_3}_{j,0}-x}{\eta^{r_3}_{k,0}-\eta^{r_2}_{k,0}}, & \text{if } x\in\left(\frac{\eta^{r_2}_{k,0}+\eta^{r_3}_{k,0}}{2}, \eta^{r_3}_{k,0}\right], \end{cases}$$

for $k = 1,2,\ldots,q$.

Let $\xi = \sum_{j=1}^{p} u_j\xi_{j,0}$ and $\eta = \sum_{k=1}^{q} v_k\eta_{k,0}$. Then

$$\begin{aligned} \mathrm{E}\left[\frac{\eta}{\xi}\right] &= \frac{1}{2}\int_0^1 \left(\left(\frac{\eta}{\xi}\right)_{\sup}(\alpha) + \left(\frac{\eta}{\xi}\right)_{\inf}(\alpha)\right)\mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left(\frac{\eta_{\sup}(\alpha)}{\xi_{\inf}(\alpha)} + \frac{\eta_{\inf}(\alpha)}{\xi_{\sup}(\alpha)}\right)\mathrm{d}\alpha \end{aligned}$$

$$= \frac{1}{2}\int_0^1 \left(\frac{\sum_{k=1}^q v_k \eta_{k,0,\sup}(\alpha)}{\sum_{j=1}^p u_j \xi_{j,0,\inf}(\alpha)} + \frac{\sum_{k=1}^q v_k \eta_{k,0,\inf}(\alpha)}{\sum_{j=1}^p u_j \xi_{j,0,\sup}(\alpha)}\right)\mathrm{d}\alpha$$
$$= \frac{1}{2}\int_0^1 (M_1 + M_2)\mathrm{d}\alpha.$$

Note that $\mu_{\xi_{j,0}}((\xi_{j,0}^{r_1} + \xi_{j,0}^{r_2})/2) = \mu_{\eta_{k,0}}((\eta_{k,0}^{r_2} + \eta_{k,0}^{r_3})/2) = (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})/2$ for each j and k. When $\alpha \in (0, (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})/2]$, $\eta_{k,0,\sup}(\alpha)$ and $\xi_{j,0,\inf}(\alpha)$ are the solutions of the following equations

$$(1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\frac{\eta_{k,0}^{r_3} - x}{\eta_{k,0}^{r_3} - \eta_{k,0}^{r_2}} - \alpha = 0 \quad \text{and} \quad (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\frac{x - \xi_{j,0}^{r_1}}{\xi_{j,0}^{r_2} - \xi_{j,0}^{r_1}} - \alpha = 0$$

respectively, and $\eta_{k,0,\inf}(\alpha)$ and $\xi_{j,0,\sup}(\alpha)$ are the solutions of the following equations

$$(1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\frac{x - \eta_{k,0}^{r_1}}{\eta_{k,0}^{r_2} - \eta_{k,0}^{r_1}} - \alpha = 0 \quad \text{and} \quad (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\frac{\xi_{j,0}^{r_3} - x}{\xi_{j,0}^{r_3} - \xi_{j,0}^{r_2}} - \alpha = 0$$

respectively. By solving the above equations, we have

$$\eta_{k,0,\sup}(\alpha) = \eta_{k,0}^{r_3} - \frac{\eta_{k,0}^{r_3} - \eta_{k,0}^{r_2}}{1+\lambda\theta_{r,0} - (1-\lambda)\theta_{l,0}}\alpha, \quad \xi_{j,0,\inf}(\alpha) = \xi_{j,0}^{r_1} + \frac{\xi_{j,0}^{r_2} - \xi_{j,0}^{r_1}}{1+\lambda\theta_{r,0} - (1-\lambda)\theta_{l,0}}\alpha,$$

$$\eta_{k,0,\inf}(\alpha) = \eta_{k,0}^{r_1} + \frac{\eta_{k,0}^{r_2} - \eta_{k,0}^{r_1}}{1+\lambda\theta_{r,0} - (1-\lambda)\theta_{l,0}}\alpha, \quad \xi_{j,0,\sup}(\alpha) = \xi_{j,0}^{r_3} - \frac{\xi_{j,0}^{r_3} - \xi_{j,0}^{r_2}}{1+\lambda\theta_{r,0} - (1-\lambda)\theta_{l,0}}\alpha.$$

Therefore, when $\alpha \in (0, (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})/2]$, we have

$$M_1 = \frac{\alpha\sum_{k=1}^q v_k(\eta_{k,0}^{r_2} - \eta_{k,0}^{r_3}) + (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\sum_{k=1}^q v_k\eta_{k,0}^{r_3}}{\alpha\sum_{j=1}^p u_j(\xi_{j,0}^{r_2} - \xi_{j,0}^{r_1}) + (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\sum_{j=1}^p u_j\xi_{j,0}^{r_1}},$$

$$M_2 = \frac{\alpha\sum_{k=1}^q v_k(\eta_{k,0}^{r_2} - \eta_{k,0}^{r_1}) + (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\sum_{k=1}^q v_k\eta_{k,0}^{r_1}}{\alpha\sum_{j=1}^p u_j(\xi_{j,0}^{r_2} - \xi_{j,0}^{r_3}) + (1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})\sum_{j=1}^p u_j\xi_{j,0}^{r_3}}.$$

