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A FULL NT-STEP INFEASIBLE INTERIOR-POINT ALGORITHM FOR SEMIDEFINITE OPTIMIZATION

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Abstract. In this paper, a full Nesterov−Todd-step infeasible interior-point algorithm is presented for semidefinite optimization (SDO) problems. In contrast of some classical interior-point algorithms for SDO problems, this algorithm does not need to perform computationally expensive calculations for centering steps which are needed for classical interior-point methods. The convergence analysis of the algorithm is shown and it is also proved that the complexity bound of the algorithm coincides with the currently best iteration bound obtained by infeasible interior-point algorithms for this class of optimization problems.

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1. Introduction

Semidefinite optimization (SDO) problems are one of the most important subclasses of optimization problems in which a linear function of a matrix variable X is minimized (maximized) over the intersection of an affine set and the cone of positive semidefinite matrices. Mathematically, the primal problem (P) of SDO in the standard form is defined as follows:

$$\begin{aligned} \min \quad & C \bullet X \\ & A_i \bullet X = b_i, \;\; i = 1, 2, \ldots, m, \\ & X \succeq 0, \end{aligned}$$

and its Lagrangian dual (D) problem is defined as

$$\begin{aligned} \max \quad & b^T y \\ & \sum_{i=1}^{m} y_i A_i + S = C, \\ & S \succeq 0, \end{aligned}$$

where $M \bullet N := \mathrm{Tr}(MN)$ is the inner product related to $\mathcal{S}^n$ (the vector space of $n \times n$ real symmetric matrices), $A_i \in \mathcal{S}^n$ are linearly independent matrices, $C \in \mathcal{S}^n$ and $b \in \mathbb{R}^m$. The SDO problem as a natural extension

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of linear optimization (LO) problem, has received considerable attention and has been one of the most active research areas in mathematical programming.

In the past decade, various approaches have been proposed for solving and finding an optimal solution of SDO problem. Among them, the interior-point methods (IPMs) are efficient algorithms in both theoretical and practical aspects. Many IPMs for LO problems have been successfully extended to SDO problems [11, 17, 22]. The primal-dual IPMs for SDO problem have been widely studied by Klerk [3] and Wolkowicz [21]. Alizadeh *et al.* [1], Helemberg *et al.* [5] and Kojima *et al.* [9] proposed some primal-dual IPMs for SDO problem.

There are two types of IPMs based on choosing the starting point. Feasible IPMs, when the initial point and subsequent iterates are in the interior of feasible region and infeasible IPMs (IIPMs) when the initial point and subsequent iterates are not necessarily feasible. Primal-dual infeasible IPMs for LO problems was analyzed by Roos [18]. After that, many authors extended the Roos's idea for LO problems to various classes of mathematical problems. Mansouri *et al.* [13] proposed a primal-dual IIPM for SDO problems and obtained the same complexity result as Roos [18] for LO problems. Liu *et al.* [10], based on a new kernel function, presented a full Nesterov−Todd (NT) step IIPM for SDO problems. Zhang *et al.* [23] proposed a simplified infeasible interior-point algorithm for SDO problems. Their algorithm is an extension of the algorithm introduced by Roos [18] for LO problems and it measures the closeness to the central path by the merit function $\delta(V) := \left\|I - V^2\right\|_F$. In 2014, Wang *et al.* [20] suggested an infeasible interior-point algorithm for SDO problems. They established a sharper quadratic convergence result for feasible IPM which leads to a slightly wider neighborhood of the feasibility step in infeasible algorithm.

In most of the aforementioned papers, each main iteration of the proposed infeasible algorithms is composed of one feasibility step and several (at most 3) centering steps to get an ε-optimal solution of SDO problem. Recently, Roos [19] and Mansouri *et al.* [14], proposed two infeasible interior-point algorithms for LO problems. They proved that both algorithms do not need centering steps and take only one step in order to get a new iterate close to the central path.

Motivated by Roos [19] and Mansouri *et al.* [14], the main goal of this paper is to propose an infeasible interior-point algorithm for SDO problems in which the algorithm does not need to perform any centering steps to obtain a new iterate close to the central path in each iteration. We prove the convergence analysis of the algorithm and derive the currently best known iteration bound for the algorithm, namely, $O\left(n \log \varepsilon^{-1}\right)$.
The paper is organized as follows. Section 2 presents some well-known concepts and definitions which will be required in our analysis. In Section 3, after some preliminary discussion on infeasible IPMs for SDO problems, we present a new class of search directions for SDO problems in Section 3.1. A modified version of the proposed algorithm in [13] will be presented in Section 3.2. Section 4 is devoted to prove the convergence analysis of the proposed algorithm in Section 3.2. In Section 4.2, we obtain the complexity bound of the proposed algorithm. Finally, the paper will end with some conclusions in Section 5.

