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AN INTRODUCTION TO THE TWIN SIGNED TOTAL k-DOMINATION NUMBERS IN DIRECTED GRAPHS

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Abstract. Let $D = (V, A)$ be a finite simple directed graph (shortly digraph), $N^-(v)$ and $N^+(v)$ denote the set of in-neighbors and out-neighbors of a vertex $v \in V$, respectively. A function $f : V \longrightarrow \{-1, 1\}$ is called a twin signed total k-dominating function (TSTkDF) if $\sum_{u \in (N^-(v))} f(u) \geq k$ and $\sum_{u \in (N^+(v))} f(u) \geq k$ for each vertex $v \in V$. The twin signed total k-domination number of D is $\gamma^*_{stk}(D) = \min\{\omega(f) \mid f$ is a TSTkDF of $D\}$, where $\omega(f) = \sum_{v \in V} f(v)$ is the weight of f. In this paper, we initiate the study of twin signed total k-domination in digraphs and present different bounds on $\gamma^*_{stk}(D)$. In addition, we determine the twin signed total k-domination number of some classes of digraphs. Our results are mostly extensions of well-known bounds of the twin signed total domination numbers of directed graphs.

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1. Introduction

Throughout this paper, D is a finite simple directed graph (digraph) with vertex set $V(D)$ and arc set $A(D)$ (briefly represented as V and A). An *oriented graph* is a digraph without directed cycles of length 2. For an arc (u, v) of D, can be stated that v is an *out-neighbor* of u and u is an *in-neighbor* of v. $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$ stand for the set of in-neighbors and out-neighbors of a vertex v, respectively. The outdegree of a vertex v and its indegree are $d^+_D(v) = |N^+(v)|$ and $d^-_D(v) = |N^-(v)|$, respectively. The *minimum* and *maximum* indegrees and *minimum* and *maximum* outdegrees of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called *regular* or r-*regular* if $\delta^-(D) = \delta^+(D) = \Delta^-(D) = \Delta^+(D) = r$. For $X \subseteq V(D)$ and $v \in V(D)$, $A(X, v)$ is the set of arcs from X to v. $A(X, Y)$ represents the set of arcs from a subset X to a subset Y. Beside, D^{-1} is used as a notation for the digraph obtained from D by reversing the arcs of D. The complete digraph of order n, K^*_n, is a digraph D provided that $(u, v), (v, u) \in A(D)$ for any two distinct vertices $u, v \in V(D)$. Let $f : V(D) \longrightarrow \mathbb{R}$ be a function. For $S \subseteq V$,

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let $f(S) = \sum_{v \in S} f(v)$. The weight of f is $w(f) = f(V)$. Notations and graph theory terminologies are based on those presented in [14].

Suppose $k \geq 1$ is an integer and $D = (V, A)$ a finite simple digraph with $\delta^-(D) \geq k$. A *signed total k-dominating function* (abbreviated STkDF) of D defined in [11] is a function $f : V \to \{-1, 1\}$ such that $f(N^-(v)) \geq k$ for every $v \in V$. The *signed total k-domination number* for a directed graph D is

$$\gamma_{stk}(D) = \min\{\omega(f) \mid f \text{ is a ST}k\text{DF of } D\}.$$

A $\gamma_{stk}(D)$-function is a STkDF of D of weight $\gamma_{stk}(D)$. This definition is analogous to that definition of the signed k-domination number in digraphs introduced by Atapour *et al.* [4]. If $k = 1$, the signed total k-domination number $\gamma_{stk}(D)$ would be the usual signed total domination number $\gamma_{st}(D)$, as introduced by Sheikholeslami [10].

Let $k \geq 1$ be an integer and D be a digraph with $\min\{\delta^-(D), \delta^+(D)\} \geq k$. A signed total k-dominating function of D which also is a signed total k-dominating function of D^{-1}, *i.e.*, $f(N^+(v)) \geq k$ for every $v \in V$, is termed a *twin signed total k-dominating function* (briefly shown as TSTkDF) of D. The *twin signed total k-domination number* for a digraph D is $\gamma^*_{stk}(D) = \min\{\omega(f) \mid f \text{ is a TST}k\text{DF of } D\}$. As $\min\{\delta^-(D), \delta^+(D)\} \geq k$ is a necessary assumption, when studying $\gamma^*_{stk}(D)$, it is assumed that $\delta^-(D) \geq k$ and $\delta^+(D) \geq k$. In the case $k = 1$, the twin signed total k-domination number $\gamma^*_{stk}(D)$ is the usual twin signed total domination number $\gamma^*_{st}(D)$, studied by Atapour *et al.* [1].

For any function $f : V(D) \to \{-1, 1\}$, we define $P = P_f = \{v \in V \mid f(v) = 1\}$ and $M = M_f = \{v \in V \mid f(v) = -1\}$. Since every TST$k$DF of D is a STkDF on both D and D^{-1}, we have

$$\max\left\{\gamma_{stk}(D), \gamma_{stk}(D^{-1})\right\} \leq \gamma^*_{stk}(D). \tag{1}$$

Let $G = (V, E)$ be a graph with vertex $V(G)$ and edge set $E(G)$ (briefly shown as V and E). For every vertex $v \in V$, the open neighborhood of v, $N(v)$, is the set $\{u \in V \mid uv \in E\}$. The degree of v in G is $\deg(v) = \deg_G(v) = |N(v)|$. A function $f : V \to \{-1, 1\}$ is called a *signed total dominating function* (STDF) of G if $f(N(v)) \geq 1$ for every $v \in V$. The *signed total domination number* of G, denoted by $\gamma_{st}(G)$, is the minimum weight of a signed total dominating function on G. The signed total domination number of a graph was introduced by Zelinka [15] and has been studied by several authors [8].

The signed total k-dominating function of a graph G defined in [13] is a function $f : V \to \{-1, 1\}$ such that $f(N(v)) \geq k$ for all $v \in V(G)$. The *signed total k-domination number* of G, denoted by $\gamma_{stk}(G)$, is the minimum weight of a signed total k-dominating function on G.

In this paper, we initiate the study of twin signed total k-domination number in directed graphs and we present some bounds on this parameter. For $k = 1$, some of our results are those recently presented in [1].

2. Basic properties of the twin signed total k-domination numbers

In this section, we study basic properties of the twin signed total k-domination number of digraphs. Obviously, the function which assigns $+1$ to every vertex of D is a TSTkDF and so $\gamma^*_{stk}(D) \leq |V(D)|$. The next proposition provides conditions to establish the equality.