Similarly, when $\alpha \in ((1 + \lambda\theta_{r,0} - (1-\lambda)\theta_{l,0})/2, 1]$, we have

$$M_1 = \frac{\alpha\sum_{k=1}^q v_k(\eta_{k,0}^{r_2} - \eta_{k,0}^{r_3}) + \sum_{k=1}^q v_k(((1-\lambda)\theta_{l,0} - \lambda\theta_{r,0})\eta_{k,0}^{r_2} + \eta_{k,0}^{r_3})}{\alpha\sum_{j=1}^p u_j(\xi_{j,0}^{r_2} - \xi_{j,0}^{r_1}) + \sum_{j=1}^p u_j(((1-\lambda)\theta_{l,0} - \lambda\theta_{r,0})\xi_{j,0}^{r_2} + \xi_{j,0}^{r_1})},$$

$$M_2 = \frac{\alpha\sum_{k=1}^q v_k(\eta_{k,0}^{r_2} - \eta_{k,0}^{r_1}) + \sum_{k=1}^q v_k(((1-\lambda)\theta_{l,0} - \lambda\theta_{r,0})\eta_{k,0}^{r_2} + \eta_{k,0}^{r_1})}{\alpha\sum_{j=1}^p u_j(\xi_{j,0}^{r_2} - \xi_{j,0}^{r_3}) + \sum_{j=1}^p u_j(((1-\lambda)\theta_{l,0} - \lambda\theta_{r,0})\xi_{j,0}^{r_2} + \xi_{j,0}^{r_3})}.$$

Hence, we have

$$\mathrm{E}\left[\frac{\eta}{\xi}\right] = \frac{1}{2}\int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} M_1\mathrm{d}\alpha + \frac{1}{2}\int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} M_2\mathrm{d}\alpha$$
$$+ \frac{1}{2}\int_{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}}^1 M_1\mathrm{d}\alpha + \frac{1}{2}\int_{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}}^1 M_2\mathrm{d}\alpha.$$

Since

$$\int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} \frac{c\alpha + d}{a\alpha + b}\mathrm{d}\alpha$$

$$
\begin{aligned}
&= \int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} \frac{\frac{c}{a}(a\alpha+b)+d-\frac{bc}{a}}{a\alpha+b}\mathrm{d}\alpha \\
&= \int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} \left(\frac{c}{a}+\frac{ad-bc}{a^2}\frac{1}{\alpha+\frac{b}{a}}\right)\mathrm{d}\alpha \\
&= \frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})c}{2a}+\frac{ad-bc}{a^2}\left(\ln\left(\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}+\frac{b}{a}\right)-\ln\frac{b}{a}\right) \\
&= \frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})c}{2a}+\frac{ad-bc}{a^2}\ln\frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})a+2b}{2b},
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} M_1\mathrm{d}\alpha &= \frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\sum_{k=1}^{q}v_k(\eta_{k,0}^{r_2}-\eta_{k,0}^{r_3})}{2\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_1})} \\
&+\frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\left(\sum_{j=1}^{p}u_j\xi_{j,0}^{r_2}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_3}-\sum_{j=1}^{p}u_j\xi_{j,0}^{r_1}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_2}\right)}{\left(\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_1})\right)^2}\ln\frac{\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_1}+\xi_{j,0}^{r_2})}{2\sum_{j=1}^{p}u_j\xi_{j,0}^{r_1}},
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\int_0^{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}} M_2\mathrm{d}\alpha &= \frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\sum_{k=1}^{q}v_k(\eta_{k,0}^{r_2}-\eta_{k,0}^{r_1})}{2\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_3})} \\
&+\frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\left(\sum_{j=1}^{p}u_j\xi_{j,0}^{r_2}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_1}-\sum_{j=1}^{p}u_j\xi_{j,0}^{r_3}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_2}\right)}{\left(\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_3})\right)^2}\ln\frac{\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}+\xi_{j,0}^{r_3})}{2\sum_{j=1}^{p}u_j\xi_{j,0}^{r_3}}.
\end{aligned}
\tag{A.2}
$$

Similarly, we have

$$
\begin{aligned}
\int_{\frac{1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0}}{2}}^{1} M_1\mathrm{d}\alpha &= \frac{(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\sum_{k=1}^{q}v_k(\eta_{k,0}^{r_2}-\eta_{k,0}^{r_3})}{2\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_1})} \\
&+\frac{(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\left(\sum_{j=1}^{p}u_j\xi_{j,0}^{r_2}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_3}-\sum_{j=1}^{p}u_j\xi_{j,0}^{r_1}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_2}\right)}{\left(\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_1})\right)^2}\ln\frac{2\sum_{j=1}^{p}u_j\xi_{j,0}^{r_2}}{\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_1}+\xi_{j,0}^{r_2})},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\int_{\frac{1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0}}{2}}^{1} M_2\mathrm{d}\alpha &= \frac{(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\sum_{k=1}^{q}v_k(\eta_{k,0}^{r_2}-\eta_{k,0}^{r_1})}{2\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_3})} \\
&+\frac{(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\left(\sum_{j=1}^{p}u_j\xi_{j,0}^{r_2}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_1}-\sum_{j=1}^{p}u_j\xi_{j,0}^{r_3}\sum_{k=1}^{q}v_k\eta_{k,0}^{r_2}\right)}{\left(\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}-\xi_{j,0}^{r_3})\right)^2}\ln\frac{2\sum_{j=1}^{p}u_j\xi_{j,0}^{r_2}}{\sum_{j=1}^{p}u_j(\xi_{j,0}^{r_2}+\xi_{j,0}^{r_3})}.
\end{aligned}
\tag{A.4}
$$

