RAIRO-Oper. Res. 52 (2018) 285–303
https://doi.org/10.1051/ro/2016065

RAIRO Operations Research
www.rairo-ro.org

A COMPARISON BETWEEN STOCHASTIC DEA AND FUZZY DEA APPROACHES: REVISITING EFFICIENCY IN ANGOLAN BANKS

P. Wanke[1,*], C.P. Barros[2] and A. Emrouznejad[3]

Abstract. Performance analysis has become a vital part of the management practices in the banking industry. There are numerous applications using DEA models to estimate efficiency in banking, and most of them assume that inputs and outputs are known with absolute precision. Here, we compare Stochastic-DEA and Fuzzy-DEA models to assess, respectively, how the underlying randomness and fuzziness impact efficiency levels. The proposed models have been demonstrated using an application in Angolan banks. Findings reveal that conclusions with respect to the ranking of DMUs may vary substantially depending upon the type of the model chosen, although efficiency scores are similar to some extent when compared within the ambits of Stochastic-DEA and Fuzzy-DEA models. Additionally, modeling choices on fuzziness, rather than on randomness, appears to be the most critical source for variations in efficiency rankings. Managerial implications for Angolan banks are also explored.

Mathematics Subject Classification. 90C05, 62F40, 62U20, 03B52.

Received May 13, 2016. Accepted September 21, 2016.

1. Introduction

One of the major research areas in banking is the measurement of the relative efficiency of banks by means of popular non-parametric techniques such as Data Envelopment Analysis [60]. In recent years, several scholars have developed new Data Envelopment Analysis (DEA) models to handle input and output uncertainty [59]. A possible path to handle input/output uncertainty in DEA relies on the use of probability distributions to model their inherent randomness. These distributions are subsequently employed in stochastic DEA models [23, 41, 87, 110]. Alternatively, however, uncertainty in input/output may be related to imprecision or vagueness, rather than to randomness. This being the case, imprecision or vagueness in input/output values can be expressed by membership functions within the ambit of fuzzy logic [36].

Angolan banks have been the object of a few recent researches, conducted using different approaches rather than DEA. While Wanke *et al.* [115] used TOPSIS and neural networks in a two-stage approach, Barros *et al.* [16] employed a B-Convexity model to estimate efficiency levels. Therefore, this paper innovates and builds upon the current literature by revisiting the relative efficiency of Angolan banks, comparing the results obtained using

Keywords. Data envelopment analysis, Angola, Fuzzy-DEA, Stochastic-DEA, KL-Divergence.

[1] Center for Studies in Logistics, Infrastructure and Management, COPPEAD Graduate Business School, Federal University of Rio de Janeiro, Rua Paschoal Lemme, 355. 21949900 Rio de Janeiro, Brisil.
[*] Corresponding author: peter@coppead.ufrj.br
[2] ISEG – Lisbon School of Economics and Management; ULisboa and CEsA – Research Centre on African, Asian and Latin American Studies; Rua Miguel Lupi, 20. 1249-078 Lisboa, Portugal.
[3] Aston Business School, Aston University, Birmingham, B4 7ET, UK.

Article published by EDP Sciences

three major Fuzzy DEA (FDEA) models, based on the α-level approach, with those derived from Stochastic DEA (SDEA) models, built upon different copulas (Clayton, Frank, and Gumbel) in order to incorporate alternative dependence structure between outputs. Thus far, applications of SDEA and FDEA to measure bank efficiency have been scarce and conducted in an isolated fashion, while both of these applications focused on ranking DMUs [29, 62, 69, 95, 96, 113, 117, 119]. In summary, the novelty of this paper resides in the practical application and comparison of different SDEA and FDEA models. Besides, according to Wanke *et al.* [117], the combination of fuzzy and probabilistic approaches represents a contribution to the emerging literature on possible analytical venues within the ambit of 2-Dimentional Fuzzy Monte Carlo Analysis (2D FMCA).

Specifically, the motivations for the present research are related to the following issues. The first one relates to the evaluation of the relative efficiency of Angolan banks using, simultaneously for the first time, copulas to model output dependence within the ambit of SDEA models and the α-level approach, within the ambit of FDEA models. With respect to randomness, one should note that a common assumption in DEA models is the independence of inputs and outputs. Very few studies, however, have challenged the assumption so as to assess the impact of different dependence structures or tail dependence of outputs and inputs within the ambit of efficiency measurement models [116]. As a matter of fact, the concept of tail dependence can be embedded within the copula theory [99]. On the other hand, as regards fuzziness, despite the existence of different types of fuzzy approaches for handling vagueness within the ambit of DEA models – see Emrouznejad and Tavana [42] for a comprehensive literature review on this subject – the α-level approach was chosen here not only in terms of its popularity among researchers, but also because in this approach an FDEA model is solved by parametric programming using α-levels. Solving this model at a given level of α produces interval efficiency for the decision making unit (DMU) under assessment [127].

The second motivation pertains to expanding the literature on 2D FMCA by using the Kullback-Leibler (KL) divergence [31, 81] to assess the similarity of several SDEA and FDEA cumulative distributions for the efficiency scores under different SDEA tail dependence structures and FDEA values of α-levels. The third goal concerns the coverage of a significant time span of a representative sample of Angolan banking – 2006 to 2014 – so that uncertainty in its different forms can be assessed. As a matter of fact, the outputs and inputs of banks presents different forms of uncertainty within their relationships. For example, loans are an output embedded in fuzziness because of the ex-ante risks associated with non-performing credit granting [84]. On the other hand, the operating income of a bank, which is not a constant number, changes randomly on account of the market value of the investment target. To evaluate Angolan bank efficiency more realistically and accurately, this study compares results obtained from FDEA and SDEA models.

Therefore, this study proposes a predictive model for banking efficiency in Angola based on the financial and operational criteria commonly found in the literature and considers uncertainty in the collection of input and output data. The remainder of the paper is organized as follows: Section 2 presents the contextual setting; Section 3 reviews the literature; Section 4 presents the data source and the model; results are discussed and presented in Section 5; and Section 6 sets out the conclusion.

2. Contextual setting

Angola is presently suffering a recession based on oil price decrease. The public expenditure was adjusted and all economy shrunk responding to the oil decrease. Meanwhile the central bank continues the adjustment of inflation to one digit. This process of controlling the money supply started in 2000 with direct support from the support of IMF – International Monetary Fund. The adoption of a macro-economic stabilization program has started to achieve its aims. The direct support stopped in 2009 but the IMF continues to oversee the central bank money supply. In this contractionary context the Angola banks face intense competition alongside with market shrinkage. Therefore, some mergers appear in banking. The banks analyzed in this paper are the older ones, already in existence in the first year of observation and includes some foreign banks that have around 50% share from a local partner, fixed by law. Banks belonging to local owners includes the BIC, which belongs to the leading Angola female entrepreneurship (50%). The other 50% are spread among various Angolan businessmen. BIC is the sole Angolan bank that is also established in Portugal, and it recently bought a bankrupt Portuguese bank – BPN-Banco Portuguese de Negócios. The Keve is a small Angolan bank with many owners and the maximum shareholding is 6.95% of the capital. The Banco Sol is owned by GEFI – Sociedade de Gestão e Participações Financeiras Business Management and Finance Company, which is owned by Angolan entrepreneurs and 55% of its shares are held through its subsidiary Sansul, while other wealth influential citizens hold the remainder. GEFI owns other assets in the hotel industry, fisheries, media, construction and real estate. BPC-Banco de Poupança e Crédito is a public bank, and BAI-Banco Africano de Investimentos belongs to Sonangol. The Sonangol Oil Company accounts for 52% of the country's GDP. Sonangol also owns the Banco Economico, the former BESA that went bankrupted during the failure of Portuguese BES group in 2014. The foreign banks includes the Millennium (68.5% of the capital is held by the Portuguese bank Millennium) and BFA – Banco de Fomento Angola (50% of the capital is in the possession of the Portuguese bank BPI-Banco Portugues de Investimento). The Banco Caixa Geral Totta de Angola jointly belongs to the oil company Sonangol (49%) and to the public Portuguese bank Caixa Geral de Depósitos (51%). Other foreign-owned banks include BCP, which belongs to a South African Bank and Barclays (50%).

3. Review of the literature

Charnes *et al.* [26] first proposed DEA for the case of constant returns-to-scale, which became known as the CCR (Charnes, Cooper and Rhodes) model. Subsequently, Banker *et al.* [9] extended the model to the case of varying returns-to-scale; the model came to be known as BCC (Banker, Charnes and Cooper). Both models apply linear programming and allow output/input weighting to compute efficiency scores [66]. Nowadays, several different DEA models are employed in different circumstances, *e.g.*, industries, countries, and organizations involved in efficiency assessment [22].

Despite the numerous studies focusing on banking efficiency and productivity using DEA [3, 5, 6, 20, 49–51, 61, 70, 108], an in-depth analysis of banks in Africa is still missing [7, 8, 46, 64, 71, 75, 76, 89, 91], thus indicating a literature gap. The situation contrasts with the extensive research that has been carried out on American banks [17, 18, 20], on European banks [11, 13, 77], on Asian banks [7, 14, 21, 28, 78], and even South American banks [106, 114].

The choice of inputs and outputs is perhaps the most important task in employing DEA to measure the relative efficiency of the DMUs. Two approaches are widely used to identify a bank's inputs and outputs: the production approach and the intermediation approach (*e.g.* [2,16,19,45,86,100,104,123]). Under the production approach, banks are treated as a firm to produce loans, deposits, and other operational results by using labor and assets. However, banks are considered as financial intermediaries to transform deposits, purchase funds and labor into loans and other assets under the intermediation approach. More specifically, deposits are treated as an input under the production approach and an output under the intermediation approach.

