RAIRO-Oper. Res. 51 (2017) 709–717
DOI: 10.1051/ro/2016058

RAIRO Operations Research
www.rairo-ro.org

SUFFICIENT CONDITION FOR PARTIAL EFFICIENCY IN A BICRITERIA NONLINEAR CUTTING STOCK PROBLEM

Manuel Arana-Jiménez[1] and L.L. Salles Neto[2]

Abstract. This work presents a sufficient criteria for partial efficient solutions of the cutting stock problem with two objectives. We consider two important objectives for an industry: number of processed objects (cost of raw materials) and number of different patterns (cost of setup). These optimality results are established through a new approach based on connections between discrete optimization and continuous vector optimization.

Mathematics Subject Classification. 90C26, 90C29, 90C46.

Received November 16, 2015. Accepted August 30, 2016.

1. Introduction

The Standard Cutting Stock Problem (CSP) is characterized by cutting stock rolls of size W (called objects) into smaller pieces of size w_i (where $W > w_i, i = 1, 2, \ldots, m$), in order to satisfy the demand d_i for each one of these m items. Each combination of items in an object is called cutting pattern and each changing of a cutting pattern has a setup cost to prepare the cutting machine. In order to illustrate these first concepts, let us consider the following example which will be use along this paper to illustrate the results.

Example 1.1. Let us suppose an industry that has an unlimited number of rolls with two meters wide and three types in stock items in three different widths to be produced from these rolls master: 30 centimeters (cm), 40 cm, 50 cm. Among all the possibilities of the arrangement of the items on each roll (cutting pattern), four are considered and listed below:

- Cutting pattern 1: 6 items of 30 cm;
- Cutting pattern 2: 5 items of 30 cm and 1 item of 50 cm;
- Cutting pattern 3: 2 items of 40 cm and 2 items of 50 cm;
- Cutting pattern 4: 3 items of 50 cm and 1 item of 40 cm.

In each cutting pattern we have the following waste: trim loss is 20 cm in cutting pattern 1; trim loss is 0 in cutting pattern 2; trim loss is 20 cm in cutting pattern 3; and trim loss is 10 cm in cutting pattern 4.

Keywords. Multiple objective programming, optimality conditions, continuous optimization, cutting stock problem.

[1] Department of Statistics and Operational Research, Faculty of SSCC and Communication, University of Cádiz, Av. de la Universidad, Jerez 11406, Spain. manuel.arana@uca.es

[2] Departamento de Ciência e Tecnologia, Universidade Federal de São Paulo, Brazil. luiz.leduino@unifesp.br

Article published by EDP Sciences

The setup number plays an important role in the period of time in which the demand of items has to be satisfied. At the same time, it is important to minimize the amount of the trim loss, which is equivalent to minimize the raw material (number of the processed objects). This problem appears in the production of paper [3,8], steel [6,9,18,21], window manufacturing [19], etc.

The mathematical model to minimize the number of processed objects and cost of the setup number can be stated as:

$$\text{(CSP) Minimize } c_1 \sum_{j=1}^{n} x_j + c_2 \sum_{j=1}^{n} \delta(x_j)$$

$$\begin{aligned} \text{s.t.: } & \sum_{j=1}^{n} a_{ji} x_j \geq d_i,\ i = 1, \ldots, m. \\ & x_j \in \mathbb{N},\ j = 1, \ldots, n. \end{aligned}$$

where c_1 is the cost of the object; a_{ji} is the number of times item i appears in the jth cutting pattern; x_j is the number of objects processed with the cutting pattern j; c_2 is the setup cost and $\delta(x_j) = \begin{cases} 1 \text{ if } x_j > 0, \\ 0 \text{ if } x_j = 0. \end{cases}$