Hence, combining equations (A.1)–(A.4) together, it is obvious to obtain the result. The proof of the theorem is complete. □

Proof of Theorem 3. We only prove assertions (i)–(iv), and assertions (v)–(viii) can be proved similarly.

Since $\eta_{k,i}$ is the λ selection of interval-valued fuzzy variable $\tilde{\eta}_{k,i} = [\eta^{r_1}_{k,i}, \eta^{r_2}_{k,i}, \eta^{r_3}_{k,i}; \overline{\theta}_{l,k,i}, \overline{\theta}_{r,k,i}]$, based on Theorem 2.3, its parametric possibility distribution is

$$\mu_{\eta_{k,i}}(x;\theta,\lambda) = \begin{cases} (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\dfrac{x-\eta^{r_1}_{k,i}}{\eta^{r_2}_{k,i}-\eta^{r_1}_{k,i}}, & \text{if } x \in \left[\eta^{r_1}_{k,i}, \dfrac{\eta^{r_1}_{k,i}+\eta^{r_2}_{k,i}}{2}\right] \\ \dfrac{(1-\lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})x + (\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta^{r_2}_{k,i} - \eta^{r_1}_{k,i}}{\eta^{r_2}_{k,i}-\eta^{r_1}_{k,i}}, & \text{if } x \in \left(\dfrac{\eta^{r_1}_{k,i}+\eta^{r_2}_{k,i}}{2}, \eta^{r_2}_{k,i}\right] \\ \dfrac{(-1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})x - (\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta^{r_2}_{k,i} + \eta^{r_3}_{k,i}}{\eta^{r_3}_{k,i}-\eta^{r_2}_{k,i}}, & \text{if } x \in \left(\eta^{r_2}_{k,i}, \dfrac{\eta^{r_2}_{k,i}+\eta^{r_3}_{k,i}}{2}\right] \\ (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\dfrac{\eta^{r_3}_{j,i}-x}{\eta^{r_3}_{k,i}-\eta^{r_2}_{k,i}}, & \text{if } x \in \left(\dfrac{\eta^{r_2}_{k,i}+\eta^{r_3}_{k,i}}{2}, \eta^{r_3}_{k,i}\right], \end{cases}$$

for $k = 1, 2, \ldots, q$.

Noting that $\tilde{\xi}_{j,i} = [\xi^{r_1}_{j,i}, \xi^{r_2}_{j,i}, \xi^{r_3}_{j,i}; \theta_{l,j,i}, \theta_{r,j,i}]$, then $-\tilde{\xi}_{j,i} = [-\xi^{r_3}_{j,i}, -\xi^{r_2}_{j,i}, -\xi^{r_1}_{j,i}; \theta_{l,j,i}, \theta_{r,j,i}]$. Similarly, the parametric possibility distribution of the λ selection $-\xi_{j,i}$ can be derived.

Denote $\delta = v^T\eta_i - u^T\xi_i$. If $\alpha_i < 0.5$, then we have

$$\mathrm{Cr}\left\{v^T\eta_i - u^T\xi_i \le 0\right\} = \mathrm{Cr}\left\{\delta \le 0\right\} = \frac{1}{2}\left(1 + \sup_{x\le 0}\nu_\delta(x;\theta,\lambda) - \sup_{x>0}\nu_\delta(x;\theta,\lambda)\right) = \frac{1}{2}\sup_{x\le 0}\nu_\delta(x;\theta,\lambda).$$

Thus the credibility constraint of model (2.4) is equivalent to $\sup_{x\le 0}\nu_\delta(x;\theta,\lambda) \ge 2\alpha_i$. If we denote

$$\delta_{\inf}(\alpha) = \inf\left\{s \mid \sup_{x\le 0}\nu_\delta(x;\theta,\lambda) \ge \alpha\right\}$$

for $\alpha \in (0,1]$, then $\delta_{\inf}(2\alpha_i) \le 0$.

