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CAPACITATED TWO-STAGE TIME MINIMIZATION TRANSPORTATION PROBLEM WITH RESTRICTED FLOW

Prabhjot Kaur[1], Vanita Verma[2] and Kalpana Dahiya[1]

Abstract. This paper discusses a capacitated time minimization transportation problem in which transportation operation takes place in two stages. In the first stage, due to some constraints, only a specified flow F_1 is transported from available sources to various destinations and in the second stage, a flow F_2 is transported depending upon the total demand of the destinations. The current problem is motivated by a production system of a steel industry where semi-finished jobs, initially located at various bins in its warehouse, are transported to various manufacturing facilities by transporters for the final processing and finishing. Due to some additional constraints, it is not possible to transport the number of semi-finished jobs equal to the exact number of final products to be manufactured, in one go. Therefore the transportation operation has to take place in two stages. Further, a capacity is associated with each bin-machine link which makes the current problem, a capacitated, two stage time minimization transportation problem with restricted flow. The objective is to minimize the sum of the transportation times for Stage-I and Stage-II. A polynomial time iterative algorithm is proposed that within each iteration, solves a restricted version of a related standard capacitated time minimization transportation problem and generates a pair of Stage-I and Stage-II times with Stage-II time strictly less than the Stage-II time of the previous iteration, whereas Stage-I time may increase. Out of these generated pairs, a pair with the minimum sum of transportation times of Stage-I and Stage-II is selected that gives the global optimal solution. Numerical illustration is included in the support of the theory.

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1. Introduction

In quantity analysis of many business problems, transportation problem has been one of the most important and successful applications which deals in the physical distribution of the products. Basically, the purpose is to minimize the cost of shipping goods from one location to another so that the needs of each arrival area are met and every shipping location operates within its capacity. The standard cost minimization transportation

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[1] University Institute Of Engineering and Technology, Panjab University, 160014 Chandigarh, India. kalpanas@pu.ac.in

[2] Department of Mathematics, Panjab University, 160014 Chandigarh, India.

problem (CMTP) is the following:

"A homogeneous product is to be transported from m sources to n destinations. The supplies available at the sources are $a_1, a_2, \ldots, a_m$ and the requirements of the destinations are $b_1, b_2, \ldots, b_n$. The sum of the requirements at the destinations is equal to the sum of amounts available at the sources. The per unit cost of transportation of the product from ith source to the jth destination is c_{ij}. The problem is to determine the amount $x_{ij},\ \forall\ (i,j) \in I \times J$ to be transported from ith source to the jth destination minimizing the total cost of transportation". Mathematically, the standard CMTP is:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

subject to

$$\sum_j x_{ij} = a_i,\ \forall\ i = 1, 2, \ldots, m.$$

$$\sum_i x_{ij} = b_j,\ \forall\ j = 1, 2, \ldots, n.$$

$$x_{ij} \geq 0, \quad \forall\ i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

This type of statement for CMTP was initially given by Hitchcock [15]. In fact in 1781, Monge stated a similar problem and derived some remarkable results using Euclidean geometry (see, Berge [8] for a description of Monge's work). Although the transportation problem can be solved by regular simplex method which was first developed by Danzing [10], its special structure offers a more convenient procedure to solve this type of problem. This procedure is based on the same theory as that of simplex method but it makes use of shadow prices which yields a simpler computational scheme. This problem is widely studied and has many applications. Appa [2] studied some useful variants of CMTP. The CMTP with mixed constraints was discussed by Brigden [6] and Klingman and Russel [21]. Khanna *et al.* [19, 20] presented a study on a flow constrained CMTP. In 1959, Harvey [14] studied the transportation problems having certain types of capacity constraints on the flows between origins and destinations called the capacitated CMTP (CCMTP). Dahiya and Verma [12] studied CCMTP with bounds on Rim conditions in 2007.