Proposition 2.1. *Let D be a digraph of order n. Then $\gamma^*_{stk}(D) = n$ if and only if every vertex has either an out-neighbor with indegree at most $k + 1$ or an in-neighbor with outdegree at most $k + 1$.*

Proof. The sufficiency is clear. Thus, we verify the necessity of the condition. Assume that $\gamma^*_{stk}(D) = n$. Suppose to the contrary that there exists a vertex $v \in V(D)$ such that $d^-(u) \geq k+2$ for each $u \in N^+(v)$ and $d^+(w) \geq k+2$ for each $w \in N^-(v)$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a TSTkDF of D of weight less than n, a contradiction. This completes the proof. □

Proposition 2.2 [11]**.** *For any digraph D of order $n \geq 2$, $\gamma_{stk}(D) \equiv n \pmod 2$.*

Observation 2.3. For any digraph D of order $n \geq 2$, $\gamma^*_{stk}(D) \equiv n \pmod 2$.

Proof. Let f be a $\gamma^*_{stk}(D)$-function. Since $n = |P|+|M|$ and $\gamma^*_{stk}(D) = |P|-|M|$, we deduce that $n-\gamma^*_{stk}(D) = 2|M|$ and hence $\gamma^*_{stk}(D) \equiv n \pmod 2$. □

Corollary 2.4. For any digraph D of order $n \geq 2$, $\gamma^*_{stk}(D) \equiv \gamma_{stk}(D) \pmod 2$.

The next corollary is an immediate consequence of Proposition 2.1 and Observation 2.3.

Corollary 2.5. *For any digraph D of order $n \geq 2$ and $\delta^+(D), \delta^-(D) \geq k+2$, $\gamma^*_{stk}(D) \leq n-2$.*

Remark 2.6. Let D be a digraph and j, k be two integers such that $1 \leq j \leq k$. Since every TSTkDF of D is also a TSTjDF of D, we have $\gamma^*_{stj}(D) \leq \gamma^*_{stk}(D)$.

A tournament is, in effect, a digraph D where for every pair u and v of distinct vertices, either $(u,v) \in A(D)$ or $(v,u) \in A(D)$, but not both. Let $n = 2r+1$ for some positive integer r. A circulant tournament $\mathrm{CT}(n)$ with n vertices is a tournament with vertex set $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$ and arc set $A(\mathrm{CT}(n)) = \{(u_i, u_{i+1}), \ldots, (u_i, u_{i+r}) \mid 0 \leq i \leq n-1\}$, where the indices are taken modulo n. The proof of the next result can be found in [11].

Proposition 2.7. *Let $r \geq k \geq 1$ be integers and $n = 2r+1$. Then*

$$\gamma_{stk}(\mathrm{CT}(n)) = \begin{cases} 2k+1 & \text{if} \quad r \equiv k \pmod 2 \\ 2k+3 & \text{if} \quad r \equiv k+1 \pmod 2. \end{cases}$$

The next proposition shows that $\gamma^*_{stk}(\mathrm{CT}(n)) = \gamma_{stk}(\mathrm{CT}(n))$.

Proposition 2.8. *Let $r \geq k \geq 1$ be integers and $n = 2r+1$. Then $\gamma^*_{stk}(\mathrm{CT}(n)) = \gamma_{stk}(\mathrm{CT}(n))$.*

Proof. By (1) and Proposition 2.7, we have

$$\gamma^*_{stk}(\mathrm{CT}(n)) \geq \begin{cases} 2k+1 & \text{if} \quad r \equiv k \pmod 2 \\ 2k+3 & \text{if} \quad r \equiv k+1 \pmod 2. \end{cases}$$

Let $s = \lfloor \frac{r-k}{2} \rfloor$, $V^- = \{u_1, \ldots, u_s, u_{r+1}, \ldots, u_{r+s}\}$ and $V^+ = V(\mathrm{CT}(n)) - V^-$. For any vertex $v \in V(\mathrm{CT}(n))$, we have $|N^+(v) \cap V^-| \leq s$ and $|N^-(v) \cap V^-| \leq s$. Let $f : V(\mathrm{CT}(n)) \to \{-1, 1\}$ be a function assigning $+1$ to every vertex $v \in V^+$ and -1 to every vertex $v \in V^-$. Obviously, $f(N^-(v)) \geq r-2s \geq k$ and $f(N^+(v)) \geq r-2s \geq k$ for each $v \in V$. Therefore f is a TSTkDF on $\mathrm{CT}(n)$ of weight $2k+1$ if $r \equiv k \pmod 2$ and $2k+3$ when $r \equiv k+1 \pmod 2$. Hence

$$\gamma^*_{stk}(\mathrm{CT}(n)) \leq \omega(f) = \begin{cases} 2k+1 & \text{if} \quad r \equiv k \pmod 2 \\ 2k+3 & \text{if} \quad r \equiv k+1 \pmod 2 \end{cases}$$

and the proof is complete. □

As we observed in (1), $\gamma^*_{stk}(D) \geq \max\{\gamma_{stk}(D), \gamma_{stk}(D^{-1})\}$. Now we show that the difference $\gamma^*_{stk}(D) - \max\{\gamma_{stk}(D), \gamma_{stk}(D^{-1})\}$ can be arbitrarily large.

Theorem 2.9. *For every positive integers $k, t \geq 1$, there exists a digraph D such that*

$$\gamma^*_{stk}(D) - \max\{\gamma_{stk}(D), \gamma_{stk}(D^{-1})\} \geq 2t.$$

Proof. Let $k, t \geq 1$ be integers. For $1 \leq i \leq 2t+1$, let D_i be a circulant tournament of order $2k+1$ with vertex set $\{u_0^i \dots u_{2k}^i\}$. Further, let D be obtained from the disjoint union of D_i's, $1 \leq i \leq 2t+1$, by adding the set $\{w^i \mid 1 \leq i \leq 2t\}$ of new vertices and the set

$$\{(u_j^{2t+1}, u_j^i), (u_j^{i+t}, u_j^{2t+1}) \mid \; 0 \leq j \leq 2k, \qquad \text{and} \qquad 1 \leq i \leq t\}$$

$$\cup \left\{(w^i, u_j^i), (u_{j+k}^i, w^i) \mid 1 \leq i \leq 2t \qquad \text{and} \qquad 1 \leq j \leq k\right\}$$

of new arcs. Then the order of D is $n = 4kt + 2k + 4t + 1$. Obviously, $D \cong D^{-1}$ and so, $\gamma_{stk}(D) = \gamma_{stk}(D^{-1})$. By Proposition 2.1, $\gamma_{stk}^*(D) = n$. On the other hand, it can be verified that the function $f : V(D) \to \{-1, 1\}$ defined by $f(x) = -1$, for $x \in \{w^i \mid 1 \leq i \leq t\}$ and $f(x) = +1$ for the remaining vertices, is a STkDF of D and thus $\gamma_{stk}(D) \leq n - 2t$. Hence, $\gamma_{stk}^*(D) - \max\{\gamma_{stk}(D), \gamma_{stk}(D^{-1})\} \geq n - (n - 2t) = 2t$, and the proof is complete. □