Fortin and Leclerc [47], however, showed that with an incomplete list of assets and liabilities, the ratio between assets and liabilities included in the model of banking production strongly influences the efficiency score under the intermediation approach. In fact, the authors found that the average score varies significantly according to the definition of inputs and outputs, thus biasing the analysis. Therefore, taking into consideration the risk of biasing the analysis for the Angolan banks under the intermediation approach, the production approach in banking is adopted in this research.

Traditionally, DEA models consider that output and input are measurements with no embedded fuzziness of randomness. As a matter of fact, one disadvantage of conventional DEA is its deterministic nature [24]. Some authors claim that DEA, unable to deal with stochastic noise in inputs and outputs, cannot distinguish between actual inefficiency and short-lived negative effects or plain bad luck. Since DEA is an extreme point technique, noise (even symmetrical noise with zero mean) measurement error can cause significant problems, because the frontier is sensitive to these errors [37, 92, 118]. By working with random variables (*i.e.*, considering the possibility for occurrence of unforeseen events), different aspects of the information can be detected. One advantage of working with random data in DEA is a clear understanding on how efficiencies behave. There are various approaches to include the stochastic element in the efficiency analysis of DEA, namely, imprecise DEA, bootstrapping [52], Monte Carlo simulation and chance constrained DEA [40], Banker's F tests, chance constrained programming, Varian's statistical test of cost minimization [97]. SDEA applications in banking are scarce and include Chen [27] and Kao and Liu [69] who used SDEA to assess the efficiency of Taiwanese commercial banks. Research on Angola banks includes Barros *et al.* [16] study of efficiency in the banking sector with the B-convexity model; and Wanke *et al.* [115], who analyzed efficiency in the banking sector with a two-stage TOPSIS and neural networks approach. Furthermore Barros and Mendes [12] analyzed the competition of Angola banks.

On the other hand, if input and output values were fuzzy, traditional DEA could not be able to assess efficiency levels in a proper manner. This being the case, several researchers [33,38,55,65,68] started structuring FDEA models, allowing for the measurement of outputs and inputs as fuzzy numbers. Particularly with respect to FDEA applications on banking, studies to assess efficiency in the financial sector still remain scarce, while their major focus tends to relate to ranking of DMUs based on computed fuzzy efficiencies [29,62,95,96,113,119].

4. Methodology

This section presents the major methodological steps adopted in this research. After presenting in Section 4.1 the data collected in terms of inputs, and outputs, the SDEA and FDEA models used in this research are explained in detail. Section 4.2 is devoted to discussing the application of the three major FDEA models used in this research. In turn, Section 4.3 is focused on the backgrounds of SDEA models, establishing the major links with respect to the use copulas for modeling the dependence structure between productive outputs. Section 4.4 concludes with the importance of comparing FDEA and SDEA scores.

4.1. The data

The data on fifteen Angolan banks was obtained from KPM and Deloitte reports on Angola banks, published yearly and usually available in web and encompassed the period from 2006 to 2014. The inputs and the outputs considered observed not only those commonly found in the literature review but also the availability of data. As regards the lack of differentiation in efficiency scores, one of the most common problems in DEA is caused by an excessive number of input and output variables with respect to the number of DMUs [1]; this research observes the convention that the minimal number of DMUs should be three times greater than the sum of the number of inputs and outputs [15]. In fact, there are 135 observations (15 DMUs $\times$ 9 years), which is greater than the total number of inputs and outputs multiplied by three, as detailed next.

The inputs and the outputs considered observed not only those commonly found in the literature review but also the availability of data. The input variables included operating costs – excluding labor costs – (USD/year), assets (USD), and number of employees. Output variables included total deposits (USD), operating income (USD/year), operational results (USD/year), securities (USD/year) and loans (USD/year). Their descriptive statistics are presented in Table 1. Readers should be aware of the fact that FDEA models are adequate to assess efficiency levels when inputs and outputs are subject to uncertainty or vagueness [68]. In this research, given the difficulty in obtaining reliable and consistent data sources on Angolan banks over the course of time, we decided to treat all the inputs and outputs as triangular fuzzy numbers, with their lower and upper values defined by an offset of 20% from their respective crisp mean values. On the other hand, considering that the conversion of the values originally expressed in Angolas's national currency (kwanzas) to US Dollars is often subject to financial crisis and/or currency board controls, we used the power purchase parity criteria to adjust the cost related contextual variables accordingly (source: Angola Central Bank). Besides, in this research, the sensitivity analysis conducted on fuzzy efficiency scores complies with the combined probabilistic-fuzzy approach advocated by Arunraj *et al.* [4], where both randomness and uncertainty are jointly considered as fuzzy methods. More precisely, randomness is treated here using the copula theory for treating the tail dependence in the outputs, as is discussed next, in light of their correlation coefficients.

Correlation analyses presented in Tables 2 and 3 indicate significant positive relationships between the input and the output variables, which are, therefore, isotonic, thus justifying their inclusion in the model [111]. Readers

TABLE 1. Descriptive statistics for the inputs and outputs.

	Variables	Mean	SD	Min	Max	CV
Inputs	Assets (1000 USD)	\$ 468 611.10	\$ 628 812.14	\$ 21 521.00	\$ 3 459 458.00	1.342
	Operating Costs (1000 USD/year)	\$ 200.85	\$ 164.36	\$8.40	\$ 1160.83	0.818
	Number of Employees	2482.79	3481.38	58.00	19 787.00	1.402
Outputs	Operating Income (1000 USD/year)	\$ 79 286.10	\$ 202 224.14	\$ 242.00	\$ 845 659.00	2.551
	Operational Results 10000 (USD/year)	\$ 41 193.99	\$ 109 358.68	\$ 97.00	\$ 550 959.00	2.655
	Deposits (1000 USD)	\$ 118 743.28	\$ 157 744.74	\$ 594.00	\$ 772 447.00	1.328
	Loans (1000 USD/year)	\$ 336 942.59	\$ 801 478.59	\$ 569.00	\$ 3 388 306.00	2.379
	Securities (1000 USD/year)	\$ 80 332.77	\$ 200 237.11	\$ 232.00	\$ 843 971.00	2.493

TABLE 2. Correlations between output variables.

	Operating income	Operational results	Deposits	Loans	Securities
Operating income	1.000				
Operational results	0.970	1.000			
Deposits	0.655	0.616	1.000		
Loans	0.881	0.857	0.549	1.000	
Securities	0.876	0.854	0.537	0.978	1.000

TABLE 3. Correlations between input variables.

	Assets	Operating costs	Number of employees
Assets	1.000		
Operating costs	0.630	1.000	
Number of employees	0.264	0.524	1.000

can easily note that the average correlation between outputs is substantially higher when compared with that obtained between inputs. Therefore, an output orientation to handle this tail dependence structure is necessary within the ambit of SDEA models.

4.2. Fuzzy DEA

There are two approaches for modelling uncertainty within the ambit of DEA: fuzzy and stochastic. The latter uses probability distributions to model the error process and is discussed next section [103]. The former, however, departs from the fuzzy set algebra [124] and this is the cornerstone that permits fuzziness and vagueness to be treated in uncertain circumstances. FDEA models found in literature are usually classified according to four general approaches [43,57,82]: (i) tolerance; (ii) α-level; (iii) fuzzy ranking; and (iv) possibility. Here we will confine the focus to the major α-level approaches found in literature, as compiled in Hatami-Marbini *et al.* [58].

The α-level approach is possibly the most popular, given the numerous papers produced using its variations, despite the fact that their models are not computationally efficient. This is so because α-level models demand more linear programs to be solved for each α value [105]. Within the α-level approach, the FDEA model is first converted into a pair of parametric programs so that the lower and upper bounds of the efficiency scores can be computed next for a given value of α [42].

The rationale behind the selection of the α-level approach in this study is related to a number of aspects. First, when using this approach, fuzzy inputs and outputs may be expressed as crisp numbers representing the limiting bounds of the intervals for different α-levels [29], thus allowing the uncertainty of the data collected from Mozambican banks to be easily modelled as triangular fuzzy numbers. Second, in the situation of various α-levels for the inputs and the outputs, FDEA may be translated into traditional DEA (crisp) models in light of the Extension Principle, thus making solving their respective linear programs simpler [121,125,128]. Third, owing to the input and output data being fuzzy numbers, the efficiency scores are also fuzzy numbers [95]. Moreover, as long as the efficiency values considered here are the upper and lower "crisp" bounds computed for various α-levels, the membership functions for the true fuzzy efficiency cannot be reconstructed, which has a number of implications on how fuzzy efficiencies should be ranked [29, 62, 95]. These bounds, however, can be treated as crisp values and incorporated into statistical modelling as efficiency scores subjected to certain fixed effects or treatments in order to properly assess the impact of different contextual variables.

Kao and Liu [67] developed a procedure to measure the efficiencies when inputs and outputs are fuzzy, starting out with a modified BCC model. This model is solved first at a given level of α-level and leads to an interval efficiency – lower and upper bounds – for each DMU. Let $(w_p)_\alpha^L$ be the lower bound and $(w_p)_\alpha^U$ be the upper bound of the fuzzy efficiency score for a specific α-level. Furthermore, let $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ denote, respectively, the input and output values for the DMU$_p$. The pair of mathematical models proposed in Kao and Liu [67] is given as follows:

$$
\begin{aligned}
(w_p)_\alpha^L &= \max \sum_{r=1}^{s} u_r (Y_{rp})_\alpha^L + u_0 \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r (Y_{rp})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ip})_\alpha^U + u_0 \leqslant 0, \\
& \sum_{r=1}^{s} u_r (Y_{rj})_\alpha^U - \sum_{i=1}^{m} v_i (X_{ij})_\alpha^L + u_0 \leqslant 0, \quad j, \quad \forall\, j \neq p \\
& \sum_{i=1}^{m} v_i (X_{ip})_\alpha^U = 1, \quad u_r, v_i \geqslant 0, \quad \forall r, i
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
(w_p)_\alpha^U &= \max \sum_{r=1}^{s} u_r (Y_{rp})_\alpha^U + u_0 \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r (Y_{rp})_\alpha^U - \sum_{i=1}^{m} v_i (X_{ip})_\alpha^L + u_0 \leqslant 0, \\
& \sum_{r=1}^{s} u_r (Y_{rj})_\alpha^L - \sum_{i=1}^{m} v_i (X_{ij})_\alpha^U + u_0 \leqslant 0, \quad \forall j, \quad j \neq p \\
& \sum_{i=1}^{m} v_i (X_{ip})_\alpha^L = 1, \quad u_r, v_i \geqslant 0, \forall r, i
\end{aligned} \tag{4.2}
$$

where $\left[(X_{ij})_\alpha^L, (X_{ij})_\alpha^U\right]$ and $\left[(Y_{rj})_\alpha^L, (Y_{rj})_\alpha^U\right]$ are α-level specifications for the respective inputs/outputs in their fuzzy form.