Recently, Prestwich *et al.* [16] have dealt with these cutting problems. But, in the literature, Diegel *et al.* [4] were the only to mention real-life values for c_1 and c_2. According to them, an exact relation between c_1 and c_2 depends on several factors as: demand, deadlines, labor costs, etc. The major advantage of a multiobjective approach (see [13]) is to give a set of solutions which are not evaluated by a common scalar function, *i.e.*, the optimization process is, in theory, not based toward a particular type of solution that depends on c_1 and c_2. The literature contains some studies using heuristics to generate solutions to (CSP) with multiples objectives [7,10,20], but there are no papers containing optimality conditions for this problem. More generally, multiobjective combinatorial optimization has not been studied widely and very few theoretical results are available about the properties of this type of problems [2,5]. In this paper we present sufficient conditions for a solution to be partial efficient for a bicriteria cutting stock problem (BCSP):

$$\text{(BCSP) Minimize } \psi(x) = (\psi_1(x), \psi_2(x)) = \left(\sum_{j=1}^{n} x_j, \sum_{j=1}^{n} \delta(x_j) \right)$$

$$\begin{aligned} \text{s.t.: } & \sum_{j=1}^{n} a_{ji} x_j \geq d_i,\ i = 1, \ldots, m, \\ & x_j \in \mathbb{N},\ j = 1, \ldots, n, \end{aligned}$$

where $\psi_1(x) = \sum_{j=1}^{n} x_j$ is the number of processed objects and $\psi_2(x) = \sum_{j=1}^{n} \delta(x_j)$ represents the setups needed. We denote the feasible set of (BCSP) as X, that is, $X = \{x = (x_1, \ldots, x_n) \in \mathbb{N}^n : \sum_{j=1}^{n} a_{ji} x_j - d_i \geq 0,\ i = 1, \ldots, m\}$.

Liu *et al.* [11] present a multiple objective optimization model taking into account trim loss, the number of cutting patterns and usable leftovers: an improved non-dominated sorting heuristic evolutionary algorithm is developed for generating the Pareto non-dominated solutions. After, a multi-attribute decision making method is used for choosing a cutting plan from efficient solutions.

The outline of this paper is as follows. Section 2 gives a general introduction of the multiobjective optimization, some notions of solutions, and optimality conditions. In this regard, we introduce partial-i efficient solutions, as well as a new type Kuhn−Tucker optimality condition, called strict Kuhn−Tucker point. It is proposed a new property of the functions involved in a multiobjective problem by a new definition, called partial-i SKT-pseudoinvex, so as a characterization of the partial-i SKT-pseudoinvexity. In Section 3 is presented a sufficient condition for partial-1 efficiency for the bicriteria cutting stock problem through an approach with an auxiliary nonlinear problem. We prove that every strict Kuhn−Tucker point for the considered auxiliary nonlinear problem is a partial-1 efficient solution for the bicriteria cutting stock problem. Finally, in Section 4, we conclude the paper and give an overview of possible future works.

2. Optimality conditions for vector optimization problem

A classical formulation of a multiobjective mathematical programming problem is as follows:

$$\begin{array}{lll} \text{(MP) Minimize} & f(x) \\ \text{s.t.} & g_i(x) \leqq 0,\ i=1,\dots,m, \\ & x \in S, \end{array}$$

where S is an open subset of $\mathbb{R}^n$, $f=(f_1,\dots,f_p): S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ and $g=(g_1,\dots,g_m): S \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are differentiable.

For the definition of the Pareto-optimal solution [15] (efficient solution) for (MP), as well as related efficiency notions, the following conventions for equalities and inequalities are assumed:

If $x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in\mathbb{R}^n$, then

$$\begin{aligned} &x = y \ \Leftrightarrow x_i = y_i, \quad \forall i=1,\dots,n,\\ &x < y \ \Leftrightarrow x_i < y_i, \quad \forall i=1,\dots,n,\\ &x \leqq y \ \Leftrightarrow x_i \leq y_i, \quad \forall i=1,\dots,n,\\ &x \leq y \ \Leftrightarrow x_i \leq y_i, \quad \forall i=1,\dots,n, \quad \text{and there exist } j \text{ such that } x_j < y_j,\\ &x \leq_j y \Leftrightarrow x_i \leq y_i \quad \forall i=1,\dots,n, \quad \text{and } x_j < y_j, \text{ with } j\in\{1,\dots,n\}. \end{aligned}$$

Similarly, $>, \geqq, \geq, \geq_j$.

We can now define efficient solution, partial efficient solution and weakly efficient solution for (MP).