Some notations used throughout this article are as follow. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m\times n}$. Moreover, S^n denotes the set of all $n \times n$ real symmetric matrices. $S^n_{++}\left(S^n_+\right)$ denotes the set of all matrices in S^n which are positive definite (positive semidefinite). For $A \in S^n$, we write $A \succ 0\,(A \succeq 0)$ if A is positive definite (positive semidefinite). For any matrix A, $\lambda_i(A)$ denotes the ith eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$ while $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalue of the matrix A, respectively. Moreover, $\det(A) := \prod_{i=1}^n \lambda_i(A)$ and $\mathrm{Tr}(A) := \sum_{i=1}^n \lambda_i(A)$ respectively denote the determinant and trace of the matrix A whereas $\|A\|_F^2 := \mathrm{Tr}(AA^T)$ and $\|A\|_2 := \max_i |\lambda_i(A)|$ denote the Frobenius and infinity norms of the matrix A, respectively. The square root of the symmetric positive definite matrix A is denoted by $A^{\frac{1}{2}}$. The notation $A \sim B \Longleftrightarrow A = QBQ^{-1}$ for some invertible matrix Q, means the similarity between the matrices A and B.

2. Preliminaries

In this section, we recall some basic concepts and useful results concerning kernel function, matrix function and matrix barrier function, which are found in [4, 6].

Definition 2.1 (Chap. 2 in [4])**.** A twice differentiable function $\phi(t) : (0,\infty) \longrightarrow [0,\infty)$ is called a kernel function if

$$\phi(1) = \phi'(1) = 0, \quad \phi''(t) > 0, \quad \forall t > 0.$$

A kernel function is called eligible, if it satisfies some extra properties as stated by Bai *et al.* [2].

Definition 2.2. Suppose the matrix $G \in \mathbb{R}^{n\times n}$ is diagonalizable with eigen-decomposition

$$G = Q_G^{-1}\mathrm{diag}\,(\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))\, Q_G,$$

where Q_G is a nonsingular matrix. Also, let $\phi(t) : \mathbb{R} \to \mathbb{R}$ be a real-valued function. The matrix function $\phi(G)$ is defined by

$$\phi(G) := Q_G^{-1}\mathrm{diag}\,(\phi(\lambda_1(G)), \phi(\lambda_2(G)), \ldots, \phi(\lambda_n(G)))\, Q_G. \tag{2.1}$$

For any diagonalizable matrix G, we define a matrix barrier function $\Phi(G) : S^n_{++} \to \mathbb{R}$ by

$$\Phi(G) := \mathrm{Tr}\,(\phi(G)) = \sum_{i=1}^{n} \phi(\lambda_i(G)). \tag{2.2}$$

Since the derivatives $\phi'(t)$ and $\phi''(t)$ are well-defined, the definitions $\phi'(G)$ and $\phi''(G)$ are also well-defined and they are defined as follows:

$$\phi'(G) := Q_G^{-1}\mathrm{diag}\,(\phi'(\lambda_1(G)), \phi'(\lambda_2(G)), \ldots, \phi'(\lambda_n(G)))\, Q_G, \tag{2.3}$$

$$\phi''(G) := Q_G^{-1}\mathrm{diag}\,(\phi''(\lambda_1(G)), \phi''(\lambda_2(G)), \ldots, \phi''(\lambda_n(G)))\, Q_G. \tag{2.4}$$

Definition 2.3. A matrix $M(t)$ is said to be a matrix of functions if each entry of $M(t)$ is a function of t, that is $M(t) := [M_{i,j}(t)]$.

It is not difficult to check that the usual concepts of continuity, differentiability and integrability can be naturally extended to matrix of functions, by interpreting them component-wise. For more details, we refer the reader to Chapter 6 in [6].

3. Central path of perturbed problems

In this section, we first describe the idea of infeasible IPMs for solving the SDO problems and then we present a modified version of the proposed algorithm in [13] for SDO problems.

For convenience of reference, we consider the feasibility set and the strictly feasibility set of the problems (P) and (D) as follows:

$$\begin{aligned}
\mathcal{F} &:= \{(X,y,S) :\ A_i \bullet X = b_i,\ \ \sum_{i=1}^{m} y_i A_i + S = C,\ \ i = 1,2,\ldots,m, \quad X, S \succeq 0\},\\
\mathcal{F}^0 &:= \{(X,y,S) \in F :\ \ X, S \succ 0\}.
\end{aligned}$$

Let (X^*, y^*, S^*) be an optimal solution of the SDO problems (P) and (D). That is, $(X^*, y^*, S^*) \in \mathcal{F}^*$ where $\mathcal{F}^* := \{(X,y,S) \in \mathcal{F} : \mathrm{Tr}(XS) = 0\}$. We also assume that $X^* + S^* \preceq \zeta I$ and $\mu^0 = \zeta^2$ and define $\left(X^0, y^0, S^0\right) := \zeta\,(I, 0, I)$ as the initial point of the algorithm.

Letting $\nu \in (0,1]$, the perturbed primal problem (P_ν) is defined as

$$\begin{aligned}
\min \quad & \left(C - \nu R_d^0\right) \bullet X\\
& A_i \bullet X = b_i - \nu (r_p^0)_i,\ \ i = 1,2,\ldots,m,\\
& X \succeq 0,
\end{aligned}$$

while the perturbed Lagrangian dual problem (D_ν) is defined as

$$\begin{aligned} \max \quad & \left(b-\nu r_p^0\right)^T y \\ & \sum_{i=1}^m y_i A_i + S = C - \nu R_d^0, \\ & S \succeq 0, \end{aligned}$$

where $(r_p^0)_i := b_i - A_i \bullet X^0$ for $i = 1, 2, \ldots, m$ and $R_d^0 := C - \sum_{i=1}^m y_i^0 A_i - S^0$ denote the initial values of residuals related to the initial point $\left(X^0, y^0, S^0\right)$. In infeasible IPMs, the triplet (X, y, S) is called an ε-optimal solution of SDO problem if the norms of the residuals and the value of duality gap related to the iterate (X, y, S) are less than the accuracy parameter ε. That is, $\max\{\mathrm{Tr}(XS), \|r_p\|, \|R_d\|_F\} \leq \varepsilon$.