In turn, Saati *et al.* [98] presented a fuzzy CCR model in its possibilistic form, transforming this model into an interval programming by means of α-levels. The transformed model could be solved, for a given α, as a crisp linear program. More precisely, model (4.4) proposed by Saati *et al.* [98] is derived for a particular case where the inputs and outputs are triangular fuzzy numbers:

$$\begin{aligned}
\max \quad & w_p = \sum_{r=1}^{s} y'_{rp} \\
\text{s.t.} \quad & \sum_{r=1}^{s} y'_{rj} - \sum_{i=1}^{m} x'_{ij} \leqslant 0, \quad \forall j, \qquad (4.3)\\
& v_i\left(\alpha x_{ij}^m + (1-\alpha)\, x_{ij}^l\right) \leqslant x'_{ij} \leqslant v_i\left(\alpha x_{ij}^m + (1-\alpha)\, x_{ij}^u\right), \quad \forall i,\ j, \\
& u_r\left(\alpha y_{rj}^m + (1-\alpha)\, y_{rj}^l\right) \leqslant y'_{rj} \leqslant u_r\left(\alpha y_{rj}^m + (1-\alpha)\, y_{rj}^u\right), \quad \forall i,\ j, \\
& \sum_{i=1}^{m} x'_{ip} = 1, \quad u_r, v_i \geqslant 0, \quad \forall r,\ j \qquad (4.4)
\end{aligned}$$

where $\tilde{x}_{ij} = \left(x_{ij}^l, x_{ij}^m, x_{ij}^u\right)$ and $\tilde{y}_{rj} = \left(y_{rj}^l, y_{rj}^m, y_{rj}^u\right)$ are the inputs/outputs expressed in terms of triangular fuzzy numbers, and x'_{ij} and y'_{ij} are decision variables used to convert the original fuzzy model into a linear program with $\alpha \in [0,1]$.

More recently, Liu [85] proposed a FDEA model to compute efficiency within the assurance region concept. The author applied the α-level approach and Zadeh's extension principle [126, 129] to convert this model into a pair of parametric mathematical programs. Therefore, the relationship importance of the inputs and outputs is given as $\frac{L_{I\delta}}{U_{Iq}} \leqslant \frac{v_\delta}{v_q} \leqslant \frac{U_{I\delta}}{L_{Iq}}$, $\delta < q = 1, \ldots, m$; and $\frac{L_{O\delta}}{U_{Oq}} \leqslant \frac{v_\delta}{v_q} \leqslant \frac{U_{O\delta}}{L_{Oq}}$, $\delta < q = 2, \ldots, s$; respectively.

The two parametric models proposed by Liu (2008) are as follows:

$$\begin{aligned}
(W_p)_\alpha^L = \max & \sum_{r=1}^{s} u_r\, (y_{rp})_\alpha^L \\
\text{st.} \quad & \sum_{r=1}^{s} u_r\, (y_{rj})_\alpha^U - \sum_{i=1}^{m} v_i\, (x_{ij})_\alpha^L \leqslant 0 \quad \forall\ j,\ j \neq p, \qquad (4.5)\\
& -v_\delta + I_{\delta q}^L v_q \leqslant 0,\ v_\delta - I_{\delta q}^U v_q \leqslant 0, \quad \forall\ \delta < q, \\
& -u_\delta + O_{\delta q}^L u_q \leqslant 0,\ u_\delta - O_{\delta q}^U u_q \leqslant 0, \quad \forall\ \delta < q, \\
& \sum_{i=1}^{m} v_i\, (x_{ip})_\alpha^U = 1, \quad u_r v_i \geqslant 0, \quad \forall r,\ j.
\end{aligned}$$

$$\begin{aligned}
(W_p)_\alpha^U = \max & \sum_{r=1}^{s} u_r\, (y_{rp})_\alpha^U \\
\text{s.t.} \quad & \sum_{r=1}^{s} u_r\, (y_{rj})_\alpha^L - \sum_{i=1}^{m} v_i\, (x_{ij})_\alpha^U \leqslant 0, \quad \forall j,\ j \neq p, \qquad (4.6)\\
& -v_\delta + I_{\delta q}^L v_q \leqslant 0, v_\delta - I_{\delta q}^U v_q \leqslant 0, \quad \forall\ \delta < q, \\
& -u_\delta + O_{\delta q}^L u_q \leqslant 0, u_\delta - O_{\delta q}^U u_q \leqslant 0,\ \forall \delta < q, \\
& \sum_{i=1}^{m} v_i\, (x_{ip})_\alpha^L = 1, \quad u_r v_i \geqslant 0, \quad \forall r,\ j. \qquad (4.7)
\end{aligned}$$

where $I_{\delta q}^L = \frac{L_{I\delta}}{U_{Iq}}, I_{\delta q}^U = \frac{U_{I\delta}}{L_{Iq}}, O_{\delta q}^L = \frac{L_{O\delta}}{U_{Oq}}$ and $O_{\delta q}^U = \frac{U_{O\delta}}{L_{Oq}}$.

4.3. Stochastic DEA

Given the need to use random data in practical models, early publications made theoretical attempts to incorporate these errors [10, 53, 101, 102]; thereafter, several researchers developed variant models for incorporating stochastic elements into DEA [32, 35, 63, 73, 74, 79, 83, 109]. SDEA is a rapidly developing research venue of DEA, and new approaches, procedures, and algorithms continue to appear. Further, SDEA is an appropriate decisional approach for assessing different decision making units within a stochastic environment and affords a more accurate assessment under randomness [94, 107, 120].

The early SDEA works are based on the theoretical paper by Land *et al.* [80], where the authors introduced the stochastic component to DEA and created chance constrained problems by introducing variability to outputs that are conditional on inputs, thus implying that only outputs were taken as normally distributed random variables. As a result, chance constrained programming is a popular approach for dealing with stochastic data in DEA. Due to the stochastic nature of the observations, the constraints that require the aggregated output to be smaller than the aggregated input are satisfied with specified probabilities. Different probability determines different efficiencies for the set of DMUs [32, 34, 93]. Under this approach, it is assumed that the efficiency of a DMU is stochastic and that the observation is just an occurrence of a random phenomenon. As a matter of fact, the chance constrained programming technique converts a linear programming problem with stochastic variables to a non-linear deterministic problem by suitably modifying the objective function to incorporate the stochastic element and changing the constraints to ensure their satisfaction, despite stochastic uncertainty, with a specified confidence level [25].

In this regard, Ray [97] modified the standard DEA model to measure relative efficiency in the presence of random variation in all of the outputs produced from the given inputs. For any input bundle, the value of the DEA estimate defines the maximum output producible from inputs under all circumstances. In the stochastic output oriented model, the inputs are assumed to be deterministic while all outputs are random, each output y_k is normally distributed with mean μ_p and variance σ_p^2 and the relation between the same stochastic output variable through different DMUs is independent, this means $\mathrm{cov}(y_k, y_p) = 0$. For practical purposes, the restriction regarding the output quantities in the DEA model translates into a random inequality that may at times be violated. Because an inequality involving a number of random variables can never be imposed with certainty, the strategy of chance constrained program is to ensure that the probability that the inequality holds for a random sample of these variables does not fall below a certain level.

In this paper, an SDEA model is applied for performance-assessment of Angolan banks, relaxing the assumption of independence between stochastic output variables. To the best of our knowledge, this is the first study that presents such novel approach for performance assessment of in the banking industry with stochastic outputs. This research departs from Ray [97]. Model (4.8) presents this modified version of the SDEA model allowing for correlated outputs:

$$
\begin{aligned}
&\text{Max } Z_p = \emptyset \\
&\text{s.t.} \\
&\sum_{i=1}^{n} \lambda_i \mu_i - \emptyset \mu_p \geqslant e \sqrt{\sum_{\substack{i=1 \\ i \neq p}}^{n} \lambda_i^2 \sigma_i^2 + \left(\lambda_p - \theta\right)^2 \sigma_p^2 + 2\mathrm{cov}(y_i, y_p)}, \quad \forall k = 1, \ldots, s \\
&\sum_{i=1}^{n} \lambda_i x_i \leqslant x_p, \quad \forall j = 1, \ldots, m \\
&\sum_{i=1}^{n} \lambda_i = 1 \\
&\lambda_i \geqslant 0, \quad (i = 1, 2, \ldots, n)
\end{aligned}
\tag{4.8}
$$

where e is the significance level, considered to be 0.05 in this research.

After designing this SDEA model, we need to incorporate the tail dependence structure within the ambit of the stochastics outputs, through the statistical measure of covariance, by means of chance constrained programming. Since all outputs are positively correlated with each other, this implies that one variable is above (below) its mean

TABLE 4. Archimedean copulas, their generators, and measures of dependence*.