In most real world problems, the complexity of the social and economic environment requires the explicit consideration of the objective function other than cost. For situations in military operations, where in the times of emergencies like war, the time of transportation of various military troops to the battlefield is of prime importance. Also, in the transportation of perishable goods, such as fresh fruits and vegetables *etc.*, a delay in transportation may result in much larger loss than any cost advantage attained by transporting at lower cost. Such kind of situations give rise to the time minimizing transportation problems (TMTP). The TMTP differs from the CMTP in a way that cost of transportation changes with variation in the quantity shipped, but the time of transportation remain unchanged if the quantity of goods to be shipped is varied within the capacity. This problem was first discussed by Hammer [13] in 1969 assuming that the shipment of goods is done in parallel. Many authors like Schwarz [30], Garfinkel *et al.* [17], Bhatia [9], Arora *et al.* [4, 5] have studied this problem and proposed various algorithms to solve it. The TMTP with mixed constraints have been studied by Khanna *et al.* [20]. Satyaparkash [24] proposed a technique to find the optimal solution to this problem by finding its lexicographic optimal solution. In 1981, Khanna *et al.* [19] studied a TMTP with flow constraint in which only a particular amount of goods is assumed to be sent optimally to the destinations and rest is to be stored at some of the sources. The mathematical structure of TMTP, if $I = \{1, 2, 3 \ldots..m\}$ is the index set of m sources, $J = \{1, 2, 3 \ldots n\}$ is the index set of n destinations and $t_{ij}(x_{ij})$ is the time of shipping an amount x_{ij} from the source i to the destination j, $\forall\ \ (i,j) \in I \times J$ is defined as follows:

$$\min_{X \in P} \left[T(X) = \max_{I \times J} (t_{ij}(x_{ij})) \right]$$

where P is defined as

$$P = \left\{ X = \{x_{ij}\} \in R^{mn} \;\middle|\; \begin{array}{l} \sum_{j \in J} x_{ij} = a_i, \;\; \forall\; i \in I \\ \sum_{i \in I} x_{ij} = b_j, \;\; \forall\; j \in J \\ \quad x_{ij} \geq 0, \qquad \forall\; (i,j) \in I \times J \end{array} \right\}.$$

The quantity shipped from the source i to the destination j is denoted by x_{ij} and

$$t_{ij}(x_{ij}) = \begin{cases} t_{ij}(\geq 0) & \text{if } \; x_{ij} > 0 \\ 0 & \text{if } \; x_{ij} = 0. \end{cases}$$

In 1980, Bansal and Puri [7] proved that the objective function in a TMTP is a concave function and hence, it belongs to the class of concave minimization problem (CMP). As the minimizer of a CMP over a polytope is attainable at an extreme point of the polytope, it is desirable to investigate only its extreme points.

In some circumstances, due to storage constraints or budget constraints or some possible political reasons, arrival areas are unable to receive the quantity of the product more than a fixed number. If the exact demand is larger than this fixed number, then it can be fulfilled only if the transportation occurs twice (once for the certain demand and then for the left over demand). Such a transportation problem becomes two-stage transportation problem. The standard two-stage transportation problem is studied as a variant of the standard transportation problem by many researchers ([16,25,29]). Two-stage time minimization transportation problem (TSTP) is also studied thoroughly in literature. In 2008, Sharma *et al.* [28] studied TSTP with capacity constraints on each source-destination link. Sharma *et al.* [26] also discussed a variant of TSTP, viz., two stage interval TMTP. This problem was further extended to the one with capacity constraints by Kaur and Dahiya [18]. The TMTP is also studied as a two level TMTP [8] where Level-I and Level-II links were specified for each level of transportation and the transportation in Level-I occurs only on Level-I links and that in Level-II occurs only on Level-II links *i.e.*, all the links are not available in different levels of transportation.

The problem considered in this paper is to study a capacitated two-stage time minimization transportation problem with restricted flow (CTPF). In the first stage, due to some constraints, only a specified flow F_1 is transported from available sources to various destinations and in the second stage, a flow F_2 is transported depending upon the total demand of the destinations. Here, CTPF is shown to be equivalent to the capacitated time minimization transportation problem with bounds on Rim conditions (discussed by Dahiya and Verma [12] for CMTP from where the similar results can be obtained for TMTP), which is further related to a standard TMTP. A polynomial time iterative algorithm is proposed, solving a restricted version of a related standard capacitated time minimization transportation problem at each iteration and generating a pair of Stage-I and Stage-II times with Stage-II time strictly less than the Stage-II time of the previous iteration, whereas Stage-I time may increase. These specific restricted versions of the standard TMTP are formed by introducing some new restrictions and partial sum constraints required to decrease the time of Stage-II [28]. The pairs of times of Stage-I and Stage-II are generated from these restricted versions and out of these generated pairs, a pair with the minimum sum of Stage-I and Stage-II times is declared as the global optimal solution of the problem CTPF.

