# THE RESOURCE CONSTRAINED PROJECT SCHEDULING PROBLEM: A THEORETICAL COMPARISON BETWEEN A RECENT FORMULATION AND THE MAIN TIME INDEXED LINEAR PROGRAMMING BASED APPROACHES

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**Abstract.** We compare, at a theoretical level, the RCPSP formulation proposed in [L. Bianco and M. Caramia, *Flexible Services and Manufacturing* **25** (2013) 6–24.] with the main time indexed linear programming based mathematical models existing in the literature. This paper was inspired by the results of the experimental comparison among these models conducted in our previous work; in fact, such results showed that the formulation proposed by Bianco and Caramia bested the competing approaches. Here, by means of a theoretical analysis, we show the reason for this behaviour.

Mathematics Subject Classification. 90C11, 90B35, 90C90.

Received April 24, 2015. Accepted May 9, 2016.

## 1. INTRODUCTION

A project consists of a set of activities that have to be carried out according to a set of precedence constraints. Activities have a fixed duration and require one or more different renewable resource types for their execution. In order to represent activities and their relationships, it is a common practice to use a project network with an *Activity-On-Node* (AON) representation, *i.e.*, a graph G = (A, E), where A denotes the set of activities and E represents the set of precedence constraints. The latter are finish-to-start relations with zero time lags, *i.e.*, an arc (i, j) in E implies that activity j cannot start before the finishing time of activity i. The objective is to properly schedule all the activities minimizing the project completion time.

If resources are unlimited, such an objective may be attained by computing the length of the *critical path*, *i.e.*, the longest path from the initial activity (source node) to the final activity (sink node) in the activity network (see, *e.g.*, [7–9]). The computation of such a path can be accomplished by means of the well-known forward pass recursion algorithm (see, *e.g.*, [4]), that is a classical label setting algorithm for longest path calculation. The computational complexity of this algorithm is O(|E|).

In this paper, we will focus on project scheduling with resources available in limited amounts, known as resource constrained project scheduling problem (RCPSP). In the basic version of this problem we have to schedule activities, without preemption, with the objective of minimizing the project completion time obeying

Keywords. Project scheduling, resource constraints, precedence constraints, mathematical model.

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precedence constraints and resource availability constraints, *i.e.*, for each resource type, the sum of the resource requirements of a set of activities scheduled in parallel must not exceed the (limited) resource availability. Unlike the case with unlimited resources, the RCPSP is NP-hard in the strong sense (see, *e.g.*, [3]).

In the following, we compare, at a theoretical level, a recent RCPSP formulation proposed in [2] with the main linear programming based approaches in the literature, with emphasis on time indexed models. The motivation of this paper stems from the experimental comparison among these models conducted in our previous work which showed that, under the same implementation environment, the formulation proposed by Bianco and Caramia had always a better performance than that of the competing approaches. Here, by means of a theoretical analysis, we show the reason for this behaviour.

The remainder of the paper is organized as follows. In Section 2, we present the state of the art, and, in Section 3, we show our theoretical results.

# 2. RCPSP mathematical models state of the art

RCPSP has been widely studied in the literature, thanks to its large number of applications, especially in the field of production management. These applications have stimulated many mathematical formulations for the RCPSP. In the following, we describe the most exploited ones in the literature and in practice, with emphasis on time indexed linear programming based ones. All the presented formulations minimize the completion time, under the hypothesis of non preemption of activities, and use an AON network representation where the source and the sink nodes are dummies.

It is assumed that there is a planning horizon within which all the activities have to be carried out. In particular, we will denote such a planning horizon as [0, T), where time instant T is an upper bound on the minimum project completion time. Moreover, we will assume, without loss of generality, that the time horizon is discretized into T unit-width time periods  $[0, 1), [1, 2), \ldots, [T - 1, T)$ , indexed by  $t = 1, \ldots, T$ . We note that, standing this definition, time period t starts at time instant t - 1. Let us define the following parameters: K, the number of renewable resources, each one available in an amount of  $b_k$  units, with  $k = 1, \ldots, K$ ;  $r_{ik}$ , the amount of units of resource k per time period necessary to carry out activity i;  $d_i$ , the duration of activity i;  $q_{ik}$ , the overall amount of units of resource k necessary to carry out activity i, *i.e.*,  $q_{ik} = r_{ik} \cdot d_i$ ; A, the set of activities (|A| = n); E, the collection of pairs of activities related by precedence constraints. The source and the sink dummy nodes are denoted with 1 and n, respectively.

In order to keep the number of variables and constraints as small as possible in the RCPSP formulations the number of periods in which an activity may potentially be processed is bounded from above. A straightforward upper bound on the minimum completion time is offered by the summation of all the activity durations. Based on this upper bound, earliest and latest start and finish times can be obtained by the well known forward and backward pass (see, e.g., [4]). Let  $ES_i$ ,  $LS_i$ ,  $EF_i$ , and  $LF_i$  be the earliest and latest start time of activity *i*, and the earliest and latest finish time of activity *i*, respectively. The interval  $[ES_i, LF_i]$  defines a time window in which activity *i* has to be processed, *i.e.*, if the minimum project completion time is bounded from above by time instant *T*, this has not to be modeled explicitly by a constraint, but can be incorporated by calculating the time windows accordingly.