Now we show that the twin signed total k-domination number of digraphs can be arbitrarily small. Recall that a complete digraph of order n, K_n^*, is a digraph in which for every pair of distinct vertices u and v, $(u, v), (v, u) \in A(K_n^*)$.

Theorem 2.10. *For any positive integers $k, t \geq 1$, there exists a digraph D such that*

$$\gamma_{stk}^*(D) \leq 4kt - 4kt^2 + 2t$$

Proof. Let $k, t \geq 1$ be integers and let D be a digraph obtained from a complete digraph of order $2kt$ with vertex set $V(K_{2kt}^*) = \{u_1^i \dots u_{2k}^i \mid \; 1 \leq i \leq t\}$ by adding the set $\{v_j^i, w_j^i \mid 1 \leq i \leq t \text{ and } 1 \leq j \leq 2kt - k - 1\}$ of new vertices and the set

$$\left\{(u_j^i, v_l^i), (v_l^i, u_{j+k}^i), (w_l^i, u_j^i), (u_{j+k}^i, w_l^i) \mid \; 1 \leq i \leq t, 1 \leq j \leq k \text{ and } 1 \leq l \leq 2kt - k - 1\right\}$$

of new arcs. It can be observe that the function $f : V(D) \to \{-1, 1\}$ defined by $f(x) = -1$ for $x \in \{v_j^i, w_j^i \mid 1 \leq j \leq 2kt - k - 1 \text{ and } 1 \leq i \leq t\}$, and $f(x) = +1$ for all other vertices x, is a TSTkDF of D and so $\gamma_{stk}^*(D) \leq w(f) = 2kt - 2t(2kt - k - 1) = 4kt - 4kt^2 + 2t$. □

Proposition 2.11. For any graph G of order n with $\delta(G) \geq 2k$, and for any orientation D of G, $\gamma_{st2k}(G) \leq \gamma_{stk}^*(D)$.

Proof. Let D be an orientation of G and let f be a $\gamma_{stk}^*(D)$-function. Since $f(N_G(v)) = f(N_D^+(v)) + f(N_D^-(v))$, $f(N_D^+(v)) \geq k$ and $f(N_D^-(v)) \geq k$ for each $v \in V$, we have $f(N_G(v)) \geq 2k$ for each $v \in V$, and so f is a ST2kDF of G. Therefore $\gamma_{st2k}(G) \leq \omega(f) = \gamma_{stk}^*(D)$ as desired. □

3. Bounds on the twin signed total k-domination numbers

In this section we establish some bounds for $\gamma_{stk}^*(D)$ in terms of the order, size, the maximum and minimum indegree and outdegree of D.

Lemma 3.1. *Let D be a digraph of order n and let f be a $\gamma_{stk}^*(D)$-function. Then*

(1) $\left\lceil \frac{\delta^- + k}{2} \right\rceil |M| \leq |A(P, M)| \leq \left\lfloor \frac{\Delta^+ - k}{2} \right\rfloor |P|$.

(2) $\left\lceil \frac{\delta^+ + k}{2} \right\rceil |M| \leq |A(M, P)| \leq \left\lfloor \frac{\Delta^- - k}{2} \right\rfloor |P|$.

(3) $|A(P, P)| \geq \max\{\left\lceil \frac{\delta^- + k}{2} \right\rceil |P|, \left\lceil \frac{\delta^+ + k}{2} \right\rceil |P|\}$.

Proof.

(1) Let $v \in M$. Since $f(N^-(v)) \geq k$, we have $|A(P,v)| \geq \left\lceil \frac{d^-(v)+k}{2} \right\rceil \geq \left\lceil \frac{\delta^-+k}{2} \right\rceil$. It follows that $|A(P,M)| \geq \left\lceil \frac{\delta^-+k}{2} \right\rceil |M|$. In addition, since $f(N^+(v)) \geq k$ for $v \in P$, we have $|A(v,M)| \leq \left\lfloor \frac{d^+(v)-k}{2} \right\rfloor \leq \left\lfloor \frac{\Delta^+-k}{2} \right\rfloor$ and so $|A(P,M)| \leq \left\lfloor \frac{\Delta^+-k}{2} \right\rfloor |P|$. Combining the inequalities, we obtain (1).
(2) The proof is similar to the proof of (1).
(3) Let $v \in P$. Since $f(N^+(v)) \geq k$ and $f(N^-(v)) \geq k$, we have

$$|A(v,P)| \geq \left\lceil \frac{d^+(v)+k}{2} \right\rceil \geq \left\lceil \frac{\delta^++k}{2} \right\rceil,$$

and

$$|A(P,v)| \geq \left\lceil \frac{d^-(v)+k}{2} \right\rceil \geq \left\lceil \frac{\delta^-+k}{2} \right\rceil.$$

Thus

$$|A(P,P)| \geq \max\left\{ \left\lceil \frac{\delta^-+k}{2} \right\rceil |P|, \left\lceil \frac{\delta^++k}{2} \right\rceil |P| \right\},$$

and the proof is complete. □

Theorem 3.2. *Let D be a digraph of order n, minimum indegree δ^-, minimum outdegree δ^+, maximum indegree Δ^- and maximum outdegree Δ^+. Then*

$$\gamma^*_{stk}(D) \geq \max\left\{ \frac{\left\lceil \frac{\delta^-+k}{2} \right\rceil - \left\lfloor \frac{\Delta^+-k}{2} \right\rfloor}{\left\lceil \frac{\delta^-+k}{2} \right\rceil + \left\lfloor \frac{\Delta^+-k}{2} \right\rfloor} n, \frac{\left\lceil \frac{\delta^++k}{2} \right\rceil - \left\lfloor \frac{\Delta^--k}{2} \right\rfloor}{\left\lceil \frac{\delta^++k}{2} \right\rceil + \left\lfloor \frac{\Delta^--k}{2} \right\rfloor} n \right\}.$$

Proof. Let f be a minimum TSTkDF of D. Using Lemma 3.1 and replacing $|M|$ and $|P|$ by $\frac{n-\gamma^*_{stk}(D)}{2}$ and $\frac{n+\gamma^*_{stk}(D)}{2}$ in (1) and (2), the desired inequality follows. □

The next corollary is a consequence of Theorem 3.2 and Proposition 2.1.