In this paper, we assume that the primal and dual problems (P) and (D) satisfy interior-point condition (IPC), *i.e.*, $\mathcal{F}^0 \neq \emptyset$. As in [3], without loss of generality, we can assume the IPC for SDO problems. In fact, we can achieve the IPC using the so-called embedding technique introduced by Klerk [3] for SDO problems.

Assuming the original problems (P) and (D) are both feasible, Lemma 4.1 in [13] guarantees the perturbed problems (P_ν) and (D_ν) satisfy the IPC for $\nu \in [0, 1]$ and therefore the central path of SDO exists. This means that the perturbed Karush–Kuhn–Tucker optimality conditions

$$\begin{aligned} b_i - A_i \bullet X &= \nu (r_p^0)_i, \quad X \succeq 0 \\ C - \sum_{i=1}^m y_i A_i - S &= \nu R_d^0, \qquad S \succeq 0 \\ XS &= \mu I, \end{aligned} \tag{3.1}$$

has a unique solution for each $\mu > 0$. Note that, under the IPC, the primal and dual problems (P) and (D) have an optimal solution with equal values and moreover, for each $\mu > 0$, system (3.1) has a unique solution [3]. These unique solutions of system (3.1), denoted by $X(\mu, \nu)$ and $(y(\mu, \nu), S(\mu, \nu))$, are called the μ-centers of the perturbed problems (P_ν) and (D_ν). The set of all μ-centers construct a curve, so-called the central path, which used as a guide line to optimal solution of SDO problem. Clearly, since $X^0 S^0 = \mu^0 I$, X^0 and $\left(y^0, S^0\right)$ are the μ^0-center of the perturbed problems (P_1) and (D_1).

3.1. Search directions

Let X and (y, S) be the strictly feasible solutions for the perturbed problems (P_ν) and (D_ν). As we mentioned before, the infeasible algorithm proceeds to generate a new feasible solution (X^+, y^+, S^+) of the perturbed problems (P_{ν^+}) and (D_{ν^+}). To this end, using Newton method, we should calculate the search directions ΔX, Δy and ΔS such that

$$\begin{aligned} A_i \bullet (X + \Delta X) &= b_i - \nu^+ (r_p^0)_i, \qquad i = 1, 2, \ldots, m, \\ \sum_{i=1}^m (y_i + \Delta y_i) A_i + (S + \Delta S) &= C - \nu^+ R_d^0. \end{aligned} \tag{3.2}$$

Since (X, y, S) is strictly feasible for the perturbed problems (P_ν) and (D_ν), it follows that the search direction $(\Delta X, \Delta y, \Delta S)$ should satisfy

$$\begin{aligned} A_i \bullet \Delta X &= \theta \nu (r_p^0)_i, \qquad i = 1, 2, \ldots, m, \\ \sum_{i=1}^m \Delta y_i A_i + \Delta S &= \theta \nu R_d^0. \end{aligned} \tag{3.3}$$

In order to obtain a unique solution for (3.3), we add the equation $\Delta XS + X\Delta S = \mu I - XS$ or equivalently $\Delta X + X\Delta SS^{-1} = \mu S^{-1} - X$ to this system, which follows the following system:

$$\begin{aligned} A_i \bullet \Delta X &= \theta\nu(r_p^0)_i, \quad i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S &= \theta\nu R_d^0, \\ \Delta X + X\Delta SS^{-1} &= \mu S^{-1} - X. \end{aligned} \tag{3.4}$$

Clearly, due to the third equation of the above system, the search direction ΔX is not necessarily symmetric because the matrix $X\Delta SS^{-1}$ may not be symmetric. Based on different symmetrization schemes, several search directions have been proposed, as presented in [3,8,22]. However, in our analysis, we use Zhang's direction [22] which is obtained by the following Newton system:

$$\begin{aligned} A_i \bullet \Delta X &= \theta\nu(r_p^0)_i, \quad i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S &= \theta\nu R_d^0, \\ \Delta X + P\Delta SP^T &= \mu S^{-1} - X, \end{aligned} \tag{3.5}$$

where P is a symmetric nonsingular matrix. The system (3.5) has a unique solution for each matrix P [21] and the obtained search direction ΔX is automatically symmetric.