Definition 2.1. A feasible point, $\bar{x}$, is said to be an efficient solution for (MP) if there does not exist another feasible point, x, such that $f(x) \leq f(\bar{x})$.

Definition 2.2. A feasible point, $\bar{x}$, is said to be a partial-i efficient solution for (MP), with $i\in\{1,\dots,n\}$, if there does not exist another feasible point, x, such that $f(x) \leq_i f(\bar{x})$.

Definition 2.3. A feasible point, $\bar{x}$, is said to be a weakly efficient solution for (MP) if there does not exist another feasible point, x, such that $f(x) < f(\bar{x})$.

It is easy to see that any efficient solution is a partial-i efficient solution for some $i\in\{1,\dots,p\}$, and a partial-i efficient solution is a weakly efficient solution. In general, the reverse in not true, such as the following examples show.

Example 2.4.

$$\begin{array}{ll} \text{Minimize} & f(x_1,x_2) = \left(x_1^2, x_1+x_2\right) \\ \text{s.t.} & -10 \leqq x_1 \leqq 10 \\ & -10 \leqq x_2 \leqq 10, \end{array}$$

Let us consider $\bar{x}=(\bar{x}_1,\bar{x}_2)=(0,1)$. We can check that $\bar{x}$ is not an efficient solution of the previous problem, since there exists another feasible point, $(0,-1)$, such that

$$f(0,-1) = (0,-1) \leq (0,1) = f(0,1) = f(\bar{x}).$$

On the other hand, $\bar{x}$ is a partial-1 efficient solution, that is,

$$f(x_1,x_2) = (x_1^2, x_1+x_2) \nleq_1 (0,1) = f(0,1) = f(\bar{x}),$$

for all feasible point (x_1,x_2), since $x_1^2 > 0$.

Example 2.5.

$$\begin{aligned} &\text{Minimize } f(x_1,x_2) = (x_1^2, x_2^2) \\ &\text{s.t.} \qquad -1 \leqq x_1 \leqq 5 \\ &\qquad\qquad -2 \leqq x_2 \leqq 6, \end{aligned}$$

Let us consider $\bar{x} = (\bar{x}_1, \bar{x}_2) = (2,0)$. It is easy to prove that $\bar{x}$ is a weakly efficient solution of the previous optimization problem, since there does not exist another feasible point (x_1, x_2) such that

$$f(x_1,x_2) = (x_1^2, x_1^2) \not< (4,0) = f(2,0) = f(\bar{x}),$$

since $x_1^2 > 0$. However, $\bar{x}$ is not a partial-1 efficient solution:

$$f(1,0) = (1,0) \leq_1 (4,0) = f(2,0) = f(\bar{x}).$$

Kuhn−Tucker conditions (see [14], for instance) allow us to obtain efficient and weakly efficient solutions for (MP):

Definition 2.6. A feasible point $\bar{x}$ for (MP) is said to be a Kuhn−Tucker point if there exist $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$ such that

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \tag{2.1}$$

$$\mu^T g(\bar{x}) = 0 \tag{2.2}$$

$$\mu \geqq 0 \tag{2.3}$$

$$\lambda \geq 0 \tag{2.4}$$

To obtain the sufficient condition established in the next section we need the following definition.

Definition 2.7. A feasible point $\bar{x}$ for (MP) is said to be a strict Kuhn−Tucker (SKT for short) point if there exist $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^m$ such that

$$\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \tag{2.5}$$

$$\mu^T g(\bar{x}) = 0 \tag{2.6}$$

$$\mu \geqq 0 \tag{2.7}$$

$$\lambda > 0 \tag{2.8}$$

From previous definitions, it is derived that every strict Kuhn−Tuker point for the multiobjective mathematical programming problem (MP) is a Kuhn−Tuker point. Osuna *et al.* [14] proposed a necessary and sufficient condition for a Kuhn−Tucker point to be a weakly efficient solution, based on generalized convexity properties. Later, Arana *et al.* [1] introduced a new formulation for this properties, as well as they extended it to the location of efficient solutions. To do it, Arana *et al.* [1] introduced KT-pseudoinvex-I and KT-pseudoinvex-II problems, as follows.