Corollary 3.3. *If D is a r-regular digraph with $r \geq k$, then $\gamma^*_{stk}(D) \geq (k+1)n/r$ when $r+k$ is odd and $\gamma^*_{stk}(D) \geq kn/r$ when $r+k$ is even. This bound is sharp for k-regular and $(k+1)$-regular digraphs.*

Next proposition gives an upper bound on twin signed total k-domination number.

Proposition 3.4. *If D is a digraph of order n with $\delta^+ \geq \delta^- \geq k+2$, then*

$$\gamma^*_{stk}(D) \leq n - 2\left\lfloor \frac{\delta^- - k}{2} \right\rfloor$$

and

$$\gamma^*_{stk}(D) \leq n - 2\left\lfloor \frac{\delta^+ - k}{2} \right\rfloor.$$

These bounds are sharp for K^*_n.

Proof. Define $t = \lfloor \frac{\delta^- - k}{2} \rfloor$. Let $v \in V(D)$ be a vertex, and let $\{u_1, u_2, \ldots, u_t\}$ be a set of t out-neighbors of v. Define the function $f : V(D) \to \{-1, 1\}$ by $f(x) = -1$ for $x \in \{u_1, u_2, \ldots, u_t\}$ and $f(x) = +1$ for all other vertices x. Then

$$f(N^-(x)) \geq -t + (\delta^- - t) = \delta^- - 2t = \delta^- - 2\left\lfloor \frac{\delta^- - k}{2} \right\rfloor \geq k$$

and

$$f(N^+(x)) \geq -t + (\delta^+ - t) = \delta^+ - 2t \geq \delta^- - 2\left\lfloor \frac{\delta^- - k}{2} \right\rfloor \geq k$$

for each vertex $x \in V(D)$. Therefore f is a TSTkDF on D of weight $1 - t + (n - t - 1) = n - 2t$ and thus $\gamma^*_{stk}(D) \leq n - 2t = n - 2\lfloor \frac{\delta^- - k}{2} \rfloor$.

Supposing $t = \lfloor \frac{\delta^+ - k}{2} \rfloor$, the proof of the first inequality leads to the second inequality.

Now we prove that $\gamma^*_{stk}(K^*_n) = k + 2$ when $n + k - 1$ is odd and $\gamma^*_{stk}(K^*_n) = k + 1$ when $n + k - 1$ is even. Since K^*_n is a $(n-1)$-regular digraph, Corollary 3.3 implies that

$$\gamma^*_{stk}(K^*_n) \geq \begin{cases} k+1 & \text{if} \quad n+k-1 \quad \text{is even} \\ k+2 & \text{if} \quad n+k-1 \quad \text{is odd.} \end{cases}$$

On the other hand, from the above inequalities we have $\gamma^*_{stk}(K^*_n) \leq n - 2\left\lfloor \frac{n-1-k}{2} \right\rfloor$. It follows that $\gamma^*_{stk}(K^*_n) = k+1$ when $n + k - 1$ is even and $\gamma^*_{stk}(K^*_n) = k + 2$ when $n + k - 1$ is odd. □

Proposition 3.5. If D is a digraph of order n and maximum indegree Δ^-, then

$$\gamma^*_{stk}(D) \geq 2\left\lceil \frac{\Delta^- + k}{2} \right\rceil - n.$$

Proof. Let $u \in V(D)$ be a vertex of maximum indegree $d^-(u) = \Delta^-$, and let f be a $\gamma^*_{stk}(D)$-function. Since $f(N^-(u)) \geq k$, we deduce that $|A(P, u)| \geq \left\lceil \frac{\Delta^- + k}{2} \right\rceil$. It follows that

$$\frac{n + \gamma^*_{stk}(D)}{2} = |P| \geq |A(P, u)| \geq \left\lceil \frac{\Delta^- + k}{2} \right\rceil,$$

and this leads to the desired inequality. □

The condition $f(N^+(v)) \geq k$ for each vertex v, yields analogously the next result.

Proposition 3.6. If D is a digraph of order n and maximum outdegree Δ^+, then $\gamma^*_{stk}(D) \geq 2\lceil \frac{\Delta^+ + k}{2} \rceil - n$.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same end vertices as e. Since $N^-_{D(G)}(v) = N^+_{D(G)}(v) = N_G(v)$ for each $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 3.7. If $D(G)$ is the associated digraph of a graph G, then $\gamma^*_{stk}(D(G)) = \gamma_{stk}(G)$.

Observation 3.7 has many interesting applications such as the following results.

Corollary 3.8 ([13]). *Let G be a r-regular graph with $r \geq k$. Then $\gamma_{stk}(G) \geq (k+1)n/r$ when $r + k$ is odd and $\gamma_{stk}(G) \geq kn/r$ when $r + k$ is even.*

Proof. Let $D(G)$ be the associated digraph of G. Then $D(G)$ is a r-regular digraph of order $n = n(D(G))$. Since $\gamma_{stk}(G) = \gamma^*_{stk}(D(G))$, the result follows from Corollary 3.3 and Observation 3.7. □

Proposition 3.9. *If G is a graph of order n and maximum degree Δ, then* $\gamma_{stk}(G) \geq 2\left\lceil \frac{\Delta+k}{2} \right\rceil - n$.

Proof. Since $\Delta(G) = \Delta^-(D(G))$ and $n = n(D(G))$, it follows from Proposition 3.5 and Observation 3.7 that

$$\gamma_{stk}(G) = \gamma^*_{stk}(D(G)) \geq 2\left\lceil \frac{\Delta^- + k}{2} \right\rceil - n = 2\left\lceil \frac{\Delta + k}{2} \right\rceil - n.$$

□

The next known corollary is a consequence of Theorem 3.2 and Observation 3.7.