This paper is organized as follows: in next section, the instance of motivation for the current problem is discussed. In Section 2, mathematical formulation of CTPF is given. The relation of CTPF with standard TMTP is established in Section 3. In Section 4, various theoretical results are justified. Based upon these results, an algorithm to solve the current problem is developed in Section 5. Numerical illustration is given in Section 6 and Section 7 contains concluding remarks.

2. Motivation

The current problem is motivated by the applications of transportation problem in iron and steel industry which manufactures a homogeneous product. The semi-finished jobs, which take the form of this homogeneous product after final processing and finishing on the various manufacturing facilities or machines, are initially stored at various bins located at different places in the warehouse of the industry. The manufactured products are then supplied to the market. This whole system involves the transportation of semi-finished jobs from bins to the machines and then the transportation of finished jobs from the industry to the market. In order to meet the market demand at the earliest, it is required to manufacture the product in the minimum possible time. Therefore, the time of transportation of semi-finished jobs from bins to the machines is of prime importance and is to be minimized. As the jobs to be carried out are big and heavy in a steel industry and their transportation may be cumbersome and expensive, it is assumed that there is capacity associated to each bin-machine link, *i.e.*, at most a particular number of jobs can be transported on each of the bin-machine links. Of course, these constraints are due to various factors affecting the transportation process, fuel consumption and consequently the cost of transportation, but it does not mean that the maximum number of units should be assigned to the cheapest route. Parallel to the cost constraints, there is a constraint on the capacity of the vehicles that are available for the transportation. So while putting a capacity restriction on each link, both the factors are taken into account. Also, the maximum amount F_1 (say), of semi-finished jobs that can be processed and finished in one stage is fixed due to additional constraints like, limited allowed units of power for consumption, working time, manpower *etc.* (which result in the shipping of only a particular number of jobs in one go) and rest of the jobs have to be processed and finished (hence shipped) later in the second stage. In the current problem, the exact demand is assumed to be less than or equal to $2F_1$, which leads the current problem to be a two stage transportation problem. Our objective in the current problem is to minimize the sum of times of transportation of Stage-I and Stage-II. Also, we assume the exact demand $F_1 + F_2$ to be less than or equal to $\sum_i a_i$ which is further less than or equal to $2F_1$. Initially, all the machines are kept involved in the processing. Inspite of this, due to the additional constraints discussed above, it is possible to manufacture only F_1 units in one stage and F_2 units have to be manufactured in the second stage. As there is no storage allowed near the machines, only F_1 units can be transported in first stage and F_2 units need to be transported in second stage. Note that in second stage, all the machines need not be involved in processing, some of them may be kept vacant depending upon the time of transportation that is consumed from various bins to these machines. As all the machines are working in first stage, so in order to utilize them to their maximum, each one of them is assigned a minimum number of jobs to be essentially processed on them because if a machine is working for some time and consuming power, then keeping the budget constraint in mind, their should be a minimum number of jobs to be processed essentially on it. This number can be different for different machines because though the machines are identical but the newer machines can work faster or consume less power than the older ones.

3. Mathematical formulation

Let $I = \{1, 2, \ldots, m\}$ denotes the index set of m bins located in the warehouse at different places and $J = \{1, 2, \ldots, n\}$ be the index set of n manufacturing facilities (machines) available for final processing and finishing. Let $a_i,\ i \in I$ be the availability of semi-finished jobs at ith bin and $b_j,\ j \in J$ be the minimum number of jobs to be essentially processed on jth machine.

Here we assume that:

(i) $\sum_{i\in I} a_i \geq \sum_{j\in J} b_j$.
(ii) In the first stage, only $F_1 \geq (\sum_{j\in J} b_j)$ units can be shipped but as the total demand is assumed to be $F_1 + F_2,\ F_2$ units should be shipped in the second stage.