Different choices have been proposed for the nonsingular matrix P. For instance, Alizadeh *et al.* [21] used $P = I$ while Kojima *et al.* [8] and Monterio [15] respectively suggested $P = X^{-\frac{1}{2}}$ and $P = S^{\frac{1}{2}}$. However, in our analysis, we choose the matrix P proposed by Nesterov and Todd [16], which is as follows:

$$P := X^{\frac{1}{2}} \left(X^{\frac{1}{2}} S X^{\frac{1}{2}}\right)^{-\frac{1}{2}} X^{\frac{1}{2}} = S^{-\frac{1}{2}} \left(S^{\frac{1}{2}} X S^{\frac{1}{2}}\right)^{\frac{1}{2}} S^{-\frac{1}{2}}. \tag{3.6}$$

Let $D := P^{\frac{1}{2}}$. The matrix D can be used to rescale X and S to the same matrix V [3], defined by

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD. \tag{3.7}$$

In each iteration of infeasible interior-point algorithm, we measure the closeness of the current iterate (X, y, S) to the μ-center by using the quantity

$$\delta(X, S; \mu) := \delta(V) := \|I - V\|_F . \tag{3.8}$$

Defining the notations

$$D_X := \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \quad D_S := \frac{1}{\sqrt{\mu}} D\Delta SD, \quad D_{XS} := \frac{1}{2}(D_X D_S + D_S D_X), \tag{3.9}$$

and using them in system (3.5), this system can be redefined as follows:

$$\begin{aligned} \bar{A}_i \bullet D_X &= \frac{1}{\sqrt{\mu}} \theta\nu(r_p^0)_i, \quad i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} \frac{\Delta y_i}{\sqrt{\mu}} \bar{A}_i + D_S &= \frac{1}{\sqrt{\mu}} \theta\nu D R_d^0 D, \\ D_X + D_S &= V^{-1} - V, \end{aligned} \tag{3.10}$$

where $\bar{A}_i = DA_iD$.

Considering the so-called classical kernel function

$$\psi(t) := \frac{t^2-1}{2} - \log t, \tag{3.11}$$

the right-hand side of the third equation in system (3.10) obviously equals to $-\psi'(V)$. Thus, the system (3.10) can be rewritten as

$$\begin{aligned} \bar{A}_i \bullet D_X &= \frac{1}{\sqrt{\mu}}\theta\nu(r_p^0)_i, \qquad i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} \frac{\Delta y_i}{\sqrt{\mu}} \bar{A}_i + D_S &= \frac{1}{\sqrt{\mu}}\theta\nu D R_d^0 D, \\ D_X + D_S &= -\psi'(V). \end{aligned} \tag{3.12}$$

One of the main differences between our algorithm with the classical interior-point algorithms for SDO problems such as [12,13,20] is the way that the search directions are generated. To compute the search directions, similar to Liu *et al.* [10], we use the kernel function $\phi(t) := \frac{1}{2}(t-1)^2$ instead of the classical logarithmic kernel function (3.11) and we obtain the following result.

Lemma 3.1. *Let $\phi(t) := \frac{1}{2}(t-1)^2$ and V be the diagonalizable matrix defined in (3.7). Then, we have*

$$\phi'(V) = V - I.$$

Proof. Since $\phi(t) = \frac{1}{2}(t-1)^2$ and $\phi'(t) = t - 1$, due to (2.4), we have

$$\phi'(V) := Q_V^{-1} \mathrm{diag}\left(\phi'(\lambda_1(V)), \phi'(\lambda_2(V)), \ldots, \phi'(\lambda_n(V))\right) Q_V,$$

which implies

$$\phi'(V) := Q_V^{-1} \mathrm{diag}\left(\lambda_1(V) - 1, \lambda_2(V) - 1, \ldots, \lambda_n(V) - 1\right) Q_V = V - I.$$

This concludes the result and ends the proof. □

Applying Lemma 3.1 and replacing $-\psi'(V)$ in (3.12) with $-\phi'(V)$, we obtain the following system:

$$\begin{aligned} \bar{A}_i \bullet D_X &= \frac{1}{\sqrt{\mu}}\theta\nu(r_p^0)_i, \qquad i = 1, 2, \ldots, m, \\ \sum_{i=1}^{m} \frac{\Delta y_i}{\sqrt{\mu}} \bar{A}_i + D_S &= \frac{1}{\sqrt{\mu}}\theta\nu D R_d^0 D, \\ D_X + D_S &= I - V. \end{aligned} \tag{3.13}$$

According to (3.8), clearly $\delta(V) = 0$ if and only if $V = I$ and it is equivalent to this fact that (X, S) coincides with the μ-center $(X(\mu), S(\mu))$, *i.e.*, $XS = \mu I$. After a full-NT step, the new iterates are given as follows:

$$\begin{aligned} X^+ &:= X + \Delta X = \sqrt{\mu} D(V + D_X)D, \\ y^+ &:= y + \Delta y, \\ S^+ &:= S + \Delta S = \sqrt{\mu} D^{-1}(V + D_S)D^{-1}. \end{aligned} \tag{3.14}$$

Due to the above discussion, after a full-NT step, the new iterate (X^+, y^+, S^+) satisfies the affine equations of the new perturbed problems (P_{ν^+}) and (D_{ν^+}). Although, the new iterate (X^+, y^+, S^+) is feasible for the new perturbed problems, it is necessary to show that X^+ and S^+ are positive definite and satisfy $\delta\left(X^+, S^+; \mu^+\right) \le \tau$.