One of the first mathematical formulations for the RCPSP, has been proposed in [8]. In this formulation, the authors introduced a binary variable  $\xi_{it}$  which equals 1 whenever activity *i* finishes at time period  $t \in [EF_i, LF_i]$ , and 0 otherwise (this type of variables is known as *impulse variables* type). With this variable the finish time of an activity *i* can be calculated as

$$\sum_{t=EF_i}^{LF_i} t \cdot \xi_{it}.$$

On the basis of the above definition, the mathematical formulation proposed by Pritsker et al. is the following

$$\min\sum_{t=EF_n}^{LF_n} t \cdot \xi_{nt} \tag{2.1}$$

$$\sum_{t=EF_i}^{LF_i} \xi_{it} = 1 \qquad \forall i \in A$$
(2.2)

$$\sum_{t=EF_i}^{LF_i} t \cdot \xi_{it} \le \sum_{t=EF_j}^{LF_j} t \cdot \xi_{jt} - d_j \qquad \forall (i,j) \in E$$

$$(2.3)$$

$$\sum_{i=1}^{n} \sum_{t'=t}^{t+d_i-1} r_{ik} \cdot \xi_{it'} \le b_k \ k = 1, \dots, K, t = 1, \dots, T$$
(2.4)

$$\xi_{it} \in \{0,1\} \quad \forall i \in A, t \in [EF_i, LF_i].$$

$$(2.5)$$

The objective function (2.1) minimizes the completion time of the dummy end activity n, and, therefore, the completion time of the project. The constraints have the following meaning: (2.2) specify that only one completion time is assigned to each activity, which must lie in the interval  $[EF_i, LF_i]$ ; (2.3) assure that the completion time of any activity i cannot exceed the start time of its successor j; (2.4) impose that, for each time period t and each resource k, the amount of resource requested by all the activities in progress during time period t must not exceed the resource availability; (2.5) specify that the decision variables  $\xi_{it}$  are binary.

The formulation by [5] has a binary variable  $s_{it}$  for each activity *i* and each time period *t* over the planning horizon. The value of  $s_{it}$  equals 1 whenever activity *i* is in progress at or has been processed before time period *t* (this type of variables is known as *step variables* type). With this definition the vector of decision variables for a certain activity consists of a series of 0's which is followed by a sequence of 1's. The switch from 0 to 1 occurs in the time period in which the activity is started. As a consequence, the starting time of an activity *i* can be written as

$$LF_i - \sum_{t=ES_i+1}^{LF_i} s_{it}.$$

Using this definition, the model by Klein is the following:

$$\min LF_n - \sum_{t=ES_n+1}^{LF_n} s_{nt} \tag{2.6}$$

$$s_{it} = 0 \ \forall i \in A, t = ES_i + 1 - d_i, \dots, ES_i$$
 (2.7)

$$s_{it} \ge s_{i,t-1} \quad \forall i \in A, t = ES_i + 1, \dots, LS_i \tag{2.8}$$

$$s_{it} = 1 \qquad \forall i \in A, t = LS_i + 1, \dots, T \tag{2.9}$$

$$s_{jt} \le s_{i,t-d_i} \ \forall (i,j) \in E, t = ES_j + 1, \dots, LF_i$$

$$(2.10)$$

$$\sum_{i=1}^{n} r_{ik} \cdot (s_{it} - s_{i,t-d_i}) \le b_k \qquad k = 1, \dots, K, t = 1, \dots, T$$
(2.11)

$$s_{it} \in \{0, 1\}$$
  $\forall i \in A, t = 0, \dots, T.$  (2.12)

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The objective function (2.6) minimizes the start time of the dummy end activity n and thus the completion time of the project. The constraints have the following meaning: (2.7), (2.8), and (2.9) assure that the vector of decision variables for each activity consists of a series of 0's, followed by a sequence of 1's; (2.10) specify the precedence relations between each pair of activities  $(i, j) \in E$ ; (2.11) satisfy the resource constraint related to activities in progress during time period t; (2.12) indicate that the decision variables  $s_{it}$  are binary.

Alvarez-Valdez and Tamarit [1] proposed a mixed-integer formulation of the RCPSP. This formulation is based on the definition of a set IS of all minimal resource incompatible sets S. S is a set of activities among which no precedence relation exists and for which at least one resource constraint is violated if scheduled in parallel. S is minimal, *i.e.*, it is impossible to remove an activity and still keep a resource incompatible set. In order to avoid a resource conflict which would be caused by concurrently processing the activities of such a set S, it is sufficient to introduce a direct or a transitive precedence relationship between a pair of activities  $\{i, j\} \in S$ . Hence, either activity i has to be completely processed before activity j can be started, or vice versa, and the corresponding resource conflict can no longer occur. This leads to the definition of 0-1  $x_{ij}$  variables that equal 1 whenever i precedes j, and is 0 otherwise. Furthermore, the formulation includes variables  $f_i$  denoting the time when i is finished. The completion time is minimized by minimizing the finish time of the dummy end activity n. The model is the following

$$\min f_n \tag{2.13}$$

$$x_{ij} = 1, x_{ji} = 0 \qquad \forall (i,j) \in E \tag{2.14}$$

 $x_{ij} + x_{ji} \le 1 \qquad \forall i, j \in A, i \ne j \tag{2.15}$ 

$$x_{ij} + x_{jl} - x_{il} \le 1 \ \forall i, j, l \in A, i \ne j, j \ne l, i \ne l$$
(2.16)

$$\sum_{i,j\in S, i\neq j} x_{ij} \ge 1 \qquad \forall S \in IS$$
(2.17)

$$f_i \le f_j - x_{ij} \cdot (d_j + M) + M \qquad \forall i, j \in A, i \ne j$$

$$(2.18)$$

$$f_i \ge 0 \qquad \forall i \in A, f_1 = 0 \tag{2.19}$$

$$x_{ij} \in \{0,1\} \qquad \forall i, j \in A, i \neq j.$$

$$(2.20)$$

The objective function (2.13) minimizes the completion time of the dummy end activity and thus the completion time of the project. The constraints have the following meaning: (2.14) introduce the precedence relation between *i* and *j* when it exists; (2.15) say that at most one precedence relation may exist between any pair of activities; (2.16) guarantee transitivity among precedence relations and avoid cycles in the network; (2.17) satisfy the resource constraints. At least one precedence relation should be specified among the activities of every set S; (2.18) satisfy the finish-to-start constraint for each precedence relation (original or introduced) between two activities *i* and *j*, *M* being a suitably large number; (2.19) impose the non negativity of variables  $f_i$  and that the dummy start activity 1 is processed at time 0; (2.20) specify that the decision variables  $x_{ij}$ should be binary.