Mathematically, Stage-I problem can be stated as

$$(P_1) \qquad \min_{X\in S} \left[\max_{I\times J}(t_{ij}(x_{ij}))\right]$$

where

$$S = \left\{ X = \{x_{ij}\} \in Z^{mn} \;\middle|\; \begin{array}{l} \sum_{j \in J} x_{ij} \leq a_i \;\forall\, i \in I \\ \sum_{i \in I} x_{ij} \geq b_j \;\forall\, j \in J \\ \sum_{i \in I} \sum_{j \in J} x_{ij} = F_1,\; 0 \leq x_{ij} \leq u_{ij},\; \forall\, (i,j) \in I \times J \end{array} \right\}$$

where u_{ij} is the capacity of the (i,j)th bin-machine link.

Corresponding to a feasible solution X of the Stage-I problem, let $S(X)$ be the set of feasible solutions of Stage-II problem. Then Stage-II problem can be formulated as

$$(P_2) \qquad \min_{Y \in S(X)} \left[\max_{I \times J}(t_{ij}(y_{ij}))\right]$$

where

$$S(X) = \left\{ Y = \{y_{ij}\} \in Z^{mn} \;\middle|\; \begin{array}{l} \sum_{j \in J} y_{ij} \leq \bar{a}_i \;\forall\, i \in I \\ \sum_{i \in I} y_{ij} \geq 0 \;\forall\, j \in J \\ \sum_{i \in I} \sum_{j \in J} y_{ij} = F_2,\; 0 \leq y_{ij} \leq \bar{u}_{ij},\; \forall\, (i,j) \in I \times J \end{array} \right\}.$$

Note that $\bar{a}_i = a_i - \sum_{j \in J} x_{ij}$ and $\bar{u}_{ij} = u_{ij} - x_{ij} \;\forall\, (i,j) \in I \times J$.

Then the CTPF can be formulated as:

$$(P_3) \qquad \min_{X \in S} \left[\max_{I \times J}(t_{ij}(x_{ij})) + \min_{Y \in S(X)} \left[\max_{I \times J}(t_{ij}(y_{ij}))\right]\right].$$

The set of feasible solutions of the problem (P_3) is the union of the set of feasible solutions of Stage-I and the corresponding optimal feasible solutions (OFS) of Stage-II problem.

To solve this problem, we are solving a related transportation problem in which the transportation matrix is unimodular. By the property of unimodularity if the availability and demands are integers then all the basic feasible solutions (of course in terms of $x'_{ij}s$) will take integral values and in the proposed solution technique, we are scanning only basic feasible solutions of the related transportation problem. So, from now onwards we have relaxed the condition of x_{ij}'s to be integers specifically.

4. Theoretical development

To find a feasible solution of Stage-I problem, we rewrite the problem (P_1) as a capacitated TMTP with bounds on Rim conditions (see [5]). Thus the Stage-I problem (P_1) becomes

$$(P_1) \qquad \min_{X \in S} \left[\max_{I \times J}(t_{ij}(x_{ij}))\right]$$

where

$$S = \left\{ X = \{x_{ij}\} \in R^{mn} \;\middle|\; \begin{array}{l} 0 \leq \sum_{j \in J} x_{ij} \leq a_i \quad \forall\, i \in I \\ b_j \leq \sum_{i \in I} x_{ij} \leq \sum_{i \in I} u_{ij} \quad \forall\, j \in J \\ \sum_{i \in I} \sum_{j \in J} x_{ij} = F_1,\; 0 \leq x_{ij} \leq u_{ij},\; \forall\, (i,j) \in I \times J \end{array} \right\}.$$

Here we assume that $\sum_{j \in J} u_{ij} \geq a_i \;\forall\, i \in I$. This kind of problem was first discussed by Dahiya and Verma [12] for a capacitated CMTP and their technique can easily be extended (discussed below) to solve (P_1), a capacitated TMTP which can further be solved by using any of the methods available in literature (see [4,9,11,13,17,22,30]). Here 0 and a_i represent the minimum and maximum availability of the semi-finished jobs at the ith bin and b_j

and $\sum_{i\in I} u_{ij}$ denote the minimum and maximum number of jobs to be processed and finished on the j^{th} machine respectively. In order to solve (P_1) (the above rewritten Stage-I problem), an amount $\sum_{i\in I}\sum_{j\in J} u_{ij} - F_1$ of machine-slack has to be retained at the various machines and the amount $\sum_{i\in I} a_i - F_1$ of bin-reserve has to be kept at various bins. This suggests the introduction of an extra bin which fills up the machine-slacks and an extra machine which processes the bin-reserves. Also as the machine-slacks (bin-reserves) are to be retained at various legitimate machines (bins), it follows that flow should be prevented from new bin to new machine and this can be done by assigning a time M (a very large positive number) to this link. Thus, the problem (P_1) transforms to an equivalent balanced capacitated TMTP.