Family	Generator $\emptyset(t)$	Dependence parameter (α) space	Kendall's τ
Clayton [30]	$t^{-\alpha}-1$	$\alpha \geqslant 0$	$\frac{\alpha}{\alpha+2}$
Frank [48]	$-\ln \frac{e^{-\alpha t}-1}{e^{-t}-1}$	$\alpha \geqslant 0$	$1-\frac{4}{\alpha}\left\{D_1(-\alpha)-1\right\}$
Gumbel [54]	$(-\ln t)^{\alpha}$	$\alpha \geqslant 1$	$1-\alpha^{-1}$

value when the other variable is above (below) its mean value. As a matter of fact, the concept of tail dependence can be embedded within the copula theory [99]. An n-dimensional distribution function C: $[0,\, 1]^n - > [0, 1]$ is called a copula if it has one-dimensional margins that are uniformly distributed on the interval [0, 1]. Copulas are functions that join or "couple" an n-dimensional distribution function F to its corresponding onedimensional marginal distribution functions F_i, $i = 1, \ldots, n$, in the following way: $F(x_1, \ldots, x_n) = C\ (F_1(x_1), \ldots, F_n(x_n))$.

Copulas have become a popular multivariate modeling tool in many fields where multivariate dependence is of interest [122]. They differ not so much in the degree of association they provide, but rather in which part of the distributions the association is strongest [88]. They are particularly useful, for instance, if one wants to model the impact of tail dependence on decision-making [112]. Tail dependence describes the amount of dependence in the tail of a bivariate distribution. In other words, tail dependence refers to the degree of dependence in the corner of the lower-left quadrant or upper-right quadrant of a bivariate distribution [99].

Precisely, a copula is a multivariate distribution whose marginals are all uniform over (0, 1]. Combined with the fact that any continuous random variable can be transformed to be uniform over (0, 1] by its probability integral transformation, copulas can be used to provide multivariate dependence structure apart from the marginal distributions [122]. According to Table 4, different choices of generators yield several important families of copulas. Similarly, different dependence parameters are closely related to some level of bivariate dependence, measured by Kendall's τ coefficient of non-parametric correlation [122].

It is worth noting that, although Archimedean copulas with dimension three or higher only allow positive association, negative association is allowed for bivariate Archimedean copulas [122]. Figure 1 compares Clayton [30], Frank [48] and Gumbel [54] copulas, with normal marginals, to the traditional bivariate normal distribution. One can easily discern that, for different families of copulas and dependence values, the resulting bivariate distribution may indicate a stronger relationship between higher (lower) values of the variables under analysis [88]. Although copulas have been often used in actuarial and finance applications in banking, such as for risk assessment, measurement and pooling [39, 44] thus far their use within the ambit of SDEA models have not been noticed.

The Archimedean copulas are often chosen for modelling purposes due to their following characteristics: (1) compatibility with "fat tail" distributions sensitive to large losses, but less sensitive to small losses; (2) simple and robust parameter estimation methods; (3) familiarity with practitioners and regulators; and (4) easy to implement. Specifically, the first criterion reflects a well-known phenomenon that large losses tend to come as surprises – the "fat tail". In this sense, Archimedean copulas outperform the elliptical class, as these copulas do not have fat tails.

4.4. On the combination of probabilistic and fuzzy approaches

Putting into perspective the issues raised in Sections 4.2 and 4.3, within the ambit of combined probabilistic-fuzzy approaches, randomness and fuzziness should have their useful properties jointly considered whenever possible [4]. A growing number of studies in the literature employ variants of combined probabilistic-fuzzy approaches in several aspects of decision-making. More specifically, 2-Dimentional Fuzzy Monte Carlo Analysis (2D FMCA) uses a combination of probability and possibility theory to include probabilistic and imprecise information in the same analytical model. Arunraj *et al.* [4] presented a comprehensive literature review on these issues and some of their key aspects are addressed next.

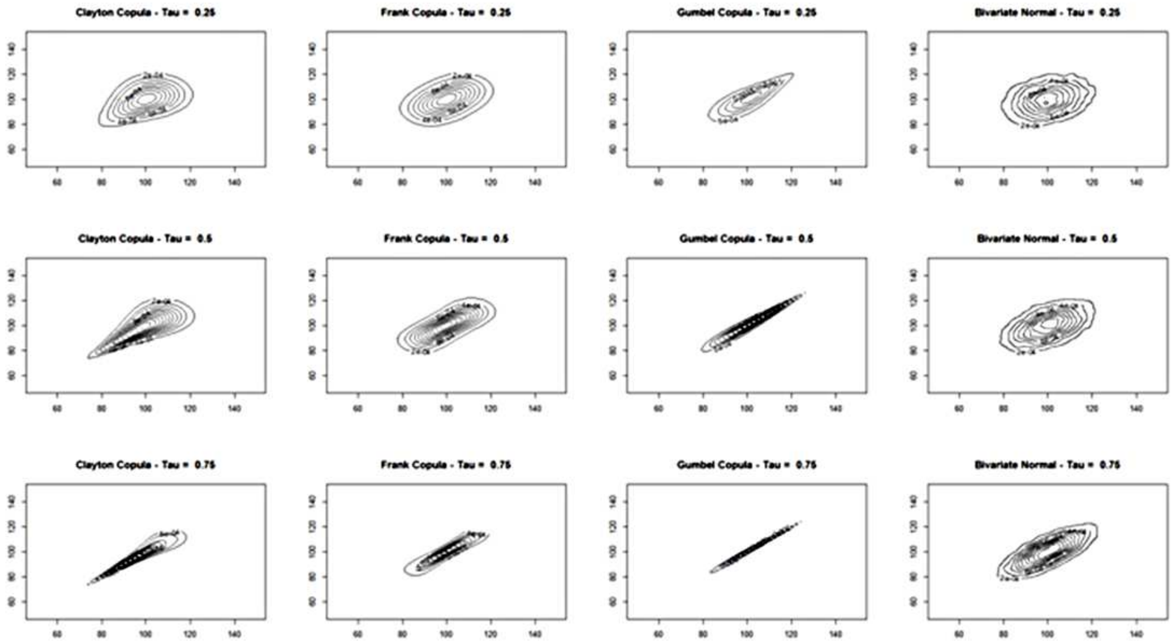

FIGURE 1. Examples of Archimedean copulas for different values of Kendall's τ.

Guyonnet *et al.* [56], for example, combined MCA with fuzzy calculations to assess uncertainty in risk management. In turn, Kentel and Aral [72] used a similar approach in order to generate the resulting combinations between probability density functions of random variables and membership functions of fuzzy variables. These resulting combinations were used for determining the fuzzy uncertainty estimates at certain percentiles of risk for given individuals or groups. Early works on 2D FMCA can be also traced back to Zonouz and Miremadi [130], who developed a fuzzy-MCA for fault tree analysis. In their approach, the variability in the random variables of the risk is treated using probability density functions and the uncertainty associated with them is treated by using fuzzy numbers for the parameters of these random variables.

In this research, a specific application of the 2D FMCA approach is developed to compare the efficiency levels in the Angolan banking industry derived from SDEA and FDEA models. The approach used here starts off from the α-level FDEA based models – where production inputs and outputs are treated as triangular fuzzy numbers (as in Puri and Yadav [95]) with a 20% offset from their central values – and culminates with different tail dependence structures – Frank, Clayton, and Gumbel – to model productive outputs within the ambit of the SDEA model. More precisely, the KL divergence between the cumulative distributions of the SDEA and the FDEA scores are computed each time for a given combination of α-level (say 0; 0.1; 0.2; ...; 1, as in Hsiao *et al.* [62]) and tail dependence structure (Frank, Clayton, and Gumbel). Readers should be aware, however, that the α-level values within this set are primarily used in the three major FDEA models – presented in Section 4.2 – so as to determine crisp values for the input and the output bounds, thus allowing the computation of the respective efficiency levels in Kao and Liu [67], Saati *et al.* [98] and Liu [85], in order to assure the proper comparability of SDEA and FDEA scores.

5. Results and discussion

The mean efficiency scores calculated for 15 selected Angolan banks from 2006 to 2014 – using a meta-frontier [90] and the previously discussed SDEA and FDEA models based, respectively on the tail dependence structure and the α-level approach – are given in Table 5. For the sake of comparison, only α-values equal

TABLE 5. Mean efficiency scores per DMU (2006–2014).

Observation	DMU	DEA BCC	SDEA Normal Bivariate	SDEA Clayton	SDEA Frank	SDEA Gumbel	FDEA Guo&Tanaka	FDEA Kao&Liu	FDEA Saati
1	BDA	0.968	0.482	0.458	0.435	0.412	0.959	0.333	0.333
2	BPA	0.965	0.189	0.197	0.214	0.168	0.950	0.337	0.333
3	BAI	0.965	0.814	0.709	0.748	0.822	0.524	0.467	0.348
4	KEVE	0.964	0.993	0.960	0.955	0.962	0.287	0.333	0.479
5	BNI	0.949	0.409	0.305	0.326	0.298	0.597	0.377	0.363
6	BFA	0.937	0.254	0.213	0.228	0.199	0.253	0.667	0.504
7	BPC	0.905	0.800	0.812	0.790	0.878	0.594	0.842	0.773
8	BESA	0.902	1.000	1.000	1.000	1.000	0.185	0.999	0.998
9	BIC	0.896	0.535	0.474	0.476	0.428	0.191	0.806	0.703
10	BMA	0.883	0.404	0.403	0.350	0.358	0.730	0.366	0.333
11	BSOL	0.851	0.747	0.756	0.701	0.810	0.108	0.973	1.000
12	BCGTA	0.848	0.175	0.205	0.166	0.144	0.086	0.990	1.000
13	BCI	0.845	0.182	0.185	0.180	0.179	0.120	0.966	0.972
14	BCA	0.769	0.091	0.066	0.110	0.089	0.052	1.000	1.000
15	BAI BMF	0.738	0.095	0.105	0.088	0.124	0.038	0.551	1.000
Mean		0.892	0.478	0.456	0.451	0.458	0.378	0.667	0.676
SD		0.072	0.322	0.316	0.311	0.337	0.322	0.283	0.299
CV		0.080	0.673	0.692	0.690	0.737	0.850	0.424	0.442