Next, we describe the formulation introduced in [6]. This formulation is based on the notion of feasible set, *i.e.*, a subset of activities among which no precedence relation exists and that, if scheduled in parallel, do not violate the resource constraints. Based on these feasible sets, each one denoted by  $R_{\ell}$ , two sets of decision variables are defined as follows. There are 0-1 variables  $y_{\ell t}$  that equal 1 whenever set  $R_{\ell}$  is processed at time period t, and 0 otherwise. Furthermore, 0-1 variables  $\zeta_{it}$ , that equal 1 whenever activity *i* starts at time period t, and 0 otherwise, are also introduced. The objective function is to find the smallest start time for the dummy end activity n. Let  $\mathcal{R} = \{1, 2, ..., r\}$  be the index set of all the feasible subsets of A, and let  $\mathcal{R}_i \subseteq \mathcal{R}$  be the index set of all feasible subsets containing activity *i*. The mathematical formulation of the RCPSP is as follows.

$$\min\sum_{t=ES_n+1}^{LS_n+1} t\dots \zeta_{nt}$$
(2.21)

$$\sum_{\ell \in \mathcal{R}_i} \left( \sum_{t=ES_i+1}^{LF_i} y_{\ell t} \right) = d_i \qquad \forall i \in A \qquad (2.22)$$
$$\sum_{\ell \in \mathcal{R}} y_{\ell t} \le 1 \qquad t = 1, \dots, T \qquad (2.23)$$

$$\sum_{\mathcal{R}} y_{\ell t} \le 1 \qquad t = 1, \dots, T \qquad (2.23)$$

$$\zeta_{it} \ge \sum_{\ell \in \mathcal{R}_i} y_{\ell t} - \sum_{\ell \in \mathcal{R}_i} y_{\ell, t-1} \ \forall i \in A, t = ES_i + 1, \dots, LS_i + 1$$

$$(2.24)$$

$$\sum_{i=1}^{N_i+1} \zeta_{it} = 1 \qquad \forall i \in A \tag{2.25}$$

$$\sum_{t=ES,\pm1}^{LS_j+1} t \cdot \zeta_{jt} - \sum_{t=ES,\pm1}^{LS_i+1} t \cdot \zeta_{it} \ge d_i \qquad \forall (i,j) \in E$$

$$(2.26)$$

$$y_{lt} \in \{0, 1\} \qquad \forall l \in \mathcal{R}, t = 1, \dots, T$$

$$(2.27)$$

$$\forall i \in \{0, 1\}$$
  $\forall i \in A, t = 1, \dots, T.$  (2.28)

The objective function (2.21) minimizes the starting time of the dummy end activity n and then the completion time of the project. The constraints have the following meaning: (2.22) guarantee that the corresponding activity is in progress for exactly its duration; (2.23) specify that, at each time period t, at most one feasible subset is in progress; (2.24) force the variables  $\zeta_{it}$  to be 1, if the activity i is contained in the feasible set in execution at time period t but not contained in the feasible set in execution at t-1; (2.25) specify that only one start time is allowed for every activity; (2.26) satisfy the precedence constraint between  $(i, j) \in E$ ; (2.27) and (2.28) specify that  $y_{lt}$  and  $\zeta_{it}$  are binary variables.

Finally, we have the RCPSP formulation in Bianco and Caramia (2013) which uses the following decision variables:  $x_{it}$ , the percentage of i executed till the end of time period t;  $s_{it}$ , a binary variable that assumes value 1 if activity i has started by the beginning of a time period  $\tau \leq t$ , and 0 otherwise;  $f_{it}$ , a binary variable that assumes value 1 if activity i has finished by the end of a time period  $\tau \leq t$ , and 0 otherwise.

Since the completion time of an activity  $i \in A$  can be expressed as  $f_i = \left(T - \sum_{t=1}^T f_{it} + 1\right)$ , denoting with n the dummy end activity of the AON network, the objective function and the constraints can be modelled as follows:

$$\min\left(T - \sum_{t=1}^{T} f_{nt} + 1\right) \tag{2.29}$$

$$s_{jt} \le f_{i,t-1} \ \forall (i,j) \in E, t = EF_i, \dots, LS_j + 1$$

$$(2.30)$$

$$x_{it} - x_{i,t-1} = \frac{1}{d_i} (s_{it} - f_{i,t-1}) \quad \forall i \in A, t = ES_i + 1, \dots, LF_i$$
(2.31)

$$s_{it} \ge s_{i,t-1} \ \forall i \in A, t = ES_i + 1, \dots, LS_i + 1$$
 (2.32)

$$f_{it} \ge f_{i,t-1} \qquad \forall i \in A, t = EF_i, \dots, LF_i \tag{2.33}$$

$$s_{it} = 1 \qquad \forall i \in A, t = LS_i + 1, \dots, T \tag{2.34}$$

$$f_{it} = 1 \qquad \forall i \in A, t = LF_i, \dots, T \tag{2.35}$$

$$s_{it} = 0 \qquad \forall i \in A, t = 0, \dots, ES_i \tag{2.36}$$

$$f_{it} = 0 \quad \forall i \in A, t = 0, \dots, EF_i - 1$$
 (2.37)

$$f_{it} \le x_{it} \le s_{it} \ \forall i \in A, t = ES_i + 1, \dots, LF_i \tag{2.38}$$

$$\sum_{i=1}^{|A|} q_{ik} \cdot (x_{it} - x_{i,t-1}) \le b_k \quad k = 1, \dots, K, t = 1, \dots, T$$
(2.39)

$$s_{it}, f_{it} \in \{0, 1\}$$
  $\forall i \in A, t = 1, \dots, T$  (2.40)