3.2. Algorithm

Let $\left(X^0, y^0, S^0\right)$ and (X, y, S) respectively denote the initial and the current iterations of the infeasible algorithm. The infeasible algorithm works as follows.

Let the current iteration (X, S) be positive and satisfy two first equations in (3.1) and $\delta\,(X, S; \mu) \leq \tau$ for some $\mu \in (0, \mu^0]$. Clearly, these certainly hold at the start of the first iteration due to the definition of initial point $\left(X^0, y^0, S^0\right)$. Reducing the parameter μ to $\mu^+ := (1-\theta)\mu$, with $\theta \in (0, 1)$, we find a new iterate (X^+, y^+, S^+) that satisfies the feasibility equations in (3.1) with ν replaced by $\nu^+ := (1-\theta)\nu$ and $\delta\,(X^+, S^+; \mu^+) \leq \tau$. This procedure is repeated until an ε-optimal solution of the SDO problem is found. A more formal description of the algorithm is summarized in bellow.

Algorithm: Infeasible interior-point algorithm for SDO problem

Step 0 (Initialization): Choose an accuracy parameter $\varepsilon > 0$, a barrier update parameter θ, $0 < \theta < 1$, a threshold parameter $\tau > 0$, an initial point $\left(X^0,\, y^0,\, s^0\right) := (\zeta I,\, 0,\, \zeta I)$ and a parameter $\mu^0 > 0$ such that $X^0 \bullet S^0 = n\mu^0$.

Step 1 (Test convergence): If $\max\{\mathrm{Tr}(XS),\, \|r_p\|,\, \|R_d\|_F\} \leq \varepsilon$, declare convergence and stop. Otherwise, proceed to the next step.

Step 2 (Computation): Calculate the search direction $(\Delta X, \Delta y, \Delta S)$ *via* (3.9) and (3.10) and compute the new iterate $(X^+,\, y^+,\, S^+)$ that satisfies $\delta\,(X^+,\, S^+;\, \mu^+) \leq \tau$. Now, update the parameters μ and ν by the factor $1-\theta$ and go to Step 3.

Step 3 (Update iterate): Set $(X,\, y,\, S) = (X^+,\, y^+,\, S^+)$ and go to Step 1.

4. Analysis of Algorithm

In this section, we present the analysis of algorithm and show that without requiring centering steps, our proposed algorithm is well-defined and finds an ε-optimal solution of SDO problem in polynomial time complexity. To this end, let us to recall some key and technical lemmas which directly required in proof of convergence analysis of algorithm. For proof and more details, we refer the reader to [3].

Lemma 4.1. *Let* $(X(\alpha), S(\alpha)) = (X, S) + \alpha\,(\Delta X, \Delta S)$ *and* $X \succ 0$ *and* $S \succ 0$. *If for* $\alpha \in [0, \bar{\alpha}]$, $\det(X(\alpha)S(\alpha)) > 0$, *then* $X(\bar{\alpha}) \succ 0$ *and* $S(\bar{\alpha}) \succ 0$.

Lemma 4.2. *Let* A *be an* $n \times n$ *skew-symmetric matrix and* B *be a positive definite one. Then*

$$\det\,(A + B) > 0,$$

and moreover if the eigenvalues of $A + B$ *are real, then*

$$0 < \lambda_{\min}(B) \leq \lambda_{\min}\,(A + B) \leq \lambda_{\max}\,(A + B) \leq \lambda_{\max}(B),$$

which means that the matrix $A + B$ *is positive definite.*

The following lemma gives some bounds for the eigenvalues of the variance matrix V.

Lemma 4.3. *Let* $\delta := \delta(V)$ *be given by* (3.8). *Then*

$$q(\delta) \leq \lambda_i(V) \leq \rho(\delta), \tag{4.1}$$

where $q(\delta) := 1 - \delta$ *and* $\rho(\delta) := 1 + \delta$.

4.1. Convergence analysis

In this subsection, we proceed to prove that algorithm is well-defined. To this end, we need to show that X^+ and S^+ are positive definite matrices and satisfy $\delta(X^+, S^+; \mu^+) \leq \tau$. That is, (X^+, y^+, S^+) belongs to a sufficiently small-neighborhood of μ^+-center. The following lemma presents a sufficient condition that guarantees the new iterates X^+ and S^+ are positive definite.

Lemma 4.4. *Let $X \succ 0$ and $S \succ 0$. Then the new iterate (X^+, y^+, S^+) are strictly feasible if $V + D_{XS} \succ 0$.*

Proof. Let $\alpha \in [0,1]$ and define $X(\alpha) := X + \alpha\Delta X$ and $S(\alpha) := S + \alpha\Delta S$. To prove the claim, considering Lemma 4.1, it suffices to show that the determinant $X(\alpha)S(\alpha)$ is positive, for $\alpha \in [0,1]$. To this end, we have

$$\begin{aligned} X(\alpha)S(\alpha) &= (X + \alpha\Delta X)(S + \alpha\Delta S) = \mu D(V + \alpha D_X)(V + \alpha D_S) D^{-1} \\ &= \mu D\Big[V^2 + \alpha(VD_S + D_X V) + \alpha^2 D_X D_S\Big] D^{-1} \\ &= \mu D\Big[V^2 + \alpha(VD_X + VD_S + D_X V - VD_X) + \alpha^2 D_X D_S\Big] D^{-1} \\ &= \mu D\Big[V^2 + \alpha(VD_X + VD_S) + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S\Big] D^{-1}. \end{aligned} \tag{4.2}$$