Corollary 3.10 [12]**.** *Let G be a graph of order n, minimum degree δ and maximum degree Δ. Then*

$$\gamma_{stk}(G) \geq \frac{\left\lceil \frac{\delta+k}{2} \right\rceil - \left\lfloor \frac{\Delta-k}{2} \right\rfloor}{\left\lceil \frac{\delta+k}{2} \right\rceil + \left\lfloor \frac{\Delta-k}{2} \right\rfloor} n.$$

Since

$$\frac{\left\lceil \frac{\delta+k}{2} \right\rceil - \left\lfloor \frac{\Delta-k}{2} \right\rfloor}{\left\lceil \frac{\delta+k}{2} \right\rceil + \left\lfloor \frac{\Delta-k}{2} \right\rfloor} n \geq \frac{\delta + 2k - \Delta}{\delta + \Delta} n,$$

Corollary 3.10 implies the following known bound.

Corollary 3.11 [11]**.** *If G is a graph of order n, minimum degree δ and maximum degree Δ, then*

$$\gamma_{stk}(G) \geq \frac{\delta + 2k - \Delta}{\delta + \Delta} n.$$

Theorem 3.12. *For any digraph D of order n, size m, minimum indegree δ^- and minimum outdegree δ^+,*

$$\gamma^*_{stk}(D) \geq \frac{n(2\left\lceil \frac{\delta^+ + k}{2} \right\rceil + \left\lceil \frac{\delta^- + k}{2} \right\rceil) - 2m}{\left\lceil \frac{\delta^- + k}{2} \right\rceil}.$$

and

$$\gamma^*_{stk}(D) \geq \frac{n(2\left\lceil \frac{\delta^- + k}{2} \right\rceil + \left\lceil \frac{\delta^+ + k}{2} \right\rceil) - 2m}{\left\lceil \frac{\delta^+ + k}{2} \right\rceil}.$$

Proof. Let f be a $\gamma^*_{stk}(D)$-function. By Lemma 3.1, we have

$$\begin{aligned} m &\geq |A(M,P)| + |A(P,M)| + |A(P,P)| \\ &\geq \left\lceil \frac{\delta^- + k}{2} \right\rceil |M| + \left\lceil \frac{\delta^+ + k}{2} \right\rceil |M| + \left\lceil \frac{\delta^+ + k}{2} \right\rceil |P| \\ &= \left\lceil \frac{\delta^+ + k}{2} \right\rceil n + \left\lceil \frac{\delta^- + k}{2} \right\rceil \left(\frac{n - \gamma^*_{stk}(D)}{2} \right). \end{aligned}$$

This leads to the first inequality. Using $|A(P,P)| \geq \left\lceil \frac{\delta^- + k}{2} \right\rceil |P|$, we obtain the second inequality as follows

$$\begin{aligned} m &\geq |A(M,P)| + |A(P,M)| + |A(P,P)| \\ &\geq \left\lceil \frac{\delta^- + k}{2} \right\rceil |M| + \left\lceil \frac{\delta^+ + k}{2} \right\rceil |M| + \left\lceil \frac{\delta^- + k}{2} \right\rceil |P| \\ &= \left\lceil \frac{\delta^- + k}{2} \right\rceil n + \left\lceil \frac{\delta^+ + k}{2} \right\rceil \left(\frac{n - \gamma^*_{stk}(D)}{2} \right). \end{aligned}$$

This completes the proof. □

Theorem 3.13. *Suppose that D is a digraph of order n, maximum indegree Δ^- and maximum outdegree Δ^+. Then*

$$\gamma^*_{stk}(D) \geq \frac{2k - \left\lfloor \frac{\Delta^- - k}{2} \right\rfloor - \left\lfloor \frac{\Delta^+ - k}{2} \right\rfloor}{2k + \left\lfloor \frac{\Delta^- - k}{2} \right\rfloor + \left\lfloor \frac{\Delta^+ - k}{2} \right\rfloor} n.$$

Proof. Let f be a $\gamma^*_{stk}(D)$-function and let $v \in M$. Since $f(N^+(v)) \geq k$ and $f(N^-(v)) \geq k$, we have $|A(v,P)| \geq k$ and $|A(P,v)| \geq k$ and thus $|A(M,P)| + |A(P,M)| \geq 2k|M|$. It follows from Lemma 3.1 (Parts 1, 2) that

$$|P| \left(\left\lfloor \frac{\Delta^- - k}{2} \right\rfloor + \left\lfloor \frac{\Delta^+ - k}{2} \right\rfloor \right) \geq 2k|M|. \tag{2}$$

Replacing $|M|$ and $|P|$ by $\frac{n - \gamma^*_{stk}(D)}{2}$ and $\frac{n + \gamma^*_{stk}(D)}{2}$ in (2), the desired bound is obtained. □

Theorem 3.14. *For any digraph D of order n and size m,*

$$\gamma^*_{stk}(D) \geq 2n - \frac{m}{k}\cdot$$

This bound is sharp for digraphs obtained in the proof of Theorem 2.10.

Proof. Let f be a $\gamma^*_{stk}(D)$-function. Following the proof of Theorem 3.13, $|A(P,M)| \geq k|M|$ and $|A(M,P)| \geq k|M|$. If $x \in P$, then it follows from $f(N^+(x)) \geq k$ that $|A(x,P)| \geq |A(x,M)| + k$, indicating that

$$|A(P,P)| \geq |A(P,M)| + k|P| \geq k|M| + k(n - |M|) = kn.$$

Hence

$$\begin{aligned} m &\geq |A(M,P)| + |A(P,M)| + |A(P,P)| \\ &\geq k|M| + k|M| + kn \\ &= 2k|M| + kn \end{aligned}$$

Since $n = |P| + |M|$, we deduce that $\gamma^*_{stk}(D) = |P| - |M| = n - 2|M| \geq 2n - \frac{m}{k}$. □

Corollary 3.15 ([13])**.** *If G is a graph of order n and size m, then*

$$\gamma_{stk}(G) \geq 2n - \frac{2m}{k}\cdot$$

Proof. Let $D(G)$ be the associated digraph of G. Since $m(D(G)) = 2m(G)$, it follows from Observation 3.7 and Theorem 3.14 that

$$\gamma_{stk}(G) = \gamma^*_{stk}(D(G)) \geq 2n(D(G)) - \frac{m(D(G))}{k} = 2n(G) - \frac{2m(G)}{k} = 2n - \frac{2m}{k}\cdot$$ □