to one were considered, so that crisp efficiencies derived from FDEA models could be directly compared to those computed using SDEA models and a sample size of 200 replications. Results for the traditional DEA BCC model and for the normal bivariate tail dependence structured were also included. As expected, results indicate that the discriminatory power of SDEA and FDEA models is higher, since their average efficiency scores are substantially lower than those computed from the traditional DEA BCC model. It is worth noting that Guo and Tanaka's presented the highest discriminatory power. On the other hand, however, SDEA scores present the highest dispersion, followed by FDEA models, which can be inferred by the coefficient of variation. Both randomness and fuzziness present a substantial impact on the efficiency rankings, illustrating that the correlation and the tail dependence structure of the outputs, as well as the underlying fuzziness in the input and output measurement cannot be neglected. As a matter of fact, although SDEA models yielded quite similar rankings when compared to each other, FDEA models presented the highest ranking discrepancy among each other, thus suggesting not only that the fuzziness modelling is a critical decision, but also that fuzziness appears to be more impacting than randomness, as suggested by Wanke *et al.* [117]. Nevertheless, since efficiency score vary substantially between different methods, a further exploration of their main properties and characteristics in terms of fuzziness and randomness is deemed necessary and conducted next.

The correlogram for the Spearman's rank test presented in Figure 2 corroborates and synthesizes these findings. Readers can easily note that SDEA scores are strongly and positively correlated with each other, while, within the ambit of FDEA models, this is only verified with respect to the models of Kao and Liu and Saati. As a matter of fact, Guo and Tanaka model is strongly and negatively correlated to Kao and Liu's and Saati's. While the correlations between SDEA and FDEA models tend to be weak, regardless of being positive or negative, thus reflecting the very different natures of fuzziness and randomness, it is interesting to note that Guo and Tanaka model is the most correlated with the classical BCC DEA model. When analyzed in light of the SDEA models, it is not possible to trace a common pattern with respect to FDEA results. This may suggest that fuzziness and randomness interact in different forms, an issue that is going to be further explored in the next paragraphs.

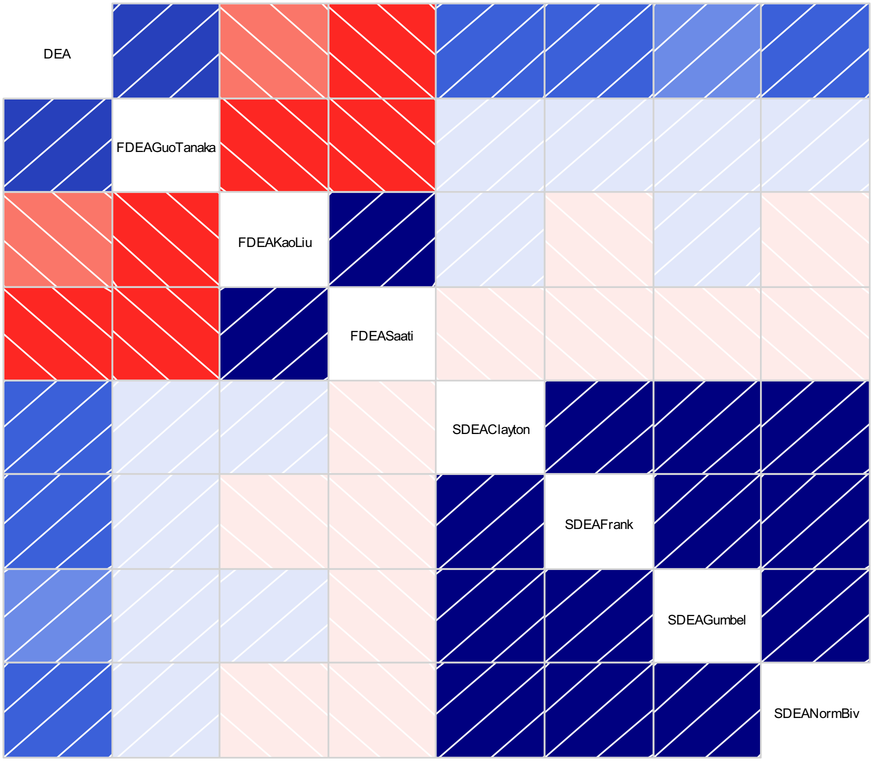

FIGURE 2. Correlogram for the FDEA and SDEA efficiency scores (α-values equal to one).

Figure 3 encompasses a panel with the cumulative distributions of the efficiency scores for the different SDEA and FDEA models, now allowing α-values to vary from zero to one in steps of 0.20. Not only can the dispersion of the FDEA and SDEA scores be now properly compared, but also it is possible to trace a parallel between higher or lower fuzziness levels – depending on the α-values – and different tail dependence structures. This is important to apprehend the practical aspects of the equivalence between FDEA and SDEA models. Putting different α-values into perspective, readers may note that while different SDEA models yield similar cumulative distributions for the efficiency scores, scores derived from Guo and Tanaka's model tend to be smaller – and thus more discriminatory – and concentrated between 0–0.50. On the other hand, Kao and Liu's and Saati's model tend to return higher efficiency scores, which are concentrated between 0.5–1.0. Although fuzziness is less impacting in the former model, since the cumulative distributions for the different α-values are narrower dispersed, both models present a smaller discriminatory power since their scores are more concentrated towards one. Results for the KL divergence, presented in Figure 4, also corroborates that lower fuzziness is more adherent to different tail dependence structures under Kao and Liu's model. Considering that, the lower the KL divergence values, the closer two cumulative distributions are, one can note that KL divergence values diminishes with higher values of α (lower fuzziness) for Kao and Liu's results under different tail dependence structures. Besides, it is interesting to observe under Guo and Tanaka's results higher fuzziness is more adherent to different tail dependence structures: not only KL divergence is minimal for different tail dependence structures when $\alpha = 0$, but also their cumulative distribution is pretty much adherent to all different copulas tested, up to $\alpha = 0.8$. Although Saati's results also presents high levels of fuzziness, since the cumulative distributions for the different α-values are broadly dispersed, it is still possible to differentiate their results in terms of different tail dependence structures for different levels of α. Putting in other words, the cumulative

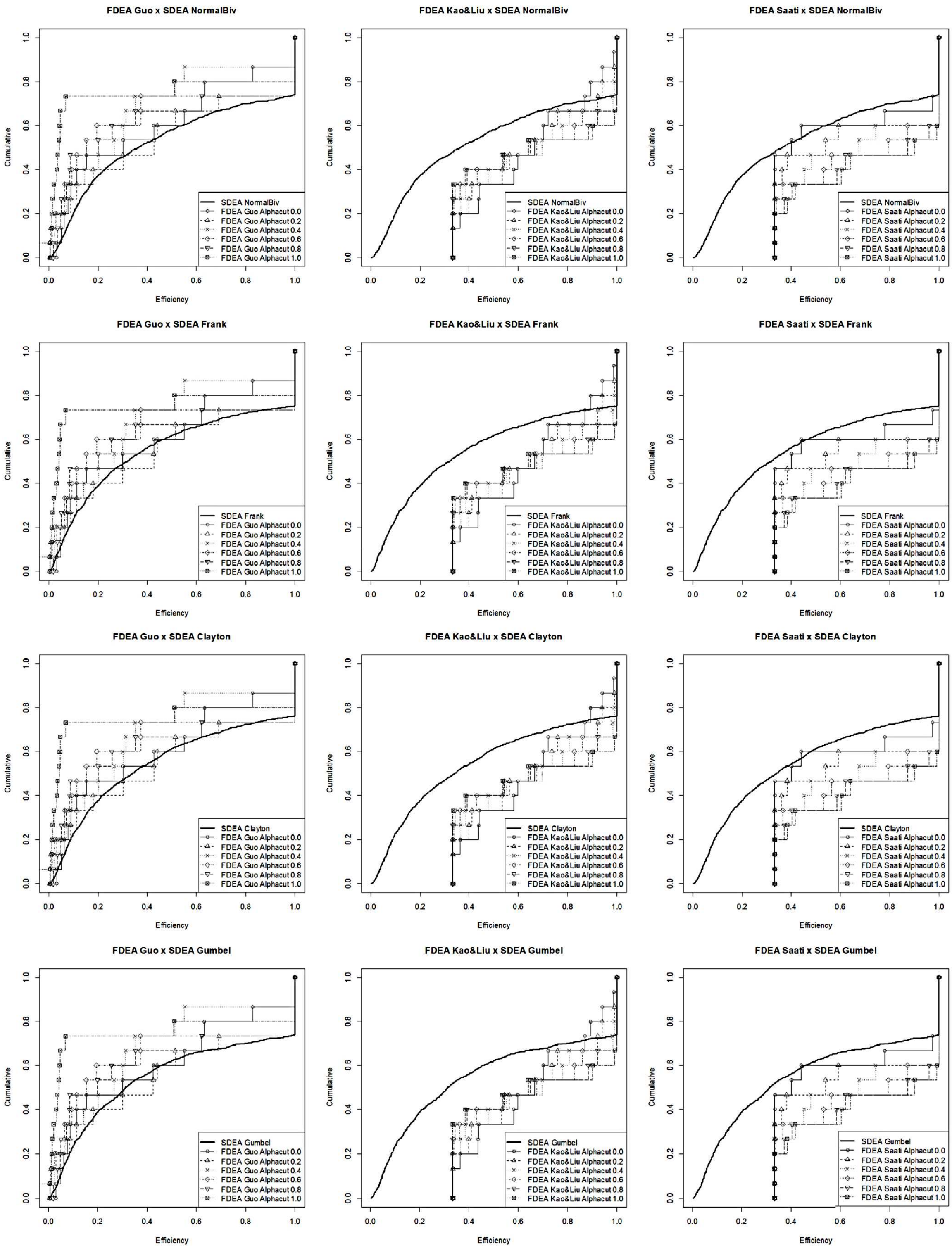

FIGURE 3. Cumulative distributions for the FDEA and SDEA efficiency scores.