$$(P_1^*) \qquad \min_{Z\in S^*}\left[\max_{I^*\times J^*}(t^*_{ij}(z_{ij}))\right]$$

where

$$S^* = \left\{ Z=\{z_{ij}\}\in R^{(m+1)(n+1)} \;\middle|\; \begin{array}{ll} \sum_{j\in J^*} z_{ij} = a_i^* & \forall\, i\in I^* \\ \sum_{i\in I^*} z_{ij} = b_j^* & \forall\, j\in J^* \\ 0\le z_{ij}\le u^*_{ij} & \forall\,(i,j)\in I^*\times J^* \end{array} \right\}$$

where

$$\begin{array}{ll} I^* = I\bigcup\{m+1\} & J^* = J\bigcup\{n+1\} \\ t^*_{ij} = t_{ij}\ \forall\,(i,j)\in I\times J & t^*_{i,n+1} = 0\ \forall\, i\in I \\ t^*_{m+1,j} = 0\ \forall\, j\in J & t^*_{m+1,n+1} = M \\ a_i^* = a_i\ \forall\, i\in I & a^*_{m+1} = \sum_{i\in I}\sum_{j\in J} u_{ij} - F_1 \\ b_j^* = \sum_{i\in I} u_{ij}\ \forall\, j\in J & b^*_{n+1} = \sum_{i\in I} a_i - F_1 \\ u^*_{ij} = u_{ij}\ \forall\,(i,j)\in I\times J & u^*_{i,n+1} = a_i\ \forall\, i\in I \\ u^*_{m+1,j} = \sum_{i\in I} u_{ij} - b_j\ \forall\, j\in J & u^*_{m+1,n+1} = \infty \end{array}$$

Definition 4.1 (Corner feasible solution)**.**
A feasible solution $\{z_{ij}\}_{I^*\times J^*}$ of the problem (P_1^*) is called a corner feasible solution (CFS) if $z_{m+1,n+1}=0$.
A feasible solution of the problem (P_1^*) which is not a CFS is called a non-corner feasible solution.

4.1. To read the feasible solution of Stage-I from the CFS of (P_1^*)

Solving the problem (P_1) is equivalent to solving the problem (P_1^*). The equivalence establishes by proving following results.

Theorem 4.2. *A non-corner feasible solution to (P_1^*) cannot provide a feasible solution to (P_1).*

Theorem 4.3. *There is a one to one correspondence between feasible solutions of (P_1) and the CFS of (P_1^*).*

This can be proved by using the relation

$$x_{ij} = z_{ij}\ \forall\,(i,j)\in I\times J \qquad (A)$$

where $Z=\{z_{ij}\}_{I^*\times J^*}$ is a CFS of the problem (P_1^*) and $X=\{x_{ij}\}_{I\times J}$ is taken to be a feasible solution of (P_1).

Theorem 4.4. *The value of the objective function of (P_1) at a feasible solution is equal to the value of the objective function of (P_1^*) at its corresponding CFS and conversely.*

Theorem 4.5. *There is one to one correspondence between optimal feasible solutions of (P_1) and optimal among the corresponding corner feasible solutions of (P_1^*).*

Theorem 4.6. *Optimizing (P_1) is exactly equivalent to optimizing (P_1^*), provided (P_1) has a feasible solution.*

All these results can be proved for TMTP on the same lines as proved by Dahiya and Verma [12] for the CMTP. After establishing the equivalence, an OFS $X = \{x_{ij}\}_{I\times J}$ of Stage-I problem (P_1) is obtained from an optimal CFS, $Z = \{z_{ij}\}_{I^*\times J^*}$ of (P_1^*) by using the relation (A) given above.

Further, an OFS of Stage-II problem corresponding to the feasible solution X of Stage-I, is obtained from an optimal CFS of (P_1^*) by using the strategy discussed in the following subsection.