Multiplying the both sides of the third equation in (3.13) from the left by V, we have

$$VD_X + VD_S = V - V^2. \tag{4.3}$$

Substituting (4.3) into (4.2), we conclude that

$$\begin{aligned} X(\alpha)S(\alpha) &= \mu D\Big[V^2 + \alpha(V - V^2) + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S\Big] D^{-1} \\ &= \mu D\Big[(1-\alpha)V^2 + \alpha V + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S\Big] D^{-1}. \end{aligned} \tag{4.4}$$

Subtracting and adding the term $\mu D\left(\frac{\alpha^2}{2} D_S D_X\right) D^{-1}$ to the right hand side of the equation (4.4), we obtain

$$\begin{aligned} X(\alpha)S(\alpha) &= \mu D\Big[(1-\alpha)V^2 + \alpha V + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S \\ &\qquad - \frac{\alpha^2}{2} D_S D_X + \frac{\alpha^2}{2} D_S D_X\Big] D^{-1} \\ &= \mu D\Big[(1-\alpha)V^2 + \alpha(1-\alpha)V + \alpha^2 (V + D_{XS}) \\ &\qquad + \alpha\left(\alpha M + (1-\alpha)(D_X V - VD_X)\right)\Big] D^{-1}, \end{aligned} \tag{4.5}$$

where M is a skew symmetric matrix defined as

$$M := (D_X V - VD_X) + \frac{1}{2}(D_X D_S - D_S D_X). \tag{4.6}$$

Now, defining the skew symmetric and symmetric matrices $M(\alpha)$ and $N(\alpha)$ respectively as

$$\begin{aligned} M(\alpha) &:= \alpha\left(\alpha M + (1-\alpha)(D_X V - VD_X)\right), \\ N(\alpha) &:= (1-\alpha)V^2 + \alpha(1-\alpha)V + \alpha^2 (V + D_{XS}), \end{aligned} \tag{4.7}$$

we can rewrite (4.5) as

$$X(\alpha)S(\alpha) = \mu D\left(N(\alpha) + M(\alpha)\right) D^{-1},$$

or equivalently

$$\frac{X(\alpha)S(\alpha)}{\mu} \sim N(\alpha) + M(\alpha). \tag{4.8}$$

Hence, to prove that the determinant $X(\alpha)S(\alpha)$ is positive, it suffices to show that the determinant $N(\alpha)+M(\alpha)$ is positive for $\alpha \in [0,1]$. To this end, using Lemma 4.2 and noticing to this fact that the matrix $M(\alpha)$ is skew symmetric for $\alpha \in [0,1]$, we only need to show that the symmetric matrix $N(\alpha)$ is positive definite. However, the later is true for $\alpha \in [0,1]$ because the matrices $V + D_{XS}$, V and V^2 are positive definite in definition of $N(\alpha)$. Thus, the determinant $X(\alpha)S(\alpha)$ is positive. On the other hand, by assumption, we have $X(0) = X \succ 0$ and $S(0) = S \succ 0$. Therefore, Lemma 4.1 implies that the matrices $X(1) = X^+$ and $S(1) = S^+$ are positive definite. This completes the proof. □

The following corollary states a key result of the above lemma.

Corollary 4.5. *Let $\delta := \delta(V)$. The new iterate (X^+, y^+, S^+) is strictly feasible if*

$$\|D_X\|_F^2 + \|D_S\|_F^2 < 2(1-\delta). \tag{4.9}$$

Proof. Due to Lemma 4.4, the new iterate (X^+, y^+, S^+) is strictly feasible if $V + D_{XS} \succ 0$ or equivalently

$$\lambda_i\left(V + D_{XS}\right) > 0, \text{ for } i = 1, 2, \ldots, n.$$

Using the definition of D_{XS}, the properties of Frobenius norm and Lemma 4.3, we have

$$\begin{aligned}
\lambda_i\left(V + D_{XS}\right) &\geq \lambda_{\min}\left(V + D_{XS}\right) \geq \lambda_{\min}(V) - \|D_{XS}\|_F\,,\\
&\geq \lambda_{\min}(V) - \|D_X\|_F \|D_S\|_F\\
&\geq \lambda_{\min}(V) - \frac{1}{2}\left(\|D_X\|_F^2 + \|D_S\|_F^2\right)\\
&\geq q(\delta) - \frac{1}{2}\left(\|D_X\|_F^2 + \|D_S\|_F^2\right)\\
&= 1 - \delta - \frac{1}{2}\left(\|D_X\|_F^2 + \|D_S\|_F^2\right).
\end{aligned}$$

Thus, $\lambda_i\left(V + D_{XS}\right) > 0$ if $1 - \delta - \frac{1}{2}\left(\|D_X\|_F^2 + \|D_S\|_F^2\right) > 0$. This implies the desired result and ends the proof. □

Let X^+ and S^+ be the generated iterations by the algorithm presented in Section 3.2. Substituting $\alpha = 1$ in (4.5), we have

$$X(1)S(1) = X^+S^+ = \mu D\Big[V + D_{XS} + M\Big]D^{-1},$$

or equivalently

$$X^+S^+ \sim \mu\Big[V + D_{XS} + M\Big], \tag{4.10}$$

where M is as defined in (4.6).