Definition 2.8. The problem (MP) is said to be KT-pseudoinvex-I(II) if there exists a vector function $\eta : S \times S \to \mathbb{R}^n$ such that for all feasible points $x, \bar{x}$

$$f(x) - f(\bar{x}) < (\leq) 0 \Rightarrow \begin{cases} \nabla f(\bar{x})\eta(x,\bar{x}) < 0 \\ \nabla g_j(\bar{x})\eta(x,\bar{x}) \leqq 0, \quad \forall j \in I(\bar{x}), \end{cases}$$

where $I(\bar{x}) = \{j = 1, \ldots, m : g_j(\bar{x}) = 0\}$.

Arana *et al.* [1] established the following characterization theorem for efficient solutions of (MP).

Theorem 2.9. *Every Kuhn−Tucker point is an (weakly) efficient solution of (MP) if and only if (MP) is KT-pseudoinvex-(I)II.*

To obtain a sufficient condition for a feasible solution to be partial-1 efficient for (BCSP), we need the following definition and the subsequent results.

Definition 2.10. Given $i \in \{1, \ldots, p\}$, the problem (MP) is said to be partial-i SKT-pseudoinvex at $\bar{x}$ if there exists a vector function $\eta : S \times S \to R^n$ such that for all feasible points x

$$f(x) - f(\bar{x}) \leq_i 0 \Rightarrow \begin{cases} \nabla f(\bar{x})\eta(x,\bar{x}) \leq 0 \\ \nabla g_j(\bar{x})\eta(x,\bar{x}) \leqq 0, \quad \forall j \in I(\bar{x}) \end{cases}$$

where $I(\bar{x}) = \{j = 1, \ldots, m : g_j(\bar{x}) = 0\}$.

We say that (MP) is partial-i SKT-pseudoinvex if (MP) is partial-i SKT-pseudoinvex at x for all $x \in X$.

Theorem 2.11. *Every strict Kuhn−Tucker point is a partial-i efficient solution of (MP) if and only if (MP) is partial-i SKT-pseudoinvex.*

Proof.

(i) Firstly, let us prove that the problem (MP) is partial-i SKT-pseudoinvex if every strict Kuhn−Tucker point is a partial-i efficient solution. To this end, let us suppose that there exist two feasible points $\bar{x}$ and x^* such that

$$f(\bar{x}) - f(x^*) \leq_i 0$$

because otherwise, by Definition 2.10, (MP) would be partial-i SKT-pseudoinvex, and the result would be proved. This means that x^* is not a partial-i efficient solution, and by using the initial hypothesis, x^* is not a strict Kuhn−Tucker point, *i.e.*,

$$\lambda^T \nabla f(x^*) + \mu^T \nabla g_{I(x^*)}(x^*) = 0$$

has no solution $\lambda > 0$ and $\mu \geqq 0$. Therefore, by Tucker's theorem [12], the system:

$$\begin{cases} \nabla f(x^*)\eta \leq 0 \\ \nabla g_j(x^*)\eta \leqq 0, \quad \forall j \in I(x^*), \end{cases}$$

has a solution $\eta(\bar{x}, x^*) \in R^n$, where $I(x^*) = \{j = 1, \ldots, m : g_j(x^*) = 0\}$. Therefore, (MP) is partial-i SKT-pseudoinvex.

(ii) Let x^* be a strict Kuhn−Tucker point and (MP) partial-i KT-pseudoinvex. Suppose that there exists a feasible $\bar{x}$ such that $f(\bar{x}) \leq_i f(x^*)$. In this case $f(\bar{x}) - f(x^*) \leq_i 0$. Since (MP) is partial-$i$ SKT-pseudoinvex, there exists $\eta : S \times S \to R^n$ such that:

$$\begin{cases} \nabla f(x^*)\eta(\bar{x}, x^*) \leq 0 \\ \nabla g_j(x^*)\eta(\bar{x}, x^*) \leqq 0, \quad \forall j \in I(x^*) \end{cases} \tag{2.9}$$

where $I(x^*) = \{j = 1, \ldots, m : g_j(x^*) = 0\}$.