$$x_{it} \ge 0 \qquad \forall i \in A, t = 1, \dots, T.$$

$$(2.41)$$

The objective function (2.29) minimizes the completion time of the dummy end activity n and thus the completion time of the project. The constraints have the following meaning: (2.30) model finish-to-start precedence constraints between i and j,  $\forall (i, j) \in E$ . In fact, if i is not completed till time slot t - 1, *i.e.*, the right hand side is zero, then activity j cannot start. After the finishing time of activity i, say t-1, activity j can start. Constraints (2.31) regulate the total amount processed of an activity  $i \in A$  over time. We note that either this value is 0 or it is equal to  $\frac{1}{d_i}$ . In particular, as long as activity *i* is not started, both sides of the equation are zero, but if activity i is started in period t then the left hand side will be equal to  $\frac{1}{d_i}$ , which is exactly the value of the right hand side of the equation. This constraint imposes also that preemption is not allowed. Indeed if activity i starts at time t then until its finishing time it must be processed without interruption. Constraints (2.32) imply that if an activity  $i \in A$  is started at time t, then variable  $s_{i\tau} = 1$  for every  $\tau \geq t$ , and, on the contrary, if activity i is not started at time t,  $s_{i\tau} = 0$  for every  $\tau \leq t$ . Constraints (2.33) are the same as constraints (2.32) when finishing times are concerned. Constraints (2.34) and (2.35) say that every activity  $i \in A$  must start and finish within the planning horizon, respectively. Constraints (2.36) and (2.37) say that activity i cannot start and finish before its earliest start time and finish time, respectively. Constraints (2.38) force  $x_{it}$  to be zero if  $s_{it} = 0$ , and  $f_{it}$  to be zero if  $x_{it} < 1$ . Resource constraints are represented by relations (2.39). Constraints (2.40) and (2.41) limit the range of variability of the variables.

# 3. Comparison of the RCPSP formulation in Bianco and Caramia [2] with the main time indexed linear programming based models

In [2] we experimentally compared all the formulations presented in the previous section. This has been done by means of the commercial solver CPLEX on PSLIB instances (http://webserver.wi.tum.de/psplib/). By an extensive experimentation it came out that the mathematical formulation by Bianco and Caramia (denoted in the following with BC) produced always better results with respect to the competing formulations. Pursuing these experimental findings, we now propose a theoretical framework to show that the BC formulation is stronger than the state of the art ones. In Sections 3.1, 3.2, 3.3 and 3.4 we compare BC with the models of Pritsker *et al.* [8] (denoted as PR), Klein [5] (denoted as KL), Alvarez-Valdes and Tamarit [1] (denoted as AT), and Mingozzi *et al.* [6], respectively.

The general technique used to show our claim is polyhedra inclusion. Assume to compare two formulations, say  $F_1$  and  $F_2$ , belonging to different solution spaces. We provide an affine transformation T, with which  $F_1$  is mapped in the same solution space of  $F_2$ , obtaining a new formulation  $F'_1$ , where  $F'_1$  is the mapping of the points in  $F_1$  by the affine transformation T. Next, we prove that  $F'_1 \subseteq F_2$ , *i.e.*, we show that  $F'_1$  is not weaker than  $F_2$ . Finally, to prove that  $F'_1$  is strictly stronger than  $F_2$ , we show, by means of gadget instances, that there exists a point p in the linear relaxation of  $F_2$  such that  $T^{-1}(p) \notin F'_1$ .

# 3.1. Comparison with the formulation by Pritsker et al. [8]

Let us examine the relation between PR and BC. To this end let us consider constraints (2.33), (2.35) and (2.37). By the latter it follows that  $\exists \tau \in \{1, 2, ..., T\}$ :  $f_{i\tau} = 1$ , and, consequently,  $f_{it} = 0, t < \tau$  and  $f_{it} = 1, t \geq \tau$ . Therefore,

$$(f_{it} - f_{i,t-1}) = 1$$
, if  $t = \tau$ 

and

$$(f_{it} - f_{i,t-1}) = 0$$
, otherwise.

That is,  $(f_{it} - f_{i,t-1})$  is a binary quantity which assumes value 1 if activity *i* finishes at time *t* and value 0 otherwise. If we limit  $t \in [EF_i, LF_i]$ , this quantity is exactly the  $\xi_{it}$  variable defined in PR. Therefore, in the following, we will use the affine transformation  $\xi_{it} = f_{it} - f_{i,t-1}$  (for all suitable *i* and *t*), and show that the constraints in PR can be obtained as a combination of the constraints defining BC.

By constraints (2.35) and (2.37), if we sum the quantity  $(f_{it} - f_{i,t-1})$  over  $t = EF_i, \ldots, LF_i$  we have

$$\sum_{t=EF_i}^{LF_i} (f_{it} - f_{i,t-1}) = 1, \quad \forall i \in A$$

that are exactly constraints (2.2), meaning that the latter are surrogate of constraints (2.33), (2.35) and (2.37).

Let us see now how precedence constraints (2.3) can be obtained by the constraints in BC.