Since $\{\tilde{\xi}_{j,i}\}$ and $\{\tilde{\eta}_{k,i}\}$ are mutually independent, then

$$\begin{aligned} \delta_{\inf}(2\alpha_i) &= (v^T\eta_i - u^T\xi_i)_{\inf}(2\alpha_i) \\ &= \left(\sum_{k=1}^{q} v_k\eta_{k,i} - \sum_{j=1}^{p} u_j\xi_{j,i}\right)_{\inf}(2\alpha_i) \\ &= \sum_{k=1}^{q} v_k\eta_{k,i,\inf}(2\alpha_i) + \sum_{j=1}^{p} u_j(-\xi_{j,i})_{\inf}(2\alpha_i) \le 0. \end{aligned}$$

Note that $\mu_{\eta_{k,i}}((\eta^{r_1}_{k,i}+\eta^{r_2}_{k,i})/2) = (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})/2$. If $0 < 2\alpha_i < (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})/2$, *i.e.*, $\alpha_i \in (0, (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})/4)$, then $\eta_{k,i,\inf}(2\alpha_i)$ is the solution of the following equation

$$(1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\frac{x-\eta^{r_1}_{k,i}}{\eta^{r_2}_{k,i}-\eta^{r_1}_{k,i}} - 2\alpha_i = 0.$$

Solving the above equation, we have

$$\eta_{k,i,\inf}(2\alpha_i) = \frac{(1-2\alpha_i+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta^{r_1}_{k,i} + 2\alpha_i\eta^{r_2}_{k,i}}{1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i}}.$$

On the other hand, if $1 > 2\alpha_i \ge (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})/2$, *i.e.*, $\beta^k_i \in [(1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})/4, 0.5)$, then $\eta_{k,i,\inf}(2\alpha_i)$ is the solution of the following equation

$$\frac{(1-\lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})x + (\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})\eta^{r_2}_{k,i} - \eta^{r_1}_{k,i}}{\eta^{r_2}_{k,i}-\eta^{r_1}_{k,i}} - 2\alpha_i = 0.$$

Solving the above equation, we have

$$\eta_{k,i,\inf}(2\alpha_i) = \frac{(1-2\alpha_i)\eta_{k,i}^{r_1} + (2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1-\lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}.$$

Similarly, we have

$$(-\xi_{j,i})_{\inf}(2\alpha_i) = \begin{cases} -\dfrac{2\alpha_i\xi_{j,i}^{r_2} + (1-2\alpha_i+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i}}, & \text{if } \alpha_i \in (0, (1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})/4) \\ -\dfrac{(2\alpha_i-\lambda\theta_{r,j,i}+(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2} + (1-2\alpha_i)\xi_{j,i}^{r_3}}{1-\lambda\theta_{r,j,i}+(1-\lambda)\theta_{l,j,i}}, & \text{if } \alpha_i \in [(1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})/4, 0.5). \end{cases}$$

Noting that $\lambda_{\xi_{j,i}} = \lambda_{\eta_{k,i}} = \lambda$, and $\lambda\overline{\theta}_{r,1,i} - (1-\lambda)\overline{\theta}_{l,1,i} \le \lambda\overline{\theta}_{r,2,i} - (1-\lambda)\overline{\theta}_{l,2,i} \le \ldots \le \lambda\overline{\theta}_{r,q,i} - (1-\lambda)\overline{\theta}_{l,q,i} \le \lambda\theta_{r,1,i} - (1-\lambda)\theta_{l,1,i} \le \lambda\theta_{r,2,i} - (1-\lambda)\theta_{l,2,i} \le \ldots \lambda\theta_{r,p,i} - (1-\lambda)\theta_{l,p,i}$, then the following results hold.

If $0 < 2\alpha_i < (1+\lambda\overline{\theta}_{r,1,i} - (1-\lambda)\overline{\theta}_{l,1,i})/2$, then $2\alpha_i < (1+\lambda\overline{\theta}_{r,k,i} - (1-\lambda)\overline{\theta}_{l,k,i})/2$ for $k = 1, 2, \ldots, q$, and $2\alpha_i < (1+\lambda\theta_{r,j,i} - (1-\lambda)\theta_{l,j,i})/2$ for $j = 1, 2, \ldots, p$. Therefore, if $\alpha_i \in (0, (1+\lambda\overline{\theta}_{r,1,i} - (1-\lambda)\overline{\theta}_{l,1,i})/4)$, then the credibility constraint of model (2.4) is equivalent to

$$\sum_{k=1}^{q} v_k \frac{(1-2\alpha_i+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_1} + 2\alpha_i\eta_{k,i}^{r_2}}{1+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{2\alpha_i\xi_{j,i}^{r_2} + (1-2\alpha_i+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i}} \le 0.$$

If there exists a k_0, $1 \le k_0 < q$ such that $(1+\lambda\overline{\theta}_{r,k_0,i} - (1-\lambda)\overline{\theta}_{l,k_0,i})/2 \le 2\alpha_i < (1+\lambda\overline{\theta}_{r,k_0+1,i} - (1-\lambda)\overline{\theta}_{l,k_0+1,i})/2$, *i.e.*, $\alpha_i \in [(1+\lambda\overline{\theta}_{r,k_0,i} - (1-\lambda)\overline{\theta}_{l,k_0,i})/4, (1+\lambda\overline{\theta}_{r,k_0+1,i} - (1-\lambda)\overline{\theta}_{l,k_0+1,i})/4)$, then the credibility constraint of model (2.4) is equivalent to

$$\sum_{k=1}^{k_0} v_k \frac{(1-2\alpha_i)\eta_{k,i}^{r_1} + (2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1-\lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$\sum_{k=k_0+1}^{q} v_k \frac{(1-2\alpha_i+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_1} + 2\alpha_i\eta_{k,i}^{r_2}}{1+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{p} u_j \frac{2\alpha_i\xi_{j,i}^{r_2} + (1-2\alpha_i+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i}} \le 0.$$