Therefore, there exists $1 \leq k \leq p$ such that $\nabla f_k(x^*)\eta(\bar{x}, x^*) < 0$.
Since x^* is a strict Kuhn−Tucker point, then there exist $\lambda > 0$ and $\mu \geqq 0$ such that:

$$\sum_{k=1}^{p} \lambda_k \nabla f_k(x^*) + \sum_{i \in I(x^*)} \mu_i \nabla g_i(x^*) = 0. \tag{2.10}$$

However, from (2.9), we have $\lambda_k \nabla f_k(x^*)\eta(\bar{x}, x^*) < 0$ and $\lambda_i \nabla f_i(x^*)\eta(\bar{x}, x^*) \leq 0$ for $i \in I(x^*)$. Therefore:

$$\sum_{k=1}^{p} \lambda_k \nabla f_k(x^*)\eta(\bar{x}, x^*) + \sum_{i \in I(x^*)} \mu_i \nabla g_i(x^*)\eta(\bar{x}, x^*) < 0$$

which leads to contradiction with (2.5). Consequently, every strict Kuhn−Tucker point is a partial-i efficient solution. □

Thus, we get an important characterization that will be used in the next section to obtain a sufficient condition for a solution to be partial-1 efficient for (BCSP).

3. Necessary and sufficient optimality conditions through a continuous auxiliary problem

The optimality conditions exposed in Section 2, based on Kuhn−Tucker points, are available for continuous optimization problems, such as the multiobjective mathematical programming problem (MP). These optimality conditions can not be applied directly to our discrete problem, the bicriteria cutting stock problem (BCSP). However, next, we propose a new approach to locate partial-1 efficient solutions for (BCSP) through the efficient solutions for an auxiliary continuous multiobjective problem. To this purpose, we present the formulation of an auxiliary problem, inspired from [17], for every M in $\mathbb{R}$, $M > 0$:

$$\begin{array}{lll} (\mathrm{BP}_{\mathrm{aux}})_M & \text{Minimize} & \varphi(x) = (\varphi_1(x), \varphi_2(x)) \\ & \text{s.t.} & x \in X_{\mathrm{aux}}, \end{array}$$

where $\varphi_1(x) = \sum_{j=1}^{n} x_j$; $\varphi_2(x) = \sum_{j=1}^{n} \phi(x_j)$, with ϕ defined as follows,

$$\phi(t) = \begin{cases} 0, & t < 0, \\ (M+1)\sin^2(\pi t), & 0 \leq t < 1/2, \\ 1 + (M)\sin^2(\pi t), & 1/2 \leq t, \end{cases}$$

and $X_{\mathrm{aux}} = \{x \in R^n,\ x \geqq 0 : \sum_{j=1}^{n} a_{ji}x_j - d_i \geqq 0, i = 1, \ldots, m\}$, the feasible set of $(\mathrm{BP}_{\mathrm{aux}})_M$.

It is easy to see that the function $\phi : \mathbb{R} \to \mathbb{R}$ is differentiable and, from definition of φ, φ_2 is differentiable. Further, $\varphi_2(x) = \psi_2(x)$, for all $x \in \mathbb{N}^n$. It is worth noting that, under a computational view point, the function φ

penalizes those non integer points with a high value. Thus, the resolution of the problem $(\mathrm{BP}_{\mathrm{aux}})_M$ can be a practical approach to get solutions for (BCSP), such as the following theorem establishes.

Theorem 3.1. *Let M be in $\mathbb{R}$, $M > 0$, and $x^* \in \mathbb{Z}^n$ a feasible point for $(BP_{\mathrm{aux}})_M$. If x^* is a partial-1 efficient solution for $(BP_{\mathrm{aux}})_M$, then x^* is a partial-1 efficient solution for (BCSP).*