For a generic activity j, summing up constraints (2.31) in BC from t = 1 to T we have

$$d_j = \sum_{t=1}^{T} (s_{jt} - f_{j,t-1})$$

Given  $(i, j) \in E$ , by constraints (2.30), we have

$$d_j = \sum_{t=1}^{T} (s_{jt} - f_{j,t-1}) \le \sum_{t=1}^{T} (f_{i,t-1} - f_{j,t-1}).$$
(3.1)

Trivially,

$$\sum_{t=1}^{T} t \cdot f_{it} = \sum_{t=1}^{T} (t-1) \cdot f_{i,t-1} + T \cdot f_{iT}$$

and

$$\sum_{t=1}^{T} t \cdot f_{jt} = \sum_{t=1}^{T} (t-1) \cdot f_{j,t-1} + T \cdot f_{jT}.$$

Since  $f_{iT}$  and  $f_{jT}$  are equal to 1 by constraints (2.35) we have

$$\sum_{t=1}^{T} t \cdot f_{it} = \sum_{t=1}^{T} (t-1) \cdot f_{i,t-1} + T$$

and

$$\sum_{t=1}^{T} t \cdot f_{jt} = \sum_{t=1}^{T} (t-1) \cdot f_{j,t-1} + T.$$

Subtracting the latter two equalities and rearranging we have

$$\sum_{t=1}^{T} t \cdot (f_{it} - f_{i,t-1}) + \sum_{t=1}^{T} (f_{i,t-1} - f_{j,t-1}) = \sum_{t=1}^{T} t \cdot (f_{jt} - f_{j,t-1})$$

By using (2.42) in the latter equality, we have

$$\sum_{t=1}^{T} t \cdot (f_{jt} - f_{j,t-1}) - d_j \ge \sum_{t=1}^{T} t \cdot (f_{it} - f_{i,t-1})$$

which by constraints (2.35) and (2.37) can be rewritten as

$$\sum_{t=EF_j}^{LF_j} t \cdot (f_{jt} - f_{j,t-1}) - d_j \ge \sum_{t=EF_i}^{LF_i} t \cdot (f_{it} - f_{i,t-1}), \ \forall (i,j) \in E.$$

Since, as it has been previously shown,  $(f_{jt} - f_{j,t-1})$  and  $(f_{it} - f_{i,t-1})$  are variables  $\xi_{jt}$  and  $\xi_{it}$ , respectively, defined by Pritsker et al. [8], it follows that this relation can be rewritten as:

$$\sum_{t=EF_i}^{LF_i} t \cdot \xi_{it} \le \sum_{t=EF_j}^{LF_j} t \cdot \xi_{jt} - d_j, \ \forall (i,j) \in E$$

that are constraints (2.3) of PR. Therefore, also constraints (2.3) can be obtained by surrogating the constraints in BC.

Let us examine now resource constraints (2.39), *i.e.*,

$$\sum_{i=1}^{|A|} q_{ik} \cdot (x_{it} - x_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T.$$

By constraints (2.31) and  $q_{ik} = r_{ik} \cdot d_i$ , we have

$$\sum_{i=1}^{|A|} r_{ik} \cdot (s_{it} - f_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

where, for each t and each k, the non zero terms at the left-hand side are only those associated to the activities in progress during time period t since  $(s_{it} - f_{i,t-1}) = 1$  if  $s_{it} = 1$  and  $f_{i,t-1} = 0$ . By  $\sum_{t=1}^{T} (f_{it} - f_{i,t-1}) = 1$ , we have that, for every  $t = 1, \ldots, T$ ,

$$\sum_{t'=t}^{t-1+d_i} (f_{it'} - f_{i,t'-1}) \le 1$$

Moreover, by constraints (2.38), *i.e.*,  $s_{it} \ge f_{it}$ , and  $(t - 1 + d_i) - t < d_i$ , we can state that

$$\sum_{t'=t}^{t-1+d_i} (f_{it'} - f_{i,t'-1}) \le s_{it} - f_{i,t-1}.$$

Therefore, we can write

$$\sum_{i=1}^{|A|} r_{ik} \cdot \sum_{t'=t}^{t-1+d_i} (f_{it'} - f_{i,t'-1}) \le \sum_{i=1}^{|A|} r_{ik} \cdot (s_{it} - f_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

that is

$$\sum_{i=1}^{|A|} r_{ik} \cdot \sum_{t'=t}^{t-1+d_i} (f_{it'} - f_{i,t'-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

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which, again, by means of the affine transformation  $\xi_{it'} = (f_{it'} - f_{i,t'-1})$ , are exactly constraints (2.4) of Pritsker *et al.* Hence, the constraints in BC imply (2.4).

This allows us to conclude that BC is not weaker than PR.

To complete the proof, let us summarize the affine transformations from PR to BC, *i.e.*,  $f_{it} = \sum_{\tau=1,...,t} \xi_{i\tau}$ ,  $s_{it} = \sum_{\tau=1,...,t+d_i-1} \xi_{i\tau}$ , and  $x_{it} = \sum_{t'=1,...,t} ((\sum_{\tau=t',...,t'+d_i-1} \xi_{i\tau})/d_i)$ , and consider, for example, an instance with two activities, say 1 and 2, with unitary durations, such that 1 precedes 2, and T = 4. Consider, moreover, for the linear relaxation of PR the following (feasible) values of the variables associated with our gadget instance:  $\xi_{11} = 1/2$ ,  $\xi_{12} = 1/2$ ,  $\xi_{13} = 0$ ,  $\xi_{14} = 0$ ,  $\xi_{21} = 0$ ,  $\xi_{22} = 3/5$ ,  $\xi_{23} = 1/10$ , and  $\xi_{24} = 3/10$ . It is easy to see that  $s_{22} = 3/5$  and  $f_{11} = 1/2$ , which violates constraint (2.30) of the linear relaxation of BC that imposes instead  $s_{22} \leq f_{11}$ . Therefore, by the above analysis, it is possible to conclude that BC is strictly stronger than PR.

### 3.2. Comparison with the formulation by Klein [5]

Let us now compare BC with KL, following a similar approach as above. In KL,  $s_{it}$  variables correspond to the  $s_{it}$  variables of BC. Therefore, we can establish the correspondence between constraints (2.7) and (2.36), between constraints (2.8) and (2.32) and between constraints (2.9) and (2.34).