Theorem 3.16. *Let D be a digraph of order n. Then*

$$\gamma^*_{stk}(D) \geq 2 \left\lceil \frac{1 + \sqrt{4kn+1}}{2} \right\rceil - n.$$

Proof. Let f be a $\gamma^*_{stk}(D)$-function. In view of the proof of Theorem 3.14, $|A(P,P)| \geq kn$. On the other hand, $|A(P,P)| \leq |P|(|P|-1)$. It follows that $|P|(|P|-1) \geq kn$ and thus $|P|^2 - |P| - kn \geq 0$, which implies that

$$|P| \geq \frac{1 + \sqrt{4kn+1}}{2},$$

and thus we obtain

$$\gamma^*_{stk}(D) = 2|P| - n \geq 2 \left\lceil \frac{1 + \sqrt{4kn+1}}{2} \right\rceil - n\cdot$$ □

Theorem 3.17. *Let D be a bipartite digraph of order n. Then*

$$\gamma^*_{stk}(D) \geq 2\left\lceil\sqrt{2kn}\right\rceil - n.$$

Proof. Let f be a $\gamma^*_{stk}(D)$-function. In view of the proof of Theorem 3.14, $|A(P,P)| \geq kn$. On the other hand, $|A(P,P)| \leq |P|^2/2$. It follows that $|P|^2/2 \geq kn$ and so $|P| \geq \sqrt{2kn}$ and thus

$$\gamma^*_{stk}(D) = 2|P| - n \geq 2\left\lceil\sqrt{2kn}\right\rceil - n.$$ □

The next well-Known result is obtained through Theorems 3.16, 3.17 and Observation 3.7.

Corollary 3.18 ([13])**.** *If G is a graph of order n, then $\gamma_{stk}(G) \geq \sqrt{4kn+1} + 1 - n$.*
If G is a bipartite graph of order n, then $\gamma_{stk}(G) \geq 2\sqrt{2kn} - n$.

Note that our proof of Corollary 3.18 is shorter than that proposed by Wang [13]. In addition, Wang [13] has provided examples with equality in the two inequalities of Corollary 3.18. The associated digraphs for these examples indicate that the bounds in Theorems 3.16 and 3.17 are sharp.

The underlying graph of a digraph D, $G(D)$, is the graph obtained by replacing each arc (u,v) of D by the edge uv.

Theorem 3.19. *Let D be a digraph of order n and let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degree sequence of the underling graph $G(D)$ of D. If s is the smallest positive integer where $\sum_{i=1}^{s} d_i - \sum_{i=s+1}^{n} d_i \geq 2kn$, then*

$$\gamma^*_{stk}(D) \geq 2s - n.$$

Furthermore, this bound is sharp.

Proof. Let f be a $\gamma^*_{stk}(D)$-function and $p = |P|$. Since $f(N^+_D(v)) \geq k$ and $f(N^-_D(v)) \geq k$ for each $v \in V(D)$, we have

$$\begin{aligned} kn &\leq \textstyle\sum_{v\in V} f(N^+_D(v)) \\ &= \textstyle\sum_{v\in V} d^-_D(v)f(v) \\ &= \textstyle\sum_{v\in P} d^-_D(v) - \sum_{v\in M} d^-_D(v) \end{aligned}$$

and

$$\begin{aligned} kn &\leq \textstyle\sum_{v\in V} f(N^-_D(v)) \\ &= \textstyle\sum_{v\in V} d^+_D(v)f(v) \\ &= \textstyle\sum_{v\in P} d^+_D(v) - \sum_{v\in M} d^+_D(v) \end{aligned}$$

Summing the above inequalities, we have

$$\begin{aligned} 2kn &\leq \textstyle\sum_{v\in P}(d^+_D(v) + d^-_D(v)) - \sum_{v\in M}(d^+_D(v) + d^-_D(v)) \\ &= \textstyle\sum_{v\in P} \deg_G(v) - \sum_{v\in M} \deg_G(v) \\ &\leq \textstyle\sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i. \end{aligned}$$

Consequently, $2kn \leq \sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i$. By the assumption on s, we must have $p \geq s$. This implies that $\gamma^*_{stk}(D) = 2p - n \geq 2s - n$.

In order to prove the sharpness, suppose that $t \geq 3$ is an integer and D is obtained from the union of k directed cycles $C_t^s := (v_1^s, \ldots, v_t^s)$, $1 \leq s \leq k$, by adding the set

$$\{u_1^j, \ldots, u_t^j \mid 1 \leq j \leq k+1\}$$

of new vertices and the set

$$\{(u_i^j, v_i^s), (v_{i+1}^s, u_i^j) \mid 1 \leq i \leq t, \ 1 \leq s \leq k, \ 1 \leq j \leq k+1\}$$

of new arcs, where the arithmetic operations are taken modulo t and $t+1$ represents 1. Then the order of D is $n = 2kt + t$ and the degree sequence of the underlying graph of D is $\underbrace{2k+4, \ldots, 2k+4}_{kt}, \underbrace{2k, \ldots, 2k}_{kt+t}$ and

$$\sum_{i=1}^{2kt} d_i - \sum_{2kt+1}^{n} d_i = (2k+4)(kt) + 2k(kt) - 2kt = 4k^2t + 2kt = 2kn.$$

It follows that $s = 2kt$ is the smallest positive integer such that $\sum_{i=1}^{s} d_i - \sum_{i=s+1}^{n} d_i \geq 2kn$ and so $\gamma_{stk}^*(D) \geq 2kt - t$. Now define $f : V(D) \to \{-1, 1\}$ with $f(x) = -1$ for $x \in \{u_1^1, \ldots, u_t^1\}$ and $f(x) = +1$ for all other vertices x. Clearly, f is a TSTkDF of D and $\omega(f) = 2kt - t$. This completes the proof. □

The special case $k = 1$ of Theorems 3.2, 3.12, 3.13, 3.14 and 3.19 was recently proved in [1].