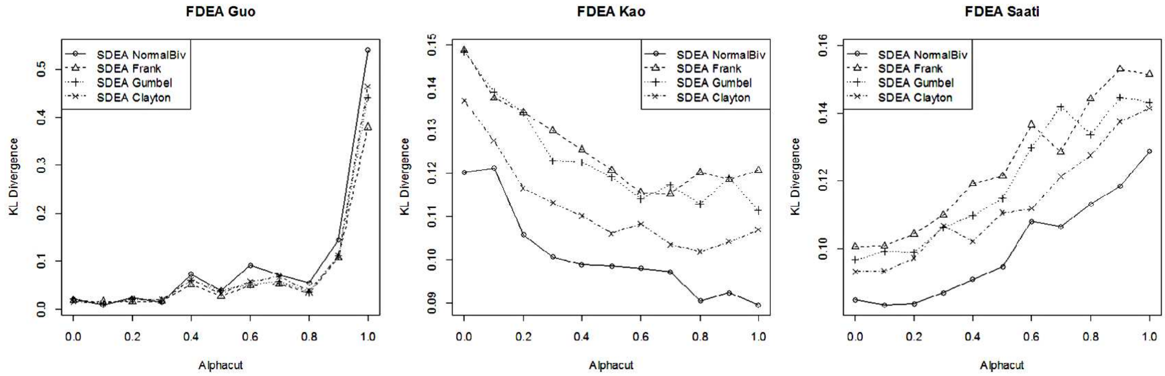

FIGURE 4. KL divergence results for cumulative distributions of SDEA and FDEA scores.

distribution of Saati's scores is not as good as Guo and Tanaka's as one being simultaneously representative of all the tail dependence structures analyzed.

The ruling implications for decision-makers based on this practical application using Angolan banks are related to very nature of the fuzziness and randomness of the problem under analysis. Considering that different tail dependence structures yield similar results for correlations between outputs that are close to zero, one should consider using SDEA model and chance constrained programming on the traditional normal bivariate dependence structure. For higher values of positive bivariate correlations between outputs the proper tail dependence should be chosen, whether Clayton, Frank, or Gumbel. On the other hand, it is important to take into account that fuzziness appears to have a dominant effect over randomness, and that is always preferable that the chosen set of DEA scores present a good discriminatory power. Therefore, putting the Angolan case into perspective and considering that not only correlation between outputs is considerably higher, decision makers should prefer Guo and Tanaka's FDEA model due to its ability to capture different aspects of tail dependence structure since their KL divergence values with SDEA models are the lowest.

Considering Guo and Tanaka's scores as cornerstones that account for fuzziness and random effects, it is possible to affirm that average efficiency in Angolan banks is pretty low (0.378) and highly dispersed between banks. Clearly, there are two distinct groups of institutions. The high efficiency group is formed by institutions like BDA, BPA, and BAI, with some links with foreign ownership; as long as the low efficiency group encompasses banks such as BCI, BCA, and BSOL. This low efficiency grouped if formed by relatively new banks owned by local Angolans. Besides, BSOL is the local bank of microcredit, but acts also on funding companies and individuals. Results therefore suggest that Angolan banks could benefit from greater levels of internationalization, while technical efficiency appears to be higher in older banks.

6. Conclusion

This paper presents an analysis of the efficiency of Angolan banks using major FDEA and SDEA models based, respectively, on the α-level approach and on different tail dependence structures. As long as FDEA enabled the treatment of fuzziness involved in the process of measuring or collecting or collecting data regarding inputs and outputs, SDEA allowed to account for the correlation between outputs using copulas theory, which is a novelty in banking research.

Results suggest that the efficiency of the Angolan banking system is impacted both by fuzziness and randomness and can be systematically represented by a FDEA model whose results are pretty much adherent to those obtained via SDEA under different tail dependence structures. Besides the practical aspect offered to decision-makers, in the sense of choosing the proper model, the contributions of this research to the current body of knowledge in the SDEA/FDEA-banking efficiency literature are fourfold: first, this research addresses a gap in the SDEA/FDEA literature by proposing a way for handling and analyzing simultaneously several models; secondly, a real case is investigated in which data collection in terms of inputs and outputs was subject to uncertainty or fuzziness because reliable sources of information on the banking industry in Angola are scarce; thirdly, the framing of the KL divergence adopted here in terms of the Fuzzy Monte Carlo Analysis, where uncertainty and randomness are supplementary parts of the analytical process, is innovative; and fourthly, a contribution to the nascent literature on the applications of SDEA/FDEA and banking efficiency has been made.

Limitations of this research are related to the type of SDEA/FDEA models adopted and to the dataset used. Further studies should be conducted in banking, both incorporating additional SDEA/FDEA approaches and eventually analyzing under different contextual variables, with the goal of interpreting their impacts in light of uncertainty and randomness. Another possible research stream should consider the use copulas altogether with chance constrained programming approaches in DEA.

References

[1] N. Adler and J. Berechman, Measuring airport quality from the airlines' viewpoint: an application of data envelopment analysis. *Transp. Policy* **8** (2001) 171–181.

[2] H.Y. Aly, R. Grabowski, C. Pasurka and N. Rangan, Technical, scale and allocative efficiencies in U.S. banking: An empirical investigation. *The Rev. Econ. Stat.* **72** (1990) 211–218.

[3] A. Amirteimoori and A. Emrouznejad, Flexible measures in production process: A DEA-based approach. *RAIRO: OR* **45** (2011) 63–74.

[4] N.S. Arunraj, S. Mandal and J. Maiti, Modeling uncertainty in risk assessment: An integrated approach with fuzzy set theory and Monte Carlo simulation. *Accid. Analysis Prev.* **55** (2013) 242–255.

[5] M. Asmild, P. Bogetoft and J. Leth Hougaard, Rationalising inefficiency: Staff utilisation in branches of a large Canadian bank. *Omega* **41** (2013) 80–87.

[6] M. Asmild, C.C. Paradi, V. Aggarwall and C. Schaffnit, Combining DEA window analysis with the Malmquist index approach in a study of the Canadian banking industry. *J. Prod. Anal.* **21** (2011) 67–89.

[7] A. Assaf, C.P. Barros and A. Ibiwoye, Performance assessment of Nigerian Banks prior and post consolidation: Evidence from a Bayesian approach. *The Serv. Industries J.* **32** (2010) 215–229.

[8] J.P. Azam, B. Biais and M. Dia, Privatisation versus Regulation in Developing Economies: The Case of West African Banks. *J. Afr Econ.* **13** (2004) 361–394.

[9] R.D. Banker, A. Charnes and W.W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Sci.* **30** (1984) 1078–1092.

[10] R. Banker, Stochastic data envelopment analysis. Carnegie-Mellon University, Pittsburgh (1986).

[11] C.P. Barros, N. Peypoch and J. Williams, A Note on Productivity Change in European Cooperative Banks: The Luenberger Indicator Approach. *Int. Rev. Appl. Econ.* **24** (2010) 137–147.

[12] C.P. Barros and Zorro Mendes, Assessing the competition in Angola's banking industry. Applied Economics (forthcoming) (2016).

[13] C.P. Barros, C. Ferreira and J. Williams, Analysing the Determinants of Performance of Best and Worst European Banks: A Mixed Logit Approach. *Journal of Banking and Finance* **31** (2007) 2189–2203.

[14] C.P. Barros, S. Managi and R. Matousek, The Technical Efficiency of the Japanese Banks: Non Radial Directional Performance Measurement with Undesirable Output. *Omega* **40** (2012) 1–8.

[15] C.P. Barros, O. Gonçalves and N. Peypoch, French regional public airports technical efficiency. *Int. J. Transp. Econ.* **39** (2012) 255–274.

[16] C.P. Barros, Q.B. Liang and N. Peypoch, Technical Efficiency in the Angolan Banking Sector with the B-Convexity Model. *South Afr. J. Econ.* **82** (2014) 443–454.

[17] P.W. Bauer, A.N. Berger and D.B. Humphrey, Efficiency and Productivity Growth in US Banking. In The Measurement of Productive Efficiency: Techniques and Applications, edited by H.O. Fried, C.A.K. Lovell and S.S. Schmidt. Oxford University Press, Oxford (1993) 386–413.

[18] A.N. Berger, G.A. Hanweck and D.B. Humphrey, Competitive Viability in Banking: Scale, Scope, and Product Mix Economies. *J. Monetary Econ.* **20** (1987) 501–520.

[19] A.N. Berger and D.B. Humphrey, Measurement and efficiency issues in commercial banking. In: Output measurement in the service sectors, edited by Z. Griliches. Chicago: University of Chicago Press, Chicago (1992).

[20] A.N. Berger and D.B. Humphrey, Efficiency of financial institutions: international survey and directions for future research. *Eur. J. Oper. Res.* **98** (1997) 175–212.

[21] A.N. Berger, I. Hasan and M. Zhou, Bank ownership and efficiency in China: what will happen in the world's largest nation? *Journal of Banking and Finance* **33** (2009) 113–130.

[22] L.C. Brandão and J.C.S. Mello, Improvements to Smooth Data Envelopment Analysis. *RAIRO: OR* **51** (2017) 157–171.

[23] F. Brázdik, Stochastic Data Envelopment Analysis: Oriented and Linearized Models, CERGE-EI. Available at: http://home.cerge-ei.cz/brazdik/download/grantmain.pdf (2004).