Now, take a  $\tau \in \{EF_i, \ldots, LF_i\}$  and consider constraints (2.31) in the BC formulation from  $t = \tau - 1$  down to  $t = \tau - d_i + 1$ , *i.e.*,

$$\begin{aligned} x_{i,\tau-1} - x_{i,\tau-2} &= \frac{1}{d_i} \left( s_{i,\tau-1} - f_{i,\tau-2} \right) \\ x_{i,\tau-2} - x_{i,\tau-3} &= \frac{1}{d_i} \left( s_{i,\tau-2} - f_{i,\tau-3} \right) \\ & \cdots \\ x_{i,\tau-d_i+2} - x_{i,\tau-d_i+1} &= \frac{1}{d_i} \left( s_{i,\tau-d_i+2} - f_{i,\tau-d_i+1} \right) \\ x_{i,\tau-d_i+1} - x_{i,\tau-d_i} &= \frac{1}{d_i} \left( s_{i,\tau-d_i+1} - f_{i,\tau-d_i} \right). \end{aligned}$$

Summing up these constraints, and simplifying terms at the left-hand side, we have

$$x_{i,\tau-1} - x_{i,\tau-d_i} = \frac{1}{d_i} \left( s_{i,\tau-1} - f_{i,\tau-2} + s_{i,\tau-2} - f_{i,\tau-3} \dots + s_{i,\tau-d_i+2} - f_{i,\tau-d_i+1} + s_{i,\tau-d_i+1} - f_{i,\tau-d_i} \right)$$

The right-hand side may be rearranged as follows

$$x_{i,\tau-1} - x_{i,\tau-d_i} = \frac{1}{d_i} \left( (s_{i,\tau-1} - f_{i,\tau-d_i}) + (s_{i,\tau-2} - f_{i,\tau-2}) + (s_{i,\tau-3} - f_{i,\tau-3}) + \dots + (s_{i,\tau-d_i+1} - f_{i,\tau-d_i+1}) \right).$$

The quantity at the right-hand side

$$(s_{i,\tau-1} - f_{i,\tau-d_i}) + (s_{i,\tau-2} - f_{i,\tau-2}) + (s_{i,\tau-3} - f_{i,\tau-3}) + \dots + (s_{i,\tau-d_i+1} - f_{i,\tau-d_i+1})$$

has exactly  $(d_i - 1)$  nonnegative terms, each one less than or equal to 1, and, therefore, we can write

$$x_{i,\tau-1} - x_{i,\tau-d_i} \le \frac{1}{d_i}(d_i - 1).$$

By constraints (2.38) we have that

$$x_{i,t-1} - x_{i,t-d_i} \ge f_{i,t-1} - s_{i,t-d_i}, \ t = ES_i + 1, \dots, LF_i.$$

By constraints (2.30) we have also that

$$f_{i,t-1} - s_{i,t-d_i} \ge s_{jt} - s_{i,t-d_i}, \ \forall (i,j) \in E, t = EF_i, \dots, LS_j + 1$$

Therefore, altogether, we have

$$s_{jt} - s_{i,t-d_i} \le f_{i,t-1} - s_{i,t-d_i} \le x_{i,t-1} - x_{i,t-d_i} \le \frac{1}{d_i}(d_i - 1)$$

that is

$$s_{jt} - s_{i,t-d_i} \le \frac{1}{d_i}(d_i - 1), \ \forall (i,j) \in E, t = ES_j + 1, \dots, LF_i.$$

Observing that  $\frac{1}{d_i}(d_i - 1) < 1$  and that the left-hand side of the above inequality contains only binary variables, we can state that

$$s_{jt} - s_{i,t-d_i} \leq 0, \ \forall (i,j) \in E, t = ES_j + 1, \dots, LF_i$$

that is

$$s_{jt} \leq s_{i,t-d_i}, \ \forall (i,j) \in E, t = ES_j + 1, \dots, LF_i$$

The latter constraints are the precedence constraints (2.10) by Klein, and, therefore, we can state that the latter are implied by constraints (2.30), (2.31) and (2.38) in BC.

Let us now examine the resource constraints. Since  $q_{ik} = r_{ik} \cdot d_i$ , constraints (2.39) can be written as:

$$\sum_{i=1}^{n} r_{ik} \cdot d_i \cdot (x_{it} - x_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

and, by constraints (2.31),

$$\sum_{i=1}^{n} r_{ik} \cdot (s_{it} - f_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T.$$

Take an activity i and a  $\tau \in \{EF_i, \ldots, LF_i\}$ ; by the previous calculations we have that,

$$x_{i,\tau-1} - x_{i,\tau-d_i} \le \frac{d_i - 1}{d_i} < 1.$$

By constraints (2.38) we have

$$f_{i,\tau-1} - s_{i,\tau-d_i} \le x_{i,\tau-1} - x_{i,\tau-d_i} \le \frac{d_i - 1}{d_i} < 1$$

which, by the binary nature of the left-most member variables, means that

$$f_{i,\tau-1} - s_{i,\tau-d_i} \le 0$$

The latter inequality leads to

$$\sum_{i=1}^{n} r_{ik} \cdot (s_{it} - s_{i,t-d_i}) \le \sum_{i=1}^{n} r_{ik} \cdot (s_{it} - f_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

which proves that constraints (2.31), (2.38) and (2.39) in BC imply constraints (2.11) in KL. Overall, we can state that BC is not weaker than KL.

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To complete the proof, similarly to what done in the comparison with PR, consider an instance with two activities, say 1 and 2, with unitary durations, such that 1 precedes 2, and T = 3. Consider, moreover, for the linear relaxation of KL the following (feasible) values of the variables associated with our gadget instance:  $s_{11} = 7/10$ ,  $s_{12} = 1$ ,  $s_{13} = 1$ ,  $s_{21} = 0$ ,  $s_{22} = 4/5$ , and  $s_{23} = 1$ . It is easy to see that with these values, by constraints (2.38) of the linear relaxation of BC (by which  $s_{11} \ge f_{11}$ ) we have  $f_{11} \le 7/10$ , and, by constraints (2.30) (by which  $s_{22} \le f_{11}$ ) we have, simultaneously,  $f_{11} \ge 4/5$ , which is not possible. Therefore, by the above analysis, it is possible to conclude that BC is strictly stronger than KL.