4. Twin signed total k-domination in oriented graphs

Suppose that G is the complete graph K_{2k+3} with vertex set $\{v_1, \ldots, v_{2k+3}\}$. Let D_1 be an orientation of G such that $D_1 \simeq CT(2k+3)$ and D_2 be an orientation of G obtained from $CT(2k+1)$ with vertex set $\{v_1, \ldots, v_{2k+1}\}$ by adding new vertices v_{2k+2} and v_{2k+3} and the set $\{(v_{2k+2}, v_i), (v_j, v_{2k+2}), (v_{2k+3}, v_j), (v_i, v_{2k+3}), (v_{2k+2}, v_{2k+3}) \mid 1 \leq i \leq k+1, \ k+2 \leq j \leq 2k+1\}$. It is easy to see that $\gamma_{stk}^*(D_1) = 2k+3$ and $\gamma_{stk}^*(D_2) = 2k+1$. Thus two distinct orientations of a graph can have distinct twin signed total k-domination numbers. Motivated by this observation, we define lower orientable twin signed total k-domination number $\mathrm{dom}_{stk}^*(G)$ and upper orientable twin signed total k-domination number $\mathrm{Dom}_{stk}^*(G)$ of a graph G with $\delta(G) \geq 2k$ as follows:

$$\mathrm{dom}_{stk}^*(G) = \min\{\gamma_{stk}^*(D) \mid D \text{ is an orientation of } G \text{ with } \delta^-(D) \geq k \text{ and } \delta^+(D) \geq k\},$$

and

$$\mathrm{Dom}_{stk}^*(G) = \max\{\gamma_{stk}^*(D) \mid D \text{ is an orientation of } G \text{ with } \delta^-(D) \geq k \text{ and } \delta^+(D) \geq k\}.$$

Corresponding concepts have been defined and studied for orientable domination (out-domination) [7], twin domination number [6], twin signed domination number [3], twin signed total domination number [1], twin minus domination number [2] and twin signed Roman domination number [5].

The next proposition shows that every graph G with $\delta(G) \geq 2k$, has an orientation D such that $\delta^-(D), \delta^+(D) \geq k$ and so the above definitions are well-defined.

Proposition 4.1. for integer $k \geq 1$, every graph G with $\delta(G) \geq 2k$, has an orientation D such that $\delta^-(D), \delta^+(D) \geq k$.

Proof. Since $\delta(G) \geq 2k \geq 2$, G contains a cycle. Let $G_1 = G$ and C_1 be a cycle in G. Assume that $G_2 = G_1 - E(C_1)$ and let C_2 be a cycle in G_2. We proceed this procedure to obtain a graph G_t such that every nontrivial component of G is a tree. Consider an orientation of G such that every cycle C_i, $1 \leq i \leq t$, be a directed cycle and every nontrivial component of G_t be a directed tree such that $|d^+(v) - d^-(v)| \leq 1$ for every vertex v. Hence we obtain an orientation D of G such that $\delta^-(D), \delta^+(D) \geq k$. □

The next corollary is an immediate consequence of Proposition 2.11.

Corollary 4.2. *For integer $k \geq 1$ and for any graph G of order n with $\delta(G) \geq 2k$, $\gamma_{st2k}(G) \leq \mathrm{dom}^*_{stk}(G)$.*

The lower orientable twin signed total k-domination numbers of several classes of graphs including complete graphs and complete bipartite graphs are determined here. The next propositions are useful in our results.

Proposition 4.3 [13]**.** For $n > k$, $\gamma_{stk}(K_n) = \begin{cases} k+2 & \text{if } n \equiv k \pmod 2 \\ k+1 & \text{if } n \equiv k+1 \pmod 2. \end{cases}$

Proposition 4.4 [9]**.** For $m, n \geq k$, $\gamma_{stk}(K_{m,n}) = \begin{cases} 2k & \text{if } m \equiv n \equiv k \pmod 2 \\ 2k+1 & \text{if } m \equiv k+1\ n \equiv k \pmod 2 \\ 2k+2 & \text{if } m \equiv n \equiv k+1 \pmod 2. \end{cases}$

The next corollaries are immediate consequences of Corollary 4.2 and Propositions 4.3 and 4.4.

Corollary 4.5. *For $k \geq 1$ and $n \geq 2k+1$,*

$$\mathrm{dom}^*_{stk}(K_n) \geq \begin{cases} 2k+1 & \text{if} \quad n \text{ is odd} \\ 2k+2 & \text{if} \quad n \text{ is even.} \end{cases}$$

Corollary 4.6. for $m, n \geq 2k$,

$$dom^*_{stk}(K_{m,n}) \geq \begin{cases} 4k & \text{if} \quad n \text{ and } m \text{ are both even} \\ 4k+1 & \text{if} \quad n \text{ and } m \text{ have different parity} \\ 4k+2 & \text{if} \quad n \text{ and } m \text{ are both odd .} \end{cases}$$

Theorem 4.7. For $k \geq 1$ and $n \geq 2k+1$,

$$\mathrm{dom}^*_{stk}(K_n) = \begin{cases} 2k+1 & \text{if} \quad n \text{ is} \quad \text{odd} \\ 2k+2 & \text{if} \quad n \text{ is} \quad \text{even.} \end{cases}$$

Proof. Let

$$V(K_n) = \left\{u_i, v_i, w_j \mid 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - (k+1) \text{ and } 1 \leq j \leq n - \left(2\left\lceil \frac{n}{2} \right\rceil - 2k - 2\right)\right\}.$$

First let n is odd. Let D be an orientation of K_n such that

$$\begin{aligned} A(D) = \{&(u_t, u_l), (u_t, v_l), (v_t, v_l), (v_r, u_s) \mid 1 \leq t < l \leq \lceil \tfrac{n}{2} \rceil - (k+1),\ 1 \leq r \leq s \leq \lceil \tfrac{n}{2} \rceil - (k+1)\} \\ &\cup \left\{(u_i, w_j), (v_i, w_j), (w_t, u_i), (w_t, v_i) \mid 1 \leq i \leq \lceil \tfrac{n}{2} \rceil - (k+1),\ 1 \leq j \leq k+1,\ k+2 \leq t \leq 2k+1\right\} \\ &\cup \{(w_t, w_{t+1}), \ldots, (w_t, w_{t+k}) \mid 1 \leq t \leq 2k+1\}, \end{aligned}$$

Where the arithmetic operations are taken modulo $2k+1$ and $2k+1$ represents 0.