Defining

$$V^+ := \frac{1}{\sqrt{\mu^+}}D^{-1}X^+D^{-1} = \frac{1}{\sqrt{\mu^+}}DS^+D = \frac{1}{\sqrt{\mu^+}}\left(D^{-1}X^+S^+D\right)^{\frac{1}{2}}, \tag{4.11}$$

we proceed to obtain a lower bound for the minimum eigenvalue of V^+.

Lemma 4.6. *Let $\delta := \delta(V)$ and V^+ be defined as (4.11). Then*

$$\lambda_{\min}\left(\left(V^+\right)^2\right) \geq \frac{1}{1-\theta}\left(1-\delta-\frac{1}{2}\left(\|D_X\|_F^2+\|D_S\|_F^2\right)\right). \tag{4.12}$$

Proof. The proof is similar to the proof of Lemma 5.7 in [13], and is therefore omitted. □

To proceed our analysis, assuming $\|D_X\|_F^2 + \|D_S\|_F^2 < 2(1-\delta)$, we need to obtain an upper bound for the term $\delta(V^+) := \delta\left(X^+, S^+; \mu^+\right)$. The following lemma tasks this goal. For proof and more details, we refer the reader to [7].

Lemma 4.7. *Let $\delta := \delta(V)$ and $\|D_X\|_F^2 + \|D_S\|_F^2 < 2(1-\delta)$. Then*

$$\delta\left(V^+\right) \leq \frac{\delta + \frac{1}{2}\left(\|D_X\|_F^2+\|D_S\|_F^2\right)+\theta\sqrt{n}}{1-\theta+\sqrt{(1-\theta)\left(1-\delta-\frac{1}{2}\left(\|D_X\|_F^2+\|D_S\|_F^2\right)\right)}}. \tag{4.13}$$

Since in our analysis we have $\delta \leq \tau$ and we need to have $\delta\left(V^+\right) \leq \tau$, using Lemma 4.7, it suffices to have

$$\frac{\tau + \frac{1}{2}\left(\|D_X\|_F^2+\|D_S\|_F^2\right)+\theta\sqrt{n}}{1-\theta+\sqrt{(1-\theta)\left(1-\tau-\frac{1}{2}\left(\|D_X\|_F^2+\|D_S\|_F^2\right)\right)}} \leq \tau. \tag{4.14}$$

Considering $t := \|D_X\|_F^2 + \|D_S\|_F^2$ as a single term, the inequality (4.14) can be reformulated as the quadratic polynomial inequality $t^2 + \alpha t + \gamma \leq 0$ where α and γ are the terms based on τ, θ and n. By some elementary calculations (using the MATLAB software), we obtain the positive root of this quadratic polynomial

$$\begin{aligned}\|D_X\|_F^2 + \|D_S\|_F^2 \leq\ & \tau\sqrt{(1-\theta)\left(\tau(1-\theta)(\tau-4)+4(1+\sqrt{n}\theta)\right)} \\ & + \tau(2-\tau)(1-\theta) - 2\tau - 2\sqrt{n}\theta.\end{aligned} \tag{4.15}$$

To proceed our analysis, we need to obtain an upper bound for the term $\|D_X\|_F^2 + \|D_S\|_F^2$. To this end, let $\mathcal{L} = \{Z \in S^n :\ \bar{A}_i \bullet Z = 0,\ i = 1, 2, \ldots, m\}$. Thus, the affine space $\{Z \in S^n :\ \bar{A}_i \bullet Z = \frac{1}{\mu}\theta\nu\left(r_p^0\right)_i\ i = 1, 2, \ldots, m\}$ equals to $D_X + \mathcal{L}$ and therefore $D_S \in \frac{1}{\sqrt{\mu}}\theta\nu R_d^0 D + \mathcal{L}^\perp$. Since $\mathcal{L} \bigcap \mathcal{L}^\perp = \{0\}$, the spaces $\mathcal{L} + D_X$ and $\mathcal{L}^\perp + D_S$ have an intersection component that we denote it by Q. The following lemma plays a key role in our analysis.

Lemma 4.8. *Let $w := \frac{1}{2}\sqrt{\|D_X\|_F^2 + \|D_S\|_F^2}$ and Q be the unique point in the intersection of the affine space $D_X + \mathcal{L}$ and $D_S + \mathcal{L}^\perp$. Then*

$$2w \leq \sqrt{\|Q\|_F^2 + \left(\|Q\|_F + \delta\right)^2}, \tag{4.16}$$

where $\delta := \delta(V)$ as defined in (3.8).

Proof. The proof is similar to the proof of Lemma 5.11 in [13], and is therefore omitted. □

Due to the above lemma, we have

$$\|D_X\|_F^2 + \|D_S\|_F^2 \leq \|Q\|_F^2 + \left(\|Q\|_F + \delta\right)^2 = 2\|Q\|_F^2 + 2\delta\|Q\|_F + \delta^2. \tag{4.17}$$

To obtain an upper bound for the term $\|D_X\|_F^2 + \|D_S\|_F^2$, we proceed to get an upper bound for the term $\|Q\|_F$. To this end, we recall the following lemma from [13].