### 3.3. Comparison with the formulation by Alvarez-Valdes and Tamarit

The finish to start precedence constraints (2.18) specify that when  $x_{ij} = 1$ , that is activity *i* precedes activity *j*, then

$$f_i \leq f_j - d_j, \ i, j = 1, \dots, n, i \neq j.$$

If  $x_{ij} = 0$  constraints (2.18) do not work.

If we consider constraints (2.30) for a generic  $(i, j) \in E$  we have

$$s_{jt} \leq f_{i,t-1},$$

and then

$$\sum_{t=1}^{T} s_{jt} \le \sum_{t=1}^{T} f_{i,t-1}$$

and

$$-\sum_{t=1}^{T} s_{jt} \ge -\sum_{t=1}^{T} f_{i,t-1}.$$

Adding T to both members, we obtain

$$(T - \sum_{t=1}^{T} f_{i,t-1}) \le \left(T - \sum_{t=1}^{T} s_{jt}\right).$$

Since  $d_j = \sum_{t=1}^T (s_{jt} - f_{j,t-1})$  and  $\sum_{t=1}^T f_{i,t-1} = \sum_{t=1}^T f_{it} - 1$  (as well as  $\sum_{t=1}^T f_{j,t-1} = \sum_{t=1}^T f_{jt} - 1$ ), we may write

$$\left(T - \sum_{t=1}^{I} f_{it} + 1\right) \le \left(T - \sum_{t=1}^{I} f_{jt} + 1\right) - d_j$$

that is, on the basis of the definition of  $f_i$  and  $f_j$ ,

$$f_i \le f_j - d_j, \ (i,j) \in E.$$

Therefore, constraints (2.18) are implied by the constraints in BC.

As for the resource constraints, let us start by substituting constraints (2.31) into (2.39) obtaining

$$\sum_{i=1}^{|A|} r_{ik} \cdot (s_{it} - f_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T.$$

Let us consider now a pair of activities i, j that are incompatible with respect to resources. The resource constraint associated with i, j is

$$r_{ik} \cdot (s_{it} - f_{i,t-1}) + r_{jk} \cdot (s_{jt} - f_{j,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

where, by hypothesis,

$$r_{ik} + r_{jk} > b_k.$$

It follows that if  $(s_{it} - f_{i,t-1}) = 1$  then  $(s_{jt} - f_{j,t-1}) = 0$  and if  $(s_{jt} - f_{j,t-1}) = 1$  then  $(s_{it} - f_{i,t-1}) = 0$ . Then

$$0 \le (s_{it} - f_{i,t-1}) + (s_{jt} - f_{j,t-1}) \le 1, \ \forall t$$

that is

$$0 \le (s_{it} - f_{j,t-1}) + (s_{jt} - f_{i,t-1}) \le 1, \quad \forall t.$$

Let us define the following binary variable

$$x_{ij} = \min_{t} [1, 1 - (s_{jt} - f_{i,t-1})]$$

which is equal to 1 if  $(s_{jt} - f_{i,t-1}) \leq 0$ ,  $\forall t$  and is equal to 0 if  $\exists t : (s_{jt} - f_{i,t-1}) = 1$ .

Since  $(s_{jt} - f_{i,t-1}) \leq 0, \forall t$  implies that *i* precedes *j*, we can write that  $x_{ij} = 1$  if *i* precedes *j* and  $x_{ij} = 0$  otherwise.

Similarly it is possible to define

$$x_{ji} = \min_{t} [1, 1 - (s_{it} - f_{j,t-1})]$$

which is equal to 1 if  $(s_{it} - f_{j,t-1}) \leq 0$ ,  $\forall t$  and is equal to 0 if  $\exists t : (s_{it} - f_{j,t-1}) = 1$ , and since  $(s_{it} - f_{j,t-1}) \leq 0, \forall t$  implies that j precedes i, we can write that  $x_{ji} = 1$  if j precedes i and  $x_{ji} = 0$  otherwise.

Then, for each resource incompatible pair of activities i, j we have that

 $x_{ij} + x_{ji} = 1$ 

that is if  $x_{ij} = 1$  then  $x_{ji} = 0$  and viceversa.

Generalizing to a set  $S \in IS$  with |S| = c the resource constraint is

$$\sum_{i=1}^{c} r_{ik} \cdot (s_{it} - f_{i,t-1}) \le b_k, \ k = 1, \dots, K, t = 1, \dots, T$$

with  $\sum_{i=1}^{c} r_{ik} > b_k$ ,  $k = 1, \dots, K$ . Since the generic term

$$0 \le (s_{it} - f_{i,t-1}) \le 1$$

the resource constraint can be satisfied, by definition of S, if

$$0 \le \sum_{i=1}^{c} (s_{it} - f_{i,t-1}) \le (c-1).$$

This condition can be transformed, by summing and subtracting an equal quantity, as follows:

$$\sum_{i,j=1:i\neq j}^{c} (s_{it} - f_{j,t-1}) \le (c-1) + \sum_{i,j=1:i\neq j}^{c} (s_{it} - f_{j,t-1}) - \sum_{i=1}^{c} (s_{it} - f_{i,t-1})$$

where the term  $(s_{it} - f_{j,t-1}) \in \{-1, 0, 1\}$  and  $|\sum_{i,j=1:i \neq j}^{c} (s_{it} - f_{j,t-1})| = c(c-1).$ 

Let us observe now that the term  $\sum_{i=1}^{c} (s_{it} - f_{i,t-1})$  can be always written as the sum of *n* terms  $(s_{it} - f_{j,t-1})$  which are a subset of the terms of  $\sum_{i,j=1:i\neq j}^{c} (s_{it} - f_{j,t-1})$ .