Proof. Absurdly suppose that there exist $M > 0$ and $x^* \in \mathbb{Z}^n$ a partial-1 efficient solution for $(\mathrm{BP}_{\mathrm{aux}})_M$, but x^* is not a partial-1 efficient solution for (BCSP). That is, there exists $\bar{x} \in X$ such that

$$(\psi_1(\bar{x}), \psi_2(\bar{x})) \leq_1 (\psi_1(x^*), \psi_2(x^*)).$$

Since $x^* \in \mathbb{Z}^n$ and $x^* \in X_{\mathrm{aux}}$, we have that $x^* \in X$, the feasible set of (BCSP), $\varphi_2(x^*) = \sum_{i=1}^n \delta(x_i^*) = \psi_2(x^*)$ and $\varphi_2(\bar{x}) = \sum_{i=1}^n \delta(\bar{x}_i) = \psi_2(\bar{x})$. Then,

$$(\varphi_1(\bar{x}), \varphi_2(\bar{x})) = (\psi_1(\bar{x}), \psi_2(\bar{x})) \leq_1 (\psi_1(x^*), \psi_2(x^*)) = (\varphi_1(x^*), \varphi_2(x^*)),$$

what implies

$$(\varphi_1(\bar{x}), \varphi_2(\bar{x})) \leq_1 (\varphi_1(x^*), \varphi_2(x^*)),$$

which stands in contradiction to the assumption that x^* is a partial-1 efficient solution for $(\mathrm{BP}_{\mathrm{aux}})_M$. Therefore, x^* is a partial-1 efficient solution for (BCSP). □

Theorem 3.2. *Let M be in $\mathbb{R}$, $M > 0$, and $x^* \in \mathbb{Z}^n$ a feasible point for $(BP_{\mathrm{aux}})_M$, then $(BP_{\mathrm{aux}})_M$ is partial-1 SKT-pseudoinvex at x^*.*

Proof. Let $x^* \in \mathbb{Z}^n$ be a feasible point for $(\mathrm{BP}_{\mathrm{aux}})_M$, with M in $\mathbb{R}$, $M > 0$. We have:

$$\nabla\varphi_1(x^*) = (1, 1, \ldots, 1). \tag{3.1}$$

For the second component of the multiobjective objective function, φ_2, it follows

$$\nabla\varphi_2(x^*) = (\phi'(x_1^*), \ldots, \phi'(x_n^*)),$$

where

$$\phi'(t) = \begin{cases} 0, & t < 0, \\ 2(M+1)\pi\sin(\pi t)\cos(\pi t), & 0 \leq t < 1/2, \\ M\pi\sin(\pi t)\cos(\pi t), & 1/2 \leq t. \end{cases}$$

Therefore, since $x^* \in \mathbb{Z}$,

$$\nabla\varphi_2(x^*) = (0, 0, \ldots, 0). \tag{3.2}$$

Let $w = (w_1, \ldots, w_m, w_{m+1}, \ldots, w_{m+n}) : \mathbb{R}^n \to \mathbb{R}^{m+n}$ be a function defined as follows:

$$w_i(x) = \begin{cases} d_i - \sum\limits_{j=1}^n a_{ij}x_j, & i = 1, \ldots, m, \\ -x_{i-m}, & i = m+1, \ldots, m+n. \end{cases}$$

With this new notation, the feasible set of $(\mathrm{BP}_{\mathrm{aux}})_M$ is defined as

$$X_{\mathrm{aux}} = \{x \in R^n, : w_i(x) \leqq 0,\ i = 1, \ldots, m+n\}.$$

To prove that $(\mathrm{BP}_{\mathrm{aux}})_M$ is partial-1 SKT-pseudoinvex let us assume that there exists $\bar{x} \in X_{\mathrm{aux}}$ such that $\varphi(\bar{x}) \leq_1 \varphi(x^*)$. Under this assumption, we have to find $\eta \in \mathbb{R}^n$ such that:

$$\begin{cases} \nabla\varphi_1(x^*)\eta \leq 0 \\ \nabla w_i(x^*)\eta \leqq 0, \quad i \in I(x^*), \end{cases} \tag{3.3}$$

where $I(x^*) = \{i = 1, \ldots, m+n : w_i(x^*) = 0\}$; $\nabla w_i(x^*) = (-a_{i1}, -a_{i2}, \ldots, -a_{in})$, for $i = 1, \ldots, m$, $\nabla w_{m+1}(x^*) = (-1, 0, \ldots, 0)$, $\nabla w_{m+2}(x^*) = (0, -1, \ldots, 0)$, $\ldots$, $\nabla w_{m+n}(x^*) = (0, 0, \ldots, -1)$.