[24] J.G. Brida, M. Deidda and M. Pulina, Tourism and transport systems in mountain environments: analysis of the economic efficiency of cableways in South Tyrol. *J. Transp. Geogr.* **36** (2014) 1–11.

[25] A. Charnes, W.W. Cooper and E. Rhodes, Chance constrained programming. *Management Sci.* **6** (1959) 73–79.

[26] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **2** (1978) 429–444.

[27] T.-Y. Chen, A comparison of chance-constrained DEA and stochastic frontier analysis: bank efficiency in Taiwan. *J. Operational Res. Soc.* **53** (2002) 492–500.

[28] X. Chen, M. Skully and K. Brown, Banking efficiency in China: Application of DEA to pre- and post-deregulations era: 1993-2000. *China Econ. Rev.* **16** (2005) 229–245.

[29] Y.-C. Chen, Y.-H. Chiu, C.-W. Huang and C.-H. Tu, The analysis of bank business performance and market risk – Applying Fuzzy DEA. *Econ. Model.* **32** (2013) 225–232.

[30] D.G. Clayton, A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika* **65** (1978) 141–152.

[31] B. Cobb, Mixture distributions for modelling demand during lead time. *J. Oper. Res. Soc.* **64** (2013) 217–228.

[32] W.W. Cooper, Z. Huang and S. Li, Satisficing DEA models under chance constraints. *Ann. Oper. Res.* **66** (1996) 259–279.

[33] W.W. Cooper, K.S. Park and G. Yu, IDEA and AR-IDEA models for dealing with imprecise data in DEA. *Management Sci.* **45** (1999) 597–607.

[34] W.W. Cooper, H. Deng, Z.M. Huang and S.X. Li, Chance constrained programming approaches to technical efficiencies and inefficiencies in stochastic data envelopment analysis. *J. Operational Res. Soc.* **53** (2002) 1347–1356.

[35] W.W. Cooper, H. Deng, Z. Huang and S.X. Li, Chance constrained programming approaches to congestion in stochastic data envelopment analysis. *Eur. J. Oper. Res.* **155** (2004) 487–501.

[36] L. Coroianu, M. Gagolewski and P. Grzegorzewski, Nearest piecewise linear approximation of fuzzy numbers. *Fuzzy Sets and Syst.* **233** (2013) 26–51.

[37] J. Cubbin and G. Tzanidakis, Regression versus data envelopment analysis for efficiency measurement: an application to the England and Wales regulated water industry. *Util. Policy* **7** (1998) 75–85.

[38] D.K. Despotis and Y.G. Smirlis, Data envelopment analysis with imprecise data. *Eur. J. Oper. Res.* **140** (2002) 24–36.

[39] J. Dobriæ, and F. Schmid, Nonparametric estimation of the lower tail dependence Î¿ l in bivariate copulas. *J. Appl. Stat.* **32** (2005) 387–407.

[40] R.G. Dyson and E.A. Shale, Data envelopment analysis, operational research and uncertainty. *J. Oper. Res. Soc.* **61** (2010) 25–34.

[41] B.E. El-Demerdash, I.A. El-Khodary and A.A. Tharwat, Developing a Stochastic Input Oriented Data Envelopment Analysis (SIODEA) Model. *Int. J. Adv. Comput. Sci. Appl.* **4** (2013) 40–44.

[42] A. Emrouznejad and M. Tavana, Performance Measurement with Fuzzy Data Envelopment Analysis. *Stud. Fuzziness and Soft Comput.* **309** (2014).

[43] L. Fang and H. Li, Multi-criteria decision analysis for efficient location-allocation problem combining DEA and goal programming. *RAIRO: OR* **49** (2015) 753–772.

[44] D. Fantazzini, The effects of misspecified marginals and copulas on computing the value at risk: A monte carlo study. *Comput. Stat. Data Anal.* **53** (2009) 2168–2188.

[45] C.A. Favero and L. Pepi, Technical efficiency and scale efficiency in Italian banking sector: A non-parametric approach. *Appl. Econ.* **27** (1995) 385–395.

[46] C. Figueira, J. Nellis and D. Parker, Does Ownership affect the Efficiency of African Banks? *J. Dev. Areas* **40** (2006) 38–63.

[47] M. Fortin and A. Leclerc, Should we Abandon the Intermediation Approach for Analyzing Banking Performance? (Working Paper No.07-01). Departement d'Economique de la Faculte d'administration à l'Universite de Sherbrooke. Available at: http://gredi.recherche.usherbrooke.ca/wpapers/GREDI-0701.pdf (2007).

[48] M.J. Frank, On the simultaneous associativity of F(x, y) and x + y − F(x, y). *Aequationes Mathematicae* **19** (1979) 194–226.

[49] H. Fukuyama and W.L. Weber, Estimating indirect allocative inefficiency and productivity change. *J. Oper. Res. Soc.* **60** (2009a) 1594–1608.

[50] H. Fukuyama and W.L. Weber, A directional slacks-based measure of technical inefficiency. *Socio-Econ. Plan Sci.* **43** (2009b) 274–287.

[51] H. Fukuyama and L.W. Weber, A slacks-based inefficiency measure for a two-stage system with bad outputs. *Omega* **38** (2010) 239–410.

[52] N.S. Gharneh, A. Nabavieh, D. Gholamiangonabadi and M. Alimoradi, Productivity change and its determinants: Application of the Malmquist index with bootstrapping in Iranian steam power plants. *Util. Policy* **31** (2014) 114–120.

[53] D. Gstach, Another approach to data envelopment analysis in noisy environments: DEA+. *J. Prod. Analysis* **9** (1998) 161–176.

[54] E.J. Gumbel, Bivariate exponential distributions. *J. Am. Statistical Assoc.* **55** (1960) 698–707.

[55] P. Guo and H. Tanaka, Fuzzy DEA: a perceptual evaluation method. *Fuzzy Sets and Syst.* **119** (2001) 149–160.

[56] D. Guyonnet, B.B. Bourgine, D. Dubois, G. Fargier, B. Côme and J.P. Chilès, Hybrid approach for addressing uncertainty in risk assessments. *J. Environ. Eng.* **129** (2003) 68–78.

[57] A. Hatami-Marbini, A. Emrouznejad and M. Tavana, A taxonomy and review of the fuzzy data envelopment analysis literature: Two decades in the making. *Eur. J. Oper. Res.*, **214** (2011) 457–472.

[58] A. Hatami-Marbini, S. Saati and M. Tavana, Data Envelopment Analysis with Fuzzy Parameters: An Interactive Approach. *Int. J. Operations Res. Inf. Syst.* **2** (2011) 39–53.

[59] A. Hatami-Marbini, M. Tavana, S. Saati and P.J. Agrell, Positive and normative use of fuzzy DEA-BCC models: A critical view on NATO enlargement. *Int. Trans. Oper. Res.* **20** (2013) 1–23.

[60] M. Hemmati, S.A. Dalghandi and H. Nazari, Measuring relative performance of banking industry using a DEA and TOPSIS. *Meas. Sci. Lett.* **3** (2013) 499–503.

[61] D. Holod and H.F. Lewis, Resolving the deposit dilemma: A new DEA bank efficiency model. *J. Bank. Finance* **35** (2011) 2801–2810.

[62] B. Hsiao, C.-C. Chern, Y.-H. Chiu and C.-R. Chiu, Using fuzzy super-efficiency slack-based measure data envelopment analysis to evaluate Taiwan's commercial bank efficiency. *Expert Syst. Appl.* **38** (2011) 9147–9156.

[63] Z. Huang and S.X. Li, Stochastic DEA models with different types of input-output disturbances. *J. Prod. Analysis* **15** (2001) 95–113.

[64] S.I. Ikhide, Measuring the operational efficiency of commercial banks in Namibia. *South Afr. J. Econ.* **76** (2008) 586–595.

[65] G.R. Jahanshahloo, M. Soleimani-damaneh and E. Nasrabadi, Measure of efficiency in DEA with fuzzy input–output levels: a methodology for assessing, ranking and imposing of weights restrictions. *Appl. Math. Comput.* **156** (2004) 175–187.

[66] G.R. Jahanshahloo, J. Sadeghi and M. Khodabakhshi, Fair ranking of the decision making units using optimistic and pessimistic weights in data envelopment analysis. *RAIRO: OR* **51** (2017) 253–260.

[67] C. Kao and S.T. Liu, Fuzzy efficiency measures in data envelopment analysis. *Fuzzy Sets and Syst.* **113** (2000a) 427–437.

[68] C. Kao and S.T. Liu, Data envelopment analysis with missing data: an application to University libraries in Taiwan. *J. Operational Res. Soc.* **51** (2000b) 897–905.

[69] C. Kao and S.T. Liu, Stochastic data envelopment analysis in measuring the efficiency of Taiwan commercial banks. *Eur. J. Oper. Res.* **196** (2009) 312–322.

[70] C. Kao and S.T. Liu, Multi-period efficiency measurement in data envelopment analysis: The case of Taiwanese commercial banks. *Omega* **47** (2014) 90–98.

[71] H.A. Kebede and B. Wassie, How Efficient Are the Ethiopian Microfinance Institutions in Extending Financial Services to the Poor? A Comparison with the Commercial Banks. *J. Afr. Econ.* **22** (2013) 112–135.

[72] E. Kentel and M.M. Aral, Probabilistic-fuzzy health risk modeling. *Stoch. Environ. Res. Risk Assess.* **18** (2004) 324–338.

[73] M. Khodabakhshi and M. Asgharian, An input relaxation measure of efficiency in stochastic data envelopment analysis. *Appl. Math. Model.* **33** (2008) 2010–2023.

[74] M. Khodabakhshi, Estimating most productive scale size with stochastic data in data envelopment analysis. *Econ. Model.* **26** (2009) 968–973.