If there exists a j_0, $1 \le j_0 < p$ such that $(1+\lambda\theta_{r,j_0,i} - (1-\lambda)\theta_{l,j_0,i})/2 \le 2\alpha_i < (1+\lambda\theta_{r,j_0+1,i} - (1-\lambda)\theta_{l,j_0+1,i})/2$, *i.e.*, $\alpha_i \in [(1+\lambda\theta_{r,j_0,i} - (1-\lambda)\theta_{l,j_0,i})/4, (1+\lambda\theta_{r,j_0+1,i} - (1-\lambda)\theta_{l,j_0+1,i})/4)$, then the credibility constraint of model (2.4) is equivalent to

$$\sum_{k=1}^{q} v_k \frac{(1-2\alpha_i)\eta_{k,i}^{r_1} + (2\alpha_i - \lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1-\lambda\overline{\theta}_{r,k,i} + (1-\lambda)\overline{\theta}_{l,k,i}}$$
$$-\sum_{j=1}^{j_0} u_j \frac{(2\alpha_i-\lambda\theta_{r,j,i}+(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2} + (1-2\alpha_i)\xi_{j,i}^{r_3}}{1-\lambda\theta_{r,j,i}+(1-\lambda)\theta_{l,j,i}}$$
$$-\sum_{j=j_0+1}^{p} u_j \frac{2\alpha_i\xi_{j,i}^{r_2} + (1-2\alpha_i+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_3}}{1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i}} \le 0.$$

If $(1+\lambda\theta_{r,p,i}-(1-\lambda)\theta_{l,p,i})/ \leq 2\alpha_i < 1$, then $(1+\lambda\overline{\theta}_{r,k,i}-(1-\lambda)\overline{\theta}_{l,k,i})/2 < 2\alpha_i$ for $k=1,2,\dots,q$, and $(1+\lambda\theta_{r,j,i}-(1-\lambda)\theta_{l,j,i})/2 < 2\alpha_i$ for $j=1,2,\dots,p$. Therefore, if $\alpha_i \in [(1+\lambda\theta_{r,p,i}-(1-\lambda)\theta_{l,p,i})/4, 0.5)$, then the credibility constraint of model (2.4) is equivalent to

$$\begin{aligned}&\sum_{k=1}^{q} v_k \frac{(1-2\alpha_i)\eta_{k,i}^{r_1}+(2\alpha_i-\lambda\overline{\theta}_{r,k,i}+(1-\lambda)\overline{\theta}_{l,k,i})\eta_{k,i}^{r_2}}{1-\lambda\overline{\theta}_{r,k,i}+(1-\lambda)\overline{\theta}_{l,k,i}}\\ &-\sum_{j=1}^{p} u_j \frac{(2\alpha_i-\lambda\theta_{r,j,i}+(1-\lambda)\theta_{l,j,i})\xi_{j,i}^{r_2}+(1-2\alpha_i)\xi_{j,i}^{r_3}}{1-\lambda\theta_{r,j,i}+(1-\lambda)\theta_{l,j,i}} \leq 0.\end{aligned}$$

The proof of assertions (i)–(iv) is complete. ☐

Appendix B. Analytical expressions for numerical example

On the one hand, the objective functions in model (4.1) are represented by

$$\begin{aligned}f_1 = &-\frac{v_1+v_2+2v_3+2v_4}{2(3u_1+u_2+2u_3+6u_4)}-\frac{2v_1+2v_2+2v_3+5v_4}{2(4u_1+2u_2+2u_3+4u_4)}\\ &+\frac{1}{2(3u_1+u_2+2u_3+6u_4)^2}((54v_1+52v_2+60v_3+48v_4)(30u_1+39u_2+40u_3+37u_4)\\ &-(55v_1+53v_2+62v_3+50v_4)(33u_1+40u_2+42u_3+43u_4))\\ &\times(2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(6.3u_1+7.9u_2+8.2u_3+8.0u_4)\\ &+(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(3.0u_1+3.9u_2+4.0u_3+3.7u_4)\\ &-(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(3.3u_1+4.0u_2+4.2u_3+4.3u_4))\\ &+\frac{1}{2(4u_1+2u_2+2u_3+4u_4)^2}((57v_1+55v_2+64v_3+55v_4)(30u_1+39u_2+40u_3+37u_4)\\ &-(55v_1+53v_2+62v_3+50v_4)(26u_1+37u_2+38u_3+33u_4))\\ &\times(2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(5.6u_1+7.6u_2+7.8u_3+7.0u_4)\\ &-(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(2.6u_1+3.7u_2+3.8u_3+3.3u_4)\\ &+(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(3.0u_1+3.9u_2+4.0u_3+3.7u_4)),\end{aligned}$$