By assumption, $\varphi(\bar{x}) \leq_1 \varphi(x^*)$, then we have that $\varphi_1(\bar{x}) < \varphi_1(x^*)$ and $\varphi_2(\bar{x}) \leqq \varphi_2(x^*)$. We define $\eta = \bar{x} - x^*$. It follows:

(i) Since $\sum_{j=1}^n \bar{x}_j < \sum_{j=1}^n x_j^*$, it follows that $\nabla\varphi_1(x^*)\eta < 0$.
(ii) If $i = 1, \ldots, m$ and $i \in I(x^*)$, then $\nabla w_i(x^*)\eta = \sum_{j=1}^n -a_{ij}(\bar{x}_j - x_j^*) = \sum_{j=1}^n -a_{ij}\bar{x}_j + d_i \leq 0$. Therefore, $\nabla w_i(x^*)\eta < 0$.
(iii) If $i = m+1, \ldots, m+n$, we have that $\nabla w_i(x^*)\eta = -\bar{x}_i + x_i^*$. If, moreover, $i \in I(x^*)$, then $x_i^* = 0$, and therefore $\nabla w_i(x^*)\eta = -\bar{x}_i \leqq 0$.

In consequence, η is a solution for (3.3). Therefore we have that $(\mathrm{BP}_{\mathrm{aux}})_M$ is partial-1 SKT-pseudoinvex at x^*. □

As a consequence of the previous result, we present a sufficient condition for a solution to be partial-1 efficient for (BCSP).

Theorem 3.3. *Let M be in $\mathbb{R}$, $M > 0$, and $x^* \in \mathbb{Z}^n$ a feasible point for $(BP_{\mathrm{aux}})_M$. If x^* is a strict Kuhn−Tucker point for $(BP_{\mathrm{aux}})_M$, then x^* is a partial-1 efficient solution for (BCSP).*

Proof. Let $x^* \in \mathbb{Z}^n$ strict Kuhn−Tucker point for $(\mathrm{BP}_{\mathrm{aux}})_M$. By Theorem 3.2, $(\mathrm{BP}_{\mathrm{aux}})_M$ is partial-1 SKT-pseudoinvex at x^*. Therefore, by Theorem 2.11, x^* is a partial-1 efficient solution for $(\mathrm{BP}_{\mathrm{aux}})_M$. Then, by Theorem 3.1, x^* is a partial-1 efficient solution for (BCSP). □

Example 3.4. Let us consider our initial Example 1.1, with the following demands: $d_1 = 102, d_2 = 200, d_3 = 150$. Its formulation is as follows:

$$\text{Minimize } \psi(x) = (\psi_1(x), \psi_2(x)) = \left(\sum_{j=1}^4 x_j, \sum_{j=1}^4 \delta(x_j)\right)$$

$$\begin{aligned} \text{s.t. } & 6x_1 + 5x_2 \geq 102 \\ & 2x_3 + x_4 \geq 200 \\ & x_2 + 2x_3 + 3x_4 \geq 150 \\ & x_1, x_2, x_3, x_4 \in \mathbb{N}. \end{aligned}$$

Now, we consider its auxiliary problem $(\mathrm{BP}_{\mathrm{aux}})_M$ with $M = 100$, and its feasible point $x^* = (x_1^*, x_2^*, x_3^*, x_4^*) = (17, 0, 100, 0)$, with $x^* \in \mathbb{Z}^4$. By calculus, we have that x^* is a strict Kuhn−Tucker point for $(\mathrm{BP}_{\mathrm{aux}})_M$, with the multipliers $\lambda = (\lambda_1, \lambda_2) = (6, 1)$, associated to the objective function $\varphi = (\varphi_1, \varphi_2)$, and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7) = (1, 3, 0, 0, 1, 0, 3)$, associated to the constraints of $(\mathrm{BP}_{\mathrm{aux}})_M$. Therefore, by Theorem 3.3, it follows that x^* is a partial-1 efficient solution for the bicriteria problem.