[75] C.H. Kirkpatrick, V. Murinde and M. Tefula, The Measurement and Determinants of X-inefficiency in Commercial Banks in Africa. *Eur. J. Finance* **14** (2007) 625–639.

[76] H. Kiyota, Efficiency of Commercial Banks in Sub-Saharan Africa: A Comparative Analysis of Domestic and Foreign Banks. *Proceeding of CSAE Conference 2009: Economic Development in Africa to be held.* University of Oxford (2009).

[77] A. Kontolaimou and K. Tsekouras, Are cooperatives the weakest link in European banking? A non-parametric metafrontier approach. *Journal of Banking and Finance* **34** (2010) 1946–1957.

[78] S.C. Kumbhakar and D. Wang, Economic reforms, efficiency and productivity in Chinese banking. State University of New York: Binghamton. Working Paper (2005).

[79] K.C. Land, C.A.K. Lovell and S. Thore, Productive efficiency under capitalism and state socialism: The chance constrained programming approach. *Public Finance in a World of Transition* **47** (1992) 109–121.

[80] K.C. Land, C.A.K. Lovell and S. Thore, Chance-constrained Data Envelopment Analysis. *Managerial and Decis. Econ.* **14** (1993) 541-554.

[81] H. Langseth, T.D. Nielsen, I. Perez-Bernabe and A. Salmeron, Learning mixtures of truncated basis functions from data. *Int. J. Approx. Reason.* **55** (2014) 940–956.

[82] S. Lertworasirikul, S.-C. Fang, J.A. Joines and H.L.W. Nutt, Fuzzy data envelopment analysis: a possibility approach. *Fuzzy Sets and Syst.*, *139* (2003) 379–394.

[83] S.X. Li, Stochastic models and variable returns to scales in data envelopment analysis. *European J. Operational Res.* **104** (1998) 532–548.

[84] Y. Li, The Asian financial crisis and non-performing loans: Evidences from commercial banks in Taiwan. *Int. J. Management* **20** (2003) 69–74.

[85] S.-T. Liu, A fuzzy DEA/AR approach to the selection of flexible manufacturing systems. *Computers and Industrial Engineering* **54** (2008) 66–76.

[86] S.M. Miller and A.G. Noulas, The technical efficiency of large bank production. *J. Bank Finance* **20** (1996) 495–509.

[87] H. Morita and L.M. Seiford, Characteristics on stochastic DEA efficiency: Reliability and probability being efficient. *J. Oper. Res. Soc. Jpn* **42** (1999) 389–404.

[88] R.B. Nelsen, An introduction to copulas. Springer, New York (1999).

[89] G.J. O'Donnell and G.V.D. Westhnizen, Regional Comparison of banking performance in South Africa. *South Afr. J. Econ.* **80** (2002) 246–263.

[90] C. O'Donnell, D. Rao and G. Battese, Metafrontier frameworks for the study of firm-level efficiencies and technology ratios. *Empir. Econ.* **34** (2007) 231–255.

[91] C.C. Okeahalam, Internationalisation and firm performance: Evidence from estimates of efficiency in banking in Namibia and Tanzania. *J. Int. Dev.* **20** (2008) 942–964.

[92] W.O. Olatubi and D.E. Dismukes, A data envelopment analysis of the levels and determinants of coal-fired electric power generation performance. *Util. Policy* **9** (2000) 47–59.

[93] O.B. Olesen and N.C. Petersen, Chance constrained efficiency evaluation. *Management Sci.* **41** (1995) 442–457.

[94] O.B. Olesen and N.C. Petersen, Stochastic Data Envelopment Analysis – A review. *Eur. J. Oper. Res.* **251** (2016) 2–21.

[95] J. Puri and S.P. Yadav, A concept of fuzzy input mix-efficiency in fuzzy DEA and its application in banking sector. *Expert Syst. Appl.* **40** (2013) 1437–1450.

[96] J. Puri and S.P. Yadav, A fuzzy DEA model with undesirable fuzzy outputs and its application to the banking sector in India. *Expert Syst. Appl.* **41** (2014) 6419–6432.

[97] S.C. Ray, Data Envelopment Analysis: Theory and Techniques for Economics and Operations Research. Cambridge University Press, UK (2004).

[98] S. Saati, A. Memariani and G.R. Jahanshahloo, Efficiency analysis and ranking of DMUs with fuzzy data. *Fuzzy Optim. Decis. Mak.* **1** (2002) 255–267.

[99] R. Schmidt and U. Stadtmuller, Nonparametric estimation of tail dependence. *Scand. J. Statistics* **33** (2006) 307–335.

[100] C. Sealey and J.T. Lindley, Inputs, outputs and a theory of production and cost at depository financial institution. *J. Finance* **32** (1977) 1251–1266.

[101] J.K. Sengupta, Efficiency measurement in stochastic input-output systems. *Int. J. Syst. Sci.* **13** (1982) 273–287.

[102] J.K. Sengupta, Data envelopment analysis for efficiency measurement in the stochastic case. *Comput. Operations Res.* **14** (1987) 117–29.

[103] J.K. Sengupta, Measuring efficiency by a fuzzy statistical approach. *Fuzzy Sets and Syst.* **46** (1992) 73–80.

[104] H.D. Sherman and F. Gold, Bank branch operating efficiency: Evaluation with data envelopment analysis. *J. Bank Finance* **9** (1985) 297–316.

[105] M. Soleimani-damaneh, G.R. Jahanshahloo and S. Abbasbandy, Computational and theoretical pitfalls in some current performance measurement techniques and a new approach. *Appl. Math. Comput.* **181** (2006) 1199–1207.

[106] R.B. Staub, G.S. Souza and B.M. Tabak, Evolution of bank efficiency in Brazil: A DEA approach. *European J. Bank. Finance* **202** (2010) 204–213.

[107] T. Sueyoshi, Stochastic DEA for restructure strategy: an application to a Japanese petroleum company. *Omega* **28** (2000) 385–98.

[108] F. Sufian, The impact of the Asian financial crisis on bank efficiency: The 1997 experience of Malaysia and Thailand. *J. Int. Dev.* **22** (2010) 866–889.

[109] S. Thore, Chance-constrained activity analysis. *Eur. J. Oper. Res.* **30** (1987) 267–269.

[110] A.Y. Vaninsky, Stochastic DEA with a Perfect Object and Its Application to Analysis of Environmental Efficiency. *Am. J. Appl. Math. Statistics* **1** (2013) 57–63.

[111] W.-K. Wang, W.-M. Lu and C.-J. Tsai, The relationship between airline performance and corporate governance amongst US Listed companies. *J. Air Transp. Management* **17** (2011) 148–152.

[112] Y. Wang and H. Pham, Modeling the dependent competing risks with multiple degradation processes and random shock using time-varying copulas. *IEEE Reliab. Soc.* **61** (2012) 13–22.

[113] W.-K. Wang, W.-M. Lu and P.-L. Liu, A fuzzy multi-objective two-stage DEA model for evaluating the performance of US bank holding companies. *Expert Syst. Appl.* **41** (2014) 4290–4297.

[114] P. Wanke and C. Barros, Two-stage DEA: An application to major Brazilian banks. *Expert Syst. Appl.* **41** (2014) 2337–2344.

[115] P. Wanke, C. Barros and N.P.J. Macanda, Predicting Efficiency in Angolan Banks: A Two-Stage TOPSIS and Neural Networks Approach. *South Afr. J. Econ.* **84** (2016) 461–483.

[116] P. Wanke, C.P. Barros and Z. Chen, An analysis of Asian airlines efficiency with two-stage TOPSIS and MCMC generalized linear mixed models. *Int. J. Production Econ.* **169** (2015b) 110–126.

[117] P. Wanke, C.P. Barros and A. Emrouznejad, Assessing productive efficiency of banks using integrated Fuzzy-DEA and bootstrapping: A case of Mozambican banks. *Eur. J. Oper. Res.* **249** (2016) 378–389.

[118] M. Welde and J. Odeck, The efficiency of Norwegian road toll companies. *Util. Policy* **19** (2011) 162–171.

[119] D. Wu, Z. Yang and L. Liang, Using DEA-neural network approach to evaluate branch efficiency of a large Canadian bank. *Expert Syst. Appl.* **31** (2006) 108–115.

[120] D.D. Wu and C.-G. Lee, Stochastic DEA with ordinal data applied to a multi-attribute pricing problem. *Eur. J. Oper. Res.* **207** (2010) 1679–1688.

[121] R. Yager, A new methodology for ordinal multiple aspect decisions based on fuzzy sets. *Decis. Sci.* **12** (1981) 589–600.

[122] J. Yan, Enjoy the joy of copulas: with a package copula. *J. Statistical Softw.* **21** (2007) 1–21.

[123] P. Yue, Data envelopment analysis and commercial bank performance: A primer with applications to Missouri banks. *Federal Reserve Bank of St. Louis* **74** (1992) 31–45.

[124] L. Zadeh, Fuzzy sets. *Information and Control* **8** (1965a) 109–141.

[125] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility. *Information and Control* **9** (1965b) 338–353.
[126] L. Zadeh, Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Syst.* **1** (1978) 3–28.
[127] M. Zerafat Angiz, A. Emrouznejad and A. Mustafa, Fuzzy assessment of performance of a decision making units using DEA: A non-radial approach. *Expert Syst. Appl.* **37** (2010) 5153–5157.
[128] H.J. Zimmermann, Description and optimization of fuzzy system. *International J. Gen. Syst.* **2** (1976) 209–216.
[129] H.J. Zimmermann, Fuzzy Set Theory – and Its Applications, 3rd edn. Kluwer Academic Publishers, Boston (1996).
[130] S.A. Zonouz and S.G. Miremadi, A fuzzy-monte carlo simulation approach for fault tree analysis. *Proceeding of the Reliability and Maintainability Symposium, 2006 – RAMS'06. Annual* (2006) 428–433.