4.2. Strategy to find an optimal feasible solution of Stage-II corresponding to a feasible solution X of Stage-I

Let $Z = \{z_{ij}\}_{I^*\times J^*}$ be a CFS of the problem (P_1^*). Define:

(1) $\bar{I} = \{i \in I | z_{i,n+1} > 0\}$.
(2) For each $i \in \bar{I}$, $\bar{J}_i = \{j \in J | \bar{u}_{ij} \neq 0\}$ where $\bar{u}_{ij} = u^*_{ij} - z_{ij} = u_{ij} - z_{ij}$.
(3) $\bar{S} = \bigcup_{i\in\bar{I}}(\{i\} \times \bar{J}_i)$.

Let there be p cells named as $(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)$ in $\bar{S}$ (*i.e.*, $|\bar{S}| = p$) such that their corresponding times are arranged in ascending order, *i.e.*, $t_{i_1 j_1} \leq t_{i_2 j_2} \leq \ldots \leq t_{i_p j_p}$. If for some a and b, $t_{i_a j_a} = t_{i_b j_b}$, the cell with the minimum index of i and then minimum index of j occurs first in this sequence. For allocation in Stage-II, select the cells in this order and proceed as follows:

Let r be a number such that $r \in \{1, 2, \ldots, p\}$.

Step 0. Set $r = 1$ and go to Step 1.

Step 1. Find $y_{i_r j_r} = \min\{\hat{F}_2, \hat{z}_{i_r,n+1}, \bar{u}_{i_r j_r}\}$
for

$$\hat{F}_2 = \begin{cases} F_2 & \text{for } r = 1, \\ F_2 - \sum_{k=1}^{r-1} y_{i_k j_k} & \text{for } r > 1. \end{cases}$$

and

$$\hat{z}_{i_r,n+1} = \begin{cases} z_{i_r,n+1} & \text{for } \quad r = 1 \\ z_{i_r,n+1} - \sum_{j\in\bar{J}_{i_r}} y_{i_r j} & \text{for } r > 1 \ \text{ and } \ t_{i_r j} \leq t_{i_r j_r}. \end{cases}$$

Step 2. If $y_{i_r j_r} = \hat{F}_2$, for some r, then go to Step 3.
If $y_{i_r j_r} = \hat{z}_{i_r,n+1}$ for some r, then send this amount to the cell (i_r, j_r) and set $r = r + 1$ and go to Step 1.
If $y_{i_r j_r} = \bar{u}_{i_r j_r}$ for some r, then send this amount to the cell (i_r, j_r) and set $r = r + 1$ and go to Step 1.

Step 3. Note $t_{i_r j_r}$ as the optimal time for Stage-II and stop.
Continue doing these steps up to an index s such that $\sum_{r=1}^{s-1} y_{i_r j_r} < F_2$ and $\sum_{r=1}^{s} y_{i_r j_r} = F_2$, then $t_{i_s j_s}$ is noted as the optimal time for Stage-II corresponding to a feasible solution of Stage-I.

Clearly $y_{i_r j_r} = 0$, $\forall\ r > s$ and $y_{ij} = 0\ \forall\ (i,j) \notin \bar{S}$.

In the above strategy, a technique is given to find an optimal solution of Stage-II problem corresponding to a given solution X of Stage-I problem systematically transporting the left over/stored amount at various bins starting with a bin-machine link corresponding to the minimum time, then to the next minimum and so on till a total flow of F_2 units is transported.

Criterion to obtain an optimal feasible solution of (P_3)

If $Z = \{z_{ij}\}_{I^*\times J^*}$ is a CFS of the problem (P_1^*), then a feasible solution $X = \{x_{ij}\}_{I\times J}$ of Stage-I, obtained by taking $x_{ij} = z_{ij}\ \forall\ (i,j) \in I \times J$ and the corresponding optimal feasible solution $Y = \{y_{ij}\}_{I\times J}$ of Stage-II, obtained by using the strategy discussed in Section 4.2, yield a feasible solution, say (X, Y), of (P_3) generating a pair (T_1^0, T_2^0) of Stage-I and Stage-II times.

In order to find an OFS of the problem (P_3), an iterative algorithm is proposed which at each iteration, solves a restricted version of the problem (P_1^*) that concentrates on decreasing the time of Stage-II, strictly. So, during this iterative algorithm, a sequence of pairs of Stage-I and Stage-II times is obtained, out of all these generated pairs, one with the minimum sum of Stage-I and Stage-II times is considered as optimal.