$$\begin{aligned}f_2 = &-\frac{2v_1+2v_2+v_3+2v_4}{2(2u_1+u_2+2u_3+u_4)}-\frac{v_1+3v_2+2v_3+v_4}{2(4u_1+2u_2+3u_3+2u_4)}\\ &+\frac{1}{2(2u_1+u_2+2u_3+u_4)^2}((40v_1+42v_2+50v_3+31v_4)(15u_1+20u_2+23u_3+20u_4)\\ &-(42v_1+44v_2+51v_3+33v_4)(17u_1+21u_2+25u_3+21u_4))\\ &\times(2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(3.2u_1+4.1u_2+4.8u_3+4.1u_4)\\ &+(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(1.5u_1+2.0u_2+2.3u_3+2.0u_4)\\ &-(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(1.7u_1+2.1u_2+2.5u_3+2.1u_4))\\ &+\frac{1}{2(4u_1+2u_2+3u_3+2u_4)^2}((43v_1+47v_2+53v_3+34v_4)(15u_1+20u_2+23u_3+20u_4)\\ &-(42v_1+44v_2+51v_3+33v_4)(11u_1+18u_2+20u_3+18u_4))\\ &\times(2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(2.6u_1+3.8u_2+4.3u_3+3.8u_4)\\ &-(1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(1.1u_1+1.8u_2+2.0u_3+1.8u_4)\\ &+(1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(1.5u_1+2.0u_2+2.3u_3+2.0u_4)),\end{aligned}$$

$$
\begin{aligned}
f_3 =& -\frac{3v_1+v_2+2v_3+3v_4}{2(u_1+3u_2+2u_3+2u_4)} - \frac{2v_1+2v_2+v_3+2v_4}{2(u_1+2u_2+3u_3+u_4)} \\
&+ \frac{1}{2(u_1+3u_2+2u_3+2u_4)^2}((42v_1+40v_2+53v_3+30v_4)(24u_1+32u_2+27u_3+28u_4) \\
&- (45v_1+41v_2+55v_3+33v_4)(25u_1+35u_2+29u_3+30u_4)) \\
&\times (2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(4.9u_1+6.7u_2+5.6u_3+5.8u_4) \\
&+ (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(2.4u_1+3.2u_2+2.7u_3+2.8u_4) \\
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(2.5u_1+3.5u_2+2.9u_3+3.0u_4)) \\
&+ \frac{1}{2(u_1+2u_2+3u_3+u_4)^2}((47v_1+43v_2+56v_3+35v_4)(24u_1+32u_2+27u_3+28u_4) \\
&- (45v_1+41v_2+55v_3+33v_4)(23u_1+30u_2+24u_3+27u_4)) \\
&\times (2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(4.7u_1+6.2u_2+5.1u_3+5.5u_4) \\
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(2.3u_1+3.0u_2+2.4u_3+2.7u_4) \\
&+ (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(2.4u_1+3.2u_2+2.7u_3+2.8u_4)),
\end{aligned}
$$

$$
\begin{aligned}
f_4 =& -\frac{2v_1+2v_2+v_3+2v_4}{2(u_1+2u_2+2u_3+2u_4)} - \frac{v_1+2v_2+2v_3+2v_4}{2(2u_1+u_2+u_3+u_4)} \\
&+ \frac{1}{2(u_1+2u_2+2u_3+2u_4)^2}((41v_1+38v_2+52v_3+35v_4)(18u_1+23u_2+28u_3+34u_4) \\
&- (43v_1+40v_2+53v_3+37v_4)(19u_1+25u_2+30u_3+36u_4)) \\
&\times (2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(3.7u_1+4.8u_2+5.8u_3+7.0u_4) \\
&+ (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(1.8u_1+2.3u_2+2.8u_3+3.4u_4) \\
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(1.9u_1+2.5u_2+3.0u_3+3.6u_4)) \\
&+ \frac{1}{2(2u_1+u_2+u_3+u_4)^2}((44v_1+42v_2+55v_3+39v_4)(18u_1+23u_2+28u_3+34u_4) \\
&- (43v_1+40v_2+53v_3+37v_4)(16u_1+22u_2+27u_3+33u_4)) \\
&\times (2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(3.4u_1+4.5u_2+5.5u_3+6.7u_4) \\
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(1.6u_1+2.2u_2+2.7u_3+3.3u_4) \\
&+ (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(1.8u_1+2.3u_2+2.8u_3+3.4u_4)),
\end{aligned}
$$

$$
\begin{aligned}
f_5 =& -\frac{3v_1+4v_2+v_3+4v_4}{2(2u_1+3u_2+2u_3+2u_4)} - \frac{2v_1+3v_2+2v_3+3v_4}{2(u_1+3u_2+2u_3+2u_4)} \\
&+ \frac{1}{2(2u_1+3u_2+2u_3+2u_4)^2}((55v_1+51v_2+66v_3+50v_4)(35u_1+25u_2+40u_3+41u_4) \\
&- (58v_1+55v_2+67v_3+54v_4)(37u_1+28u_2+42u_3+43u_4)) \\
&\times (2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(7.2u_1+5.3u_2+8.2u_3+8.4u_4) \\
&+ (1-\lambda\theta_{r,0}+(1-\lambda)\theta_{l,0})\ln 2(3.5u_1+2.5u_2+4.0u_3+4.1u_4) \\
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(3.7u_1+2.8u_2+4.2u_3+4.3u_4)) \\
&+ \frac{1}{2(u_1+3u_2+2u_3+2u_4)^2}((60v_1+58v_2+69v_3+57v_4)(35u_1+25u_2+40u_3+41u_4) \\
&- (58v_1+55v_2+67v_3+54v_4)(34u_1+22u_2+38u_3+39u_4)) \\
&\times (2(\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln(6.9u_1+4.7u_2+7.8u_3+8.0u_4) \\
&- (1+\lambda\theta_{r,0}-(1-\lambda)\theta_{l,0})\ln 2(3.4u_1+2.2u_2+3.8u_3+3.9u_4)
\end{aligned}
$$

$$+ (1 - \lambda\theta_{r,0} + (1-\lambda)\theta_{l,0}) \ln 2(3.5u_1 + 2.5u_2 + 4.0u_3 + 4.1u_4)).$$