4. Conclusions

In this paper we present sufficient conditions for partial-1 efficient solutions of the cutting stock problem with two objectives: cost of raw material and cost of setup. Such optimality conditions can be used in computational tests of methods that seek to obtain the solutions for this problem. Moreover, the auxiliary problems can be used in exact or heuristic methods.

References

[1] M. Arana, A. Rufián, R. Osuna and G. Ruiz., Pseudoinvexity, optimality conditions and efficiency in multiobjective problems; duality. *Nonlinear Anal.* **68** (2008) 24–34.

[2] A. Chinchuluun and P. Pardalos, A survey of recent developments in multiobjective optimization. *Ann. Oper. Res.* **154** (2007) 2950.

[3] A. Diegel, Cutting paper in Richards Bay: dynamic local and global optimization in the trim problem *Orion* **3** (1988) 42–55.

[4] A. Diegel, E. Montocchio, E. Walters, S. Schalkwyk and S. Naidoo, Setup minimising conditions in the trim loss problem. *Eur. J. Oper. Res.* **95** (1996) 631–640.

[5] M. Ehrgott and X. Gandibleux, A survey and annotated bibliography of multiobjective combinatorial optimization. *OR Spektrum* **22** (2000) 425–460.

[6] S. Eilon, Optimizing the shearing of steel bars. *J. Mech. Eng. Sci.* **2** (1960) 129–142.

[7] R. Golfeto, A. Moretti and L. Salles Neto, A genetic symbiotic algorithm applied to the cutting stock problem with multiple objectives. *Adv. Model. Optim.* **11** (2009) 473–501.

[8] R. Haessler, Controlling Cutting Pattern Changes in One-Dimensional Trim Problems. *Oper. Res.* **23** (1975) 483–493.

[9] C. Hardley, Optimal cutting of zinc-coated steel strip. *Oper. Res.* **4** (1976) 92–100.

[10] J. Lee, In situ column generation for a cutting-stock problem. *Comput. Oper. Res.* **34** (2007) 2345–2358.

[11] L. Liu, X. Liu, J. Pei, W. Fan and P.M. Pardalos, A study on decision making of cutting stock with frustum of cone bars. *Oper. Res.* (2015) 1–18.

[12] O.L. Mangasarian, *Nonlinear Programming*, McGraw Hill Book Company, New York (1969).

[13] B.S. Mordukhovich, Variational analysis and generalized differentiation (I and II). Springer-Verlag Berlin Heidelberg, The Netherlands (2006).

[14] R. Osuna, A. Beato and A. Rufián, Generalized Convexity in Multiobjective Programming. *J. Math. Anal. Appl.* **233** (1999) 205–220.

[15] V. Pareto, Course d'economie politique. Rouge, Lausanne (1896).

[16] S.D. Prestwich, A.O. Fajemisin, L. Climent and B. OSullivan, Solving a Hard Cutting Stock Problem by Machine Learning and Optimisation. European Conference on Machine Learning and Knowledge Discovery in Databases. IEEE (2015) 335–347.

[17] L.L. Salles, A. Rufián, M. Arana and G.Ruiz, Weak efficiency in cutting stock problem. *Anais do CNMAC* **3** (2010) 187–193.

[18] C. Sousa, E. Silva, M. Lopes and A. Ramos , The Cutting Stock Problem: A Case Study in a Manufacturer of Pet Vivaria. *Oper. Res. Big Data* **15** (2015) 221–228.

[19] H. Stadler, A one-dimensional cutting stock problem in the aluminum industry and its solution. *Eur. J. Oper. Res.* **44** (1990) 209–223.

[20] S. Umetani, M. Yagiura and T. Ibaraki, One Dimensional Cutting Stock Problem with a Given Number of Setups: A Hybrid Approach of Metaheuristics and Linear Programming. *J. Math. Model. Algorithms* **5** (2006) 43–64.

[21] G. Wascher, P. Carow and H. Muller, Entwicklung eines flexiben Verfahrens für Zuschneideprobleme in einem Kaltwalzwerk. *Z. Oper. Res.* **29** (1985) 209–230.