Remark 4.7. The current algorithm solves a balanced capacitated TMTP (P_1^*) corresponding to Stage-I problem and a strategy discussed in Section 3.2 to read the solution of Stage-II problem. Also to solve this balanced capacitated TMTP, we transform it into a balanced capacitated CMTP using lexicographic technique and the best polynomial running time for this capacitated CMTP is $o((m+1)\log n + 1((m+1) + (n+1)\log(n+1)))$, where $m+1$ and $n+1$ are number of sources and destinations respectively (Orlin, 1997). While using the strategy for Stage-II solution, only a finite time is consumed and therefore the proposed algorithm is a polynomial time algorithm.

4.3. Restricted version of (P_1^*)

Let at the kth iteration of the algorithm, Stage-I time be T_1^k and the corresponding optimal time for Stage-II be T_2^k. The restricted version of (P_1^*) concentrates on decreasing the Stage-II time strictly, therefore, it is defined at T_2^k and is denoted by $P_1^*(T_2^k)$.

Following restrictions are imposed on (P_1^*):

(i) Put $t_{i,n+1} = M$, for $i \in I$ for which $\min_j(t_{ij}) \geq T_2^k$ where M is a large positive number.

(ii) Set $t_{ij} = M \; \forall \; (i,j) \in I \times J$ for which $t_{ij} + T_2^L \geq \min_{h=0,1,\dots,k}[T_1^h + T_2^h]$ where T_2^L is the lower bound on Stage-II time (discussed in Sect. 4.2).
Let $I^k = \{i \in I | \min_j t_{ij} < T_2^k\}$. Now for each $i \in I^k$, define $J_i^k = \{j \in J | t_{ij} < T_2^k\}$ and form a set $S^k = \bigcup_{i \in I^k} (\{i\} \times J_i^k)$.
This set S^k is the set of all bin-machine links which are eligible for transportation in the restricted version for Stage-II problem.

(iii) If for some $i \in I^k$, $\sum_{j \in J_i^k} u_{ij} < a_i$, then for this i, apply the partial sum constraint defined by

$$\sum_{j \in J_i^k} z_{ij} + z_{i,n+1} \leq \sum_{j \in J_i^k} u_{ij}.$$

Mathematically, the restricted version $P_1^*(T_2^k)$ takes the following form:

$$\min_{Z \in S^*} \left[\max_{I^* \times J^*} (t_{ij}^{**}(z_{ij})) \right]$$

where

$$S^* = \left\{ Z = \{z_{ij}\} \in R^{(m+1)(n+1)} \;\middle|\; \begin{array}{ll} \sum_{j \in J^*} z_{ij} = a_i^* & \forall \; i \in I^* \\ \sum_{i \in I^*} z_{ij} = b_j^* & \forall \; j \in J^* \\ 0 \leq z_{ij} \leq u_{ij}^* & \forall \; (i,j) \in I^* \times J^* \end{array} \right\}$$

and

$$\begin{aligned} & t_{m+1,j}^{**} = 0 \; \forall \; j \in J \quad t_{m+1,n+1}^{**} = M \\ & t_{i,n+1}^{**} = 0 \; \forall \; i \in I^k \quad t_{i,n+1}^{**} = M \; \forall \; i \notin I^k \\ & t_{ij}^{**} = t_{ij}^* \; \forall \; (i,j) \in L \end{aligned}$$

where

$$L = \{(i,j) \in I \times J | t_{ij} + T_2^L < \min_{h=0,1,\dots,k} [T_1^h + T_2^h]\} \quad t_{ij}^{**} = M \; \forall \; (i,j) \notin L.$$

The partial sum constraints in restriction (iii) are treated as side constraints while solving the problem $P_1^*(T_2^k)$ as they do not occur in S^*. These side constraints can be handled by the technique discussed by Dantzing and Thapa [11].

Definition 4.8 (M-feasible solution)**.**
A feasible solution $Z = \{z_{ij}\}_{I^* \times J^*}$ of the problem $P_1^*(T_2^k)$ is called an M-feasible solution (MFS) if $z_{ij} = 0 \; \forall \; (i,j) \in I^* \times J^*$ with $t_{ij}^{**} = M$.