On the other hand, the constraint conditions in model (4.1) are given by

$$\begin{aligned}
g_1 = &- \frac{2.6(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.0(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_1 - \frac{3.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.9(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_2 \\
&- \frac{3.8(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.0(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_3 - \frac{3.3(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.7(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_4 \\
&+ \frac{5.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_1 + \frac{5.5(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_2 \\
&+ \frac{6.4(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 6.2(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_3 + \frac{5.5(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.0(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_4,
\end{aligned}$$

$$\begin{aligned}
g_2 = &- \frac{1.1(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 1.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_1 - \frac{1.8(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.0(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_2 \\
&- \frac{2(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_3 - \frac{1.8(4(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_4 \\
&+ \frac{4.3(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.2(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_1 + \frac{4.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.4(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_2 \\
&+ \frac{5.3(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.1(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_3 + \frac{3.4(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_4,
\end{aligned}$$

$$\begin{aligned}
g_3 = &- \frac{2.3(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.4(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_1 - \frac{3.0(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.2(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_2 \\
&- \frac{2.4(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.7(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_3 - \frac{2.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.8(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_4 \\
&+ \frac{4.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_1 + \frac{4.3(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.1(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_2 \\
&+ \frac{5.6(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_3 + \frac{3.5(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_4,
\end{aligned}$$

$$\begin{aligned}
g_4 = &- \frac{1.6(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 1.8(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_1 - \frac{2.2(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_2 \\
&- \frac{2.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.8(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_3 - \frac{3.3(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.4(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_4 \\
&+ \frac{4.4(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_1 + \frac{4.2(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.0(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_2 \\
&+ \frac{5.5(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.3(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_3 + \frac{3.9(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.7(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_4,
\end{aligned}$$

$$\begin{aligned}
g_5 = &- \frac{3.4(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 3.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_1 - \frac{2.2(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 2.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_2 \\
&- \frac{3.8(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.0(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_3 - \frac{3.9(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 4.1(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} u_4
\end{aligned}$$

$$+ \frac{6.0(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.8(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_1 + \frac{5.8(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.5(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_2$$
$$+ \frac{6.9(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 6.7(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_3 + \frac{5.7(2\alpha - 1 + \lambda\theta_r - (1-\lambda)\theta_l) + 5.4(2-2\alpha)}{1 + \lambda\theta_r - (1-\lambda)\theta_l} v_4.$$