Lemma 4.9 (Lem. 5.13 in [13]). *With* $\left(X^0, y^0, S^0\right) := (\zeta I, 0, \zeta I)$, *we have*

$$\|Q\|_F \leq \frac{\theta}{\zeta \lambda_{min}(V)} \operatorname{Tr}(X+S). \tag{4.18}$$

Substituting (4.18) into (4.17), we get

$$\|D_X\|_F^2 + \|D_S\|_F^2 \leq \frac{2\theta^2}{\zeta^2 \lambda_{\min}(V)^2}\left(\operatorname{Tr}(X+S)^2\right) + \frac{2\theta\delta}{\zeta\lambda_{\min}(V)} \operatorname{Tr}(X+S) + \delta^2. \tag{4.19}$$

However, to obtain an exact upper bound for the term $\|D_X\|_F^2 + \|D_S\|_F^2$, we need to present an upper bound for the term $\operatorname{Tr}(X+S)$. The following Lemma tasks this goal.

Lemma 4.10 (Lem. 5.16 in [13]). *Let* X *and* (y, S) *respectively be feasible for the perturbed problems* (P_ν) *and* (D_ν) *and let* $\left(X^0, y^0, S^0\right) := (\zeta I, 0, \zeta I)$. *Then, one has*

$$\operatorname{Tr}(X+S) \leq \left(1 + \rho(\delta)^2\right) n\zeta, \tag{4.20}$$

where $\rho(\delta)$ *as defined in Lemma* 4.3.

Using (4.1) and substituting (4.20) into (4.19), we have

$$\begin{aligned}
\|D_X\|_F^2 + \|D_S\|_F^2 &\leq \frac{2n^2\theta^2}{q(\delta)^2}\left(1+\rho(\delta)^2\right)^2 + \frac{2n\theta\delta}{q(\delta)}\left(1+\rho(\delta)^2\right) + \delta^2 \\
&= \frac{2n^2\theta^2}{(1-\delta)^2}\left(1+(1+\delta)^2\right)^2 + \frac{2n\theta\delta}{1-\delta}\left(1+(1+\delta)^2\right) + \delta^2.
\end{aligned} \tag{4.21}$$

In order to have (4.15), using (4.21) and assuming $\delta \leq \tau$, we consider the following weaker condition:

$$\begin{aligned}
&\frac{2n^2\theta^2}{(1-\tau)^2}\left(1+(1+\tau)^2\right)^2 + \frac{2n\theta\tau}{1-\tau}\left(1+(1+\tau)^2\right) + \tau^2 \\
\leq& \tau\sqrt{(1-\theta)\left(\tau(1-\theta)(\tau-4) + 4(1+\sqrt{n}\theta)\right)} + \tau(2-\tau)(1-\theta) - 2\tau - 2\sqrt{n}\theta.
\end{aligned}$$

Obviously, the left-hand side of the above inequality is increasing in τ. Using this, one may easily check that the above inequality holds if

$$\tau = \frac{1}{5}, \qquad \theta = \frac{1}{15n}, \qquad n \geq 2. \tag{4.22}$$

This means that the new iterates (X, S) are positive and $\delta(X, S; \mu) \leq \frac{1}{5}$ during algorithm . Thus, algorithm is well-defined.

4.2. Iteration bound

In previous section, we proved that algorithm is well-defined. That is, if at the start of an iteration the iterate (X, S) is positive and satisfies $\delta(X, S; \mu) \leq \frac{1}{5}$, then after the full-NT step, with θ as defined in (4.22), the new generated iterate (X^+, S^+) is also positive and satisfies $\delta(X^+, S^+; \mu^+) \leq \frac{1}{5}$. Moreover, in each main iteration of algorithm , the value of duality gap and the norms of the residuals reduce by the factor $1-\theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\left\{X^0 \bullet S^0, \left\|R_d^0\right\|_F, \left\|r_p^0\right\|\right\}}{\varepsilon}.$$

Due to (4.22), the total number of inner iterations is bounded above by

$$15n\log\frac{\max\left\{X^0\bullet S^0,\,\left\|R_d^0\right\|_F,\,\left\|r_p^0\right\|\right\}}{\varepsilon}.$$

Thus, we may state without further proof the main result of the paper as follows.

Theorem 4.11. *If (P) and (D) have the optimal solution $(X^*,\,y^*,\,S^*)$ such that $X^*+S^*\le\zeta I$, then after at most*

$$15n\log\frac{\max\left\{X^0\bullet S^0,\,\left\|R_d^0\right\|_F,\,\left\|r_p^0\right\|\right\}}{\varepsilon},$$

iterations algorithm finds an ε-optimal solution of SDO problem.

5. Concluding remarks

In this paper, we proposed an improved version of the classical infeasible interior-point algorithm proposed by Mansouri *et al.* [13] for SDO problems. Modification is based on elimination of the centering steps in each main iteration of the classical infeasible algorithm. We proved that the proposed algorithm is convergent and well-defined and its complexity coincides with the currently best known iteration bound for infeasible interior-point algorithms for this class of optimization problems.

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