If we call R the set of terms  $(s_{it} - f_{j,t-1})$  deriving from rewriting  $\sum_{i=1}^{c} (s_{it} - f_{i,t-1})$  with |R| = c, in the second member of the previous condition it is possible to write

$$\sum_{i,j=1:i\neq j}^{c} (s_{it} - f_{j,t-1}) - \sum_{i=1}^{c} (s_{it} - f_{i,t-1}) = \sum_{i,j=1:i\neq j,i,j\notin R}^{c} (s_{it} - f_{j,t-1})$$

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where

$$\sum_{i,j=1:i\neq j,i,j\notin R}^{c} (s_{it} - f_{j,t-1}) \le c \cdot (c-1) - |R| = c \cdot (c-1) - c$$

The overall condition becomes

$$\sum_{i,j=1:i\neq j}^{c} (s_{it} - f_{j,t-1}) \le (c-1) + c \cdot (c-1) - c = c \cdot (c-1) - 1.$$

Since  $c \cdot (c-1)$  is the number of all possible combinations of pairs i, j such the  $i \neq j$ , the previous condition implies that it exists at least one pair of activities, say h, l, for which the term

$$(s_{ht} - f_{l,t-1}) \le 0, \forall t.$$

Thus, for this pair h, l of resource incompatible activities the associated variable  $x_{hl}$  is equal to 1. This means that for each minimal resource incompatible set  $S \in IS$ , if we associate to every pair of activities  $i, j \in S$  a binary variable  $x_{ij}$ , as defined before, at least one of these variables must be 1. This condition, obtained starting from constraints (2.39) of BC can be formally expressed as

$$\sum_{i,j\in S} x_{ij} \ge 1, \forall S \in IS$$

where  $x_{ij} = 1$  if *i* precedes *j*, and  $x_{ij} = 0$  otherwise.

This condition is properly constraint (2.17) in AT. Thus, the resource constraints by Alvarez-Valdes and Tamarit are implied by the constraints in BC.

Constraints (2.14), (2.15) and (2.16) are implied by constraints (2.30), (2.33), and (2.38).

As for constraints (2.14), constraints (2.30) impose that activity *i* must finish before the start time of activity *j*, if *i* precedes *j* ( $x_{ij} = 1$ ).

As for the anticyclic constraints (2.15), constraints (2.38) avoid the relation *i* precedes *j* and *j* precedes *i*. In this case in fact we might have

$$s_{jt} \le f_{i,t-1} \le f_{it} \le s_{it} \le f_{j,t-1} \le f_{jt}$$

that is

 $s_{jt} \leq f_{jt}$ 

which is not possible due to constraints (2.38).

Constraints (2.30), (2.33) and (2.38) assure also that the transitivity property, imposed by constraint (2.16) in the formulation by Alvarez-Valdes and Tamarit, is verified. In fact,

$$s_{lt} \le f_{j,t-1} \le f_{jt} \le s_{jt} \le f_{i,t-1}$$

and then

 $s_{lt} \leq f_{i,t-1}$ 

that is if i precedes j and j precedes l also i precedes l. Therefore, all the constraints in AT are implied by the constraints of BC, which means that BC is not weaker than AT.

To complete the proof, consider an instance with two independent activities, say 1 and 2, which are minimal resource incompatible, *i.e.*, the set  $\{1,2\} \in S$ . Constraints (2.15) and (2.17) say that  $x_{12} + x_{21} = 1$ . Consider the linear relaxation of AT and the feasible solution  $x_{12} = x_{21} = 1/2$ . By the relation  $x_{ij} = \min_t [1, 1 - (s_{jt} - f_{i,t-1})]$  introduced before, we have  $x_{12} = \min_t [1, 1 - (s_{2t} - f_{1,t-1})]$ , *i.e.*,  $1/2 = \min_t [1, 1 - (s_{2t} - f_{1,t-1})]$ . This, in turn, means that there exists a time period  $\tau$  for which  $s_{2\tau} - f_{1,\tau-1} = 1/2$ , meaning that  $f_{1,\tau-1} = s_{2\tau} - 1/2 < s_{2\tau}$  which violates constraints (2.30) in BC. Therefore, BC is strictly stronger than AT.

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## 3.4. Comparison with the formulation by Mingozzi et al. [6]

In this formulation constraints (2.25) and (2.26) are the same as constraints (2.2) and (2.3) of the formulation of Pritsker *et al.* [8]. Therefore, as shown, they are surrogate of the constraints of BC. As for constraints (2.22), (2.23) and (2.24) we can observe that they are expressed in terms of the  $y_{\ell t}$  variable associated to a feasible set  $R_{\ell}$ . It follows that variable  $y_{\ell t}$  does not correspond to a single activity *i* and in particular to its processing over time as it happens with the decision variables of the other formulations. To establish a relation with BC, let us consider variables  $s_{it}$  and  $f_{it}$  associated to the activities of a generic feasible set  $R_{\ell}$ . If an activity  $i \in R_{\ell}$  is in progress at time t,  $(s_{it} - f_{i,t-1}) = 1$ , otherwise  $(s_{it} - f_{i,t-1}) = 0$ . Then, we can define the following quantity

$$\prod_{i \in R_{\ell}} (s_{it} - f_{i,t-1})$$

which is equal to 1 if all the activities  $i \in R_{\ell}$  are in progress at time t, and is zero otherwise. This variable is  $y_{\ell t}$ . Hence, if we want to express constraints (2.22), (2.23) and (2.24) in terms of  $s_{it}$  and  $f_{it}$  we obtain non linear constraints and the resulting formulation looses the characteristic of being a linear programming based approach.

## 4. Conclusions

In this work we presented a theoretical comparison of the RCPSP formulation proposed in [2] with the main time indexed linear programming based models in the state of the art. The results of the analysis showed that formulation BC is stronger than the competing formulations. This characteristic explains why the computational performance illustrated by the authors in the mentioned paper are always better than those of the other RCPSP considered formulations.

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