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References

[1] A. Amirteimoori and A. Emrouznejad, Flexible measures in production process: A DEA-based approach. *RAIRO: OR* **45** (2011) 63–74.
[2] R. Azizi, R.K. Matin, and R.F. Saen, Ranking units and determining dominance relations in the cost efficiency analysis. *RAIRO: OR* **49** (2015) 879–896.
[3] X. Bai, Optimization for pre-positioning emergency supplies problem under fuzzy environment. *Syst. Eng. Theory Practice* **35** (2015) 1465–1473.
[4] R.D. Banker, A. Charnes and W.W. Cooper, Some models for estimating technical and scale efficiencies in data envelopment. *Anal. Manag. Sci.* **30** (1984) 1078–1092.
[5] X. Bai and Y.K. Liu, Semideviations of reduced fuzzy variable: A possibility approach. *Fuzzy Optimiz. Decision Making* **13** (2014) 173–196.
[6] X. Bai and Y.K.Liu, CVAR reduced fuzzy variables and their second order moments. *Iranian J. Fuzzy Syst.* **12** (2015) 45–75.
[7] X. Bai and Y.K. Liu, Robust optimization of supply chain network design in fuzzy decision system. *J. Intell. Manufacturing* **27** (2016) 1131–1149.
[8] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **2** (1978) 429–444.
[9] W.W. Cooper, L.M. Seiford and K. Tone, Data envelopment analysis. Springer Science and Business Media, New York (2007).
[10] W.D. Cook and L.M. Seiford, Data envelopment analysis (DEA)–thirty years on. *Eur. J. Oper. Res.* **192** (2009) 1–17.
[11] W.W. Cooper, K.S. Park and J.T. Pastor, RAM: A range adjusted measure of inefficiency for use with additive models, and relations to other models and measures in DEA. *J. Productivity Anal.* **11** (1999) 5–24.
[12] M. Dotoli, N. Epicoco, M. Falagario and F. Sciancalepore, A cross-efficiency fuzzy data envelopment analysis technique for performance evaluation of decision making units under uncertainty. *Comput. Industrial Eng.* **79** (2015) 103–114.
[13] G. Egilmez, S. Gumus, M.Kucukvar and O. Tatari, A fuzzy data envelopment analysis framework for dealing with uncertainty impacts of input-output life cycle assessment models on eco-efficiency assessment. *J. Cleaner Production* **129** (2016) 622–636.
[14] J. Ignatius, M.R. Ghasemi, F. Zhang, A. Emrouznejad and A. Hatami−Marbini, A. Carbon efficiency evaluation: An analytical framework using fuzzy DEA. *Eur. J. Oper. Res.* **253** (2016) 428–440.
[15] C. Kao and and S.T. Liu, Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets Syst.* **119** (2000)149–160.
[16] J.S. Liu, L.Y.Y. Lu, W.M. Lu and B.J.Y. Lin, A survey of DEA applications. *Omega* **41**(2013) 893–902.
[17] J.S. Liu, L.Y.Y. Lu and W.M. Lu, Research fronts in data envelopment analysis. *Omega* **58** (2016) 33–45.
[18] B. Liu and Y. Liu, Expected value of fuzzy variable and fuzzy expected value models. *IEEE Trans. Fuzzy Syst.* **10** (2002) 445–450.
[19] J.S. Liu, L.Y.Y. Lu, W.M. Lu and B.J.Y. Lin, Data envelopment analysis 1978–2010: A citation-based literature survey. *Omega* **41** (2013) 3–15.
[20] Z. Murena, Mab and W. Cui, Generalized fuzzy data envelopment analysis methods. *Appl. Soft Comput.* **19** (2014) 215–225.
[21] N.C. Petersen, Data envelopment analysis on a relaxed set of assumptions. *Manag. Sci.* **36** (1990) 305–313.
[22] P. Guo and H. Tanaka, Fuzzy DEA: A perceptual evaluation method. *Fuzzy Sets Syst.* **119** (2001) 149–160.
[23] J.K. Sengupta, A fuzzy systems approach in data envelopment analysis. *Comput. Math. Appl.* **24** (1992) 259–266.
[24] K. Triantis and O. Girod, A mathematical programming approach for measuring technical efficiency in a fuzzy environment. *J. Productivity Anal.* **10** (1998) 85–102.
[25] K. Tone, A slack-based measure of efficiency in data envelopment analysis. *Eur. J. Oper. Res.* **130** (2001) 498–509.
[26] M.L. Wen and H.S. Li, Fuzzy data envelopment analysis (DEA): Model and ranking method. *J. Comput. Appl. Math.* **223** (2009) 872–878.
[27] M.L. Wen, Z.F. Qin and R. Kang, Sensitivity and stability analysis in fuzzy data envelopment analysis. *Fuzzy Optimiz. Decision Making* **10** (2011) 1–10.
[28] Y.M. Wang and K.S. Chin, Fuzzy data envelopment analysis: A fuzzy expected value approach. *Expert Syst. Appl.* **38** (2011)11678–11685.
[29] M.L. Zerafat, Angiz, A. Emrouznejad and A. Mustafa, Fuzzy data envelopment analysis: A discrete approach. *Expert Syst. Appl.* **39** (2012) 2263–2269.

[30] M.Q. Meng, A hybrid particle swarm optimization algorithm for satisficing data envelopment analysis under fuzzy chance constraints. *Expert Syst. Appl.* **41** (2014) 2074–2082.

[31] X.Q. Feng, M.Q. Meng and Y.K. Liu, Modeling credibilistic data envelopment analysis under fuzzy input and output data. *J. Uncertain Syst.* **9** (2015) 230–240.

[32] M.R. Ghasemi, J. Ignatius, S. Lozano, A. Emrouznejad and A. Hatami−Marbini, A fuzzy expected value approach under generalized data envelopment analysis. *Knowledge-Based Sys.* **89** (2015) 148–159.

[33] Z. Liu and Y.K. Liu, Type-2 fuzzy variables and their arithmetic. *Soft Comput.* **14** (2010) 729–747.

[34] R. Qin, Y.K. Liu and Z. Liu, Modeling fuzzy data envelopment analysis by parametric programming method. *Expert Syst. Appl.* **38** (2011a) 8648–8663.

[35] R. Qin, Y.K. Liu and Z. Liu, Methods of critical value reduction for type-2 fuzzy variables and their applications. *J. Comput. Appl. Math.* **235** (2011b) 1454–1481.

[36] Y. Liu and Y.K. Liu, The lambda selections of parametric interval-valued fuzzy variables and their numerical characteristics. *Fuzzy Optimiz. Decision Making* **15** (2016) 255–279.

[37] X. Wu and Y.K. Liu, Optimizing fuzzy portfolio selection problems by parametric quadratic programming. *Fuzzy Optimiz. Decision Making* **11** (2012) 411–449.

[38] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning–I. *Information Sci.* **8** (1975) 199–249.

[39] X. Zhou, W. Pedrycz, Y. Kuang and Z. Zhang, Type-2 fuzzy multi-objective DEA model: An application tosustainable supplier evaluation. *Appl. Soft Comput.* **46** (2016) 424–440.

[40] L.A. Zadeh, Fuzzy sets. *Information and Control* **8** (1965) 338–353.