Let now n be even. Let D be an orientation of K_n such that

$$\begin{aligned} A(D) = \{&(u_t, u_l), (u_t, v_l), (v_t, v_l), (v_r, u_s) \mid 1 \leq t < l \leq \lceil \tfrac{n}{2} \rceil - (k+1),\ 1 \leq r \leq s \leq \lceil \tfrac{n}{2} \rceil - (k+1)\} \\ \cup &\{(u_i, w_j), (v_i, w_j), (w_t, u_i), (w_t, v_i) \mid 1 \leq i \leq \lceil \tfrac{n}{2} \rceil - (k+1),\ 1 \leq j \leq k+1,\ k+2 \leq t \leq 2k+2\} \\ \cup &\{(w_t, w_{t+1}), \ldots, (w_t, w_{t+k}), (w_{2k+2}, w_j), (w_{j+k}, w_{2k+2}), (w_{2k+2}, w_{2k+1}) \mid 1 \leq t \leq 2k+1,\ 1 \leq j \leq k\}. \end{aligned}$$

Where the arithmetic operations are taken modulo $2k+1$ and $2k+1$ represents 0.

It is easy to see that the function $f : V(D) \to \{-1, +1\}$ defined by $f(x) = -1$ for $x \in \{u_i \mid 1 \leq i \leq \lceil \frac{n}{2} \rceil - (k+1)\}$ and $f(x) = +1$ otherwise, is a TSTkDF of D of weight $2k+1$ when n is odd and $2k+2$ when n is even. This implies that

$$\mathrm{dom}^*_{stk}(K_n) \leq \omega(f) = \begin{cases} 2k+1 & \text{if} \quad n \text{ is odd} \\ 2k+2 & \text{if} \quad n \text{ is even.} \end{cases}$$

Now the result follows from Corollary 4.5. □

Theorem 4.8. For $m, n \geq 2k$,

$$\mathrm{dom}^*_{stk}(K_{m,n}) = \begin{cases} 4k & \text{if } n \text{ and } m \text{ are both even} \\ 4k+1 & \text{if } n \text{ and } m \text{ have different parity} \\ 4k+2 & \text{if } n \text{ and } m \text{ are both odd .} \end{cases}$$

Proof. Let $V_1 = \{u_1, \ldots, u_m\}$ and $V_2 = \{v_1, \ldots, v_n\}$ be the partite sets of $K_{m,n}$. First assume that m and n are both even. Let D be an orientation of $K_{m,n}$ such that $A(D) = \{(u_i, v_j), (u_l, v_j), (v_j, u_r), (v_t, u_i)(u_i, v_s), (u_l, v_t), (v_s, u_l), (v_t, u_r), (u_r, v_s) \mid 2k+1 \leq i \leq m,\ 2k+1 \leq j \leq n,\ 1 \leq l, t \leq k,\ k+1 \leq r, s \leq 2k\}$. It is easy to verify that the function $f : V(G) \to \{-1, +1\}$ defined by $f(x) = -1$ for $x \in \{u_{2k+1}, \ldots, u_{\frac{m+2k}{2}}\} \cup \{v_{2k+1}, \ldots, v_{\frac{n+2k}{2}}\}$ and $f(x) = +1$ otherwise, is a TSTkDF of D of weight $4k$ and so $\mathrm{dom}^*_{stk}(K_{m,n}) \leq 4k$.

Now let m and n have different parity. Assume that m is even and n is odd. Let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_j), (u_l, v_j), (v_j, u_r), (v_t, u_i), (u_i, v_s), (u_l, v_t)(v_s, u_l), (v_t, u_r), (u_r, v_s) \mid$$
$$2k+1 \leq i \leq m,\ 2k+2 \leq j \leq n,\ 1 \leq l, t \leq k,\ k+1 \leq r \leq 2k,\ k+1 \leq s \leq 2k+1\}.$$

Define $f(x) = -1$ for $x \in \{u_{2k+1}, \ldots, u_{\frac{m+2k}{2}}\} \cup \{v_{2k+2}, \ldots, v_{\frac{n+2k+1}{2}}\}$ and $f(x) = +1$ otherwise. It is easy to verify that f is a TSTkDF of D of weight $4k+1$ and so $\mathrm{dom}^*_{stk}(K_{m,n}) \leq 4k+1$.

Finally assume that m and n are both odd. Let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_j), (u_l, v_j), (v_j, u_r), (v_t, u_i), (u_i, v_s), (u_l, v_t)(v_s, u_l), (v_t, u_r),$$
$$(u_r, v_s) \mid 2k+2 \leq i \leq m,\ 2k+2 \leq j \leq n,\ 1 \leq l, t \leq k,\ k+1 \leq r, s \leq 2k+1\}.$$

Define $f(x) = -1$ for $x \in \{u_{2k+2}, \ldots, u_{\frac{m+2k+1}{2}}\} \cup \{v_{2k+2}, \ldots, v_{\frac{n+2k+1}{2}}\}$ and $f(x) = +1$ otherwise. It is easy to verify that f is a TSTkDF of D of weight $4k+2$ and so $\mathrm{dom}^*_{stk}(K_{m,n}) \leq 4k+2$.
Now, the result follows from Corollary 4.6. □

The special case $k = 1$ of Theorems 4.7 and 4.8 was recently proved in [1]. Theorems 4.7 and 4.8 also show that $\mathrm{dom}^*_{stk}(K_n) = \gamma_{st2k}(K_n)$ and $\mathrm{dom}^*_{stk}(K_{m,n}) = \gamma_{st2k}(K_{m,n})$.

5. Conclusion

In this paper, we initiate the study of twin signed total k-domination numbers in digraphs and we present some bounds on this parameter which some of them are sharp. We observe that $\gamma_{stk}(D) \leq \gamma^*_{stk}(D)$ for all digraphs and $\gamma_{st2k}(G) \leq \mathrm{dom}^*_{stk}(G)$ for all graphs. Also, we observe that the equality $\gamma_{stk}(D) = \gamma^*_{stk}(D)$ is valid for circulant tournaments and $\gamma_{st2k}(G) = \mathrm{dom}^*_{stk}(G)$ for complete graphs and complete bipartite graphs. The problem of finding other classes of graphs and digraphs achieving the equalities is under investigation.

Also, we introduce the lower and upper orientable twin signed total k-domination numbers of graphs. It might be worthwhile to find $\mathrm{Dom}^*_{stk}(G)$ for some classes of graphs and find conditions to the equality $\mathrm{Dom}^*_{stk}(G) = |V(G)|$.

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