Definition 4.9 (M-feasible solution w.r.t (P_1))**.**
A feasible solution $Z = \{z_{ij}\}_{I^* \times J^*}$ of the problem $P_1^*(T_2^k)$ is called an M-feasible solution with respect to (P_1) if:

(i) It is CFS of the problem $P_1^*(T_2^k)$.
(ii) $z_{ij} = 0 \; \forall \; (i,j) \in I \times J$ with $t_{ij}^{**} = M$.

From an M-feasible solution w.r.t (P_1) of the problem $P_1^*(T_2^k)$, we read the solution $X = \{x_{ij}\}$ of Stage-I problem by taking $x_{ij} = z_{ij} \; \forall \; (i,j) \in I \times J$. To also find the corresponding OFS of Stage-II problem giving the time strictly less than T_2^k, we define a Special MFS of the problem $P_1^*(T_2^k)$.

Definition 4.10 (Special M-feasible solution)**.**
A feasible solution $Z = \{z_{ij}\}_{I^* \times J^*}$ of the problem $P_1^*(T_2^k)$ is called Special M-feasible solution (SMFS) if:

(i) It is MFS w.r.t. P_1.
(ii) $\sum_{i \in I^k} z_{i,n+1} \geq F_2$.

Result. For a SMFS of the problem $P_1^*(T_2^k)$, $F_2 \leq \sum\limits_{(i,j) \in} \sum\limits_{S^k} \bar{u}_{ij}$.

Proof. Let the problem $P_1^*(T_2^k)$ has a SMFS, $Z = \{z_{ij}\}_{I^* \times J^*}$. Clearly, the restriction (iii) holds, *i.e.*,

$$\sum_{j \in J_i^k} z_{ij} + z_{i,n+1} \leq \sum_{j \in J_i^k} u_{ij} \; \forall \; i \in I^k.$$

Summing over $i \in I^k$

$$\sum_{i \in I^k} \sum_{j \in J_i^k} z_{ij} + \sum_{i \in I^k} z_{i,n+1} \leq \sum_{i \in I^k} \sum_{j \in J_i^k} u_{ij}. \tag{4.1}$$

Since the solution is SMFS of $P_1^*(T_2^k)$, this implies

$$F_2 \leq \sum_{i \in I^k} z_{i,n+1}.$$

Adding $\sum\limits_{i \in I^k} \sum\limits_{j \in J_i^k} z_{ij}$ on both sides, we get

$$\sum_{i \in I^k} \sum_{j \in J_i^k} z_{ij} + F_2 \leq \sum_{i \in I^k} \sum_{j \in J_i^k} z_{ij} + \sum_{i \in I^k} z_{i,n+1} \leq \sum_{i \in I^k} \sum_{j \in J_i^k} u_{ij}$$

(By (4.1))

$$\Rightarrow \qquad \sum_{i \in I^k} \sum_{j \in J_i^k} z_{ij} + F_2 \leq \sum_{i \in I^k} \sum_{j \in J_i^k} u_{ij}$$

$$\Rightarrow F_2 \leq \sum_{i \in I^k} \sum_{j \in J_i^k} (u_{ij} - z_{ij}) = \sum_{i \in I^k} \sum_{j \in J_i^k} \bar{u}_{ij} = \sum_{(i,j) \in} \sum_{S^k} \bar{u}_{ij}$$

$$\Rightarrow \qquad F_2 \leq \sum_{(i,j) \in} \sum_{S^k} \bar{u}_{ij}.$$

□

4.4. Some important remarks

Remark 4.11. If the solution of the problem $P_1^*(T_2^k)$ is not SMFS for some k, then the proposed algorithm is terminated.

Remark 4.12. Corresponding to a SMFS, say $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ of $P_1^*(T_2^k)$, a feasible solution X^{k+1} of Stage-I is obtained by taking $x_{ij} = z_{ij}\ \forall\ (i,j) \in I \times J$ and corresponding to X^{k+1}, an OFS Y^{k+1} of Stage-II problem is obtained by using the strategy discussed in Section 3.2 for $(i,j) \in \bar{S}^k$ instead of $\bar{S}$ where $\bar{S}^k = \bigcup_{i \in \bar{I}^k}(\{i\} \times \bar{J}_i^k)$ and $\bar{I}^k = \{i \in I^k | z_{i,n+1} > 0\}$, $\bar{J}_i^k = \{j \in J_i^k | \bar{u}_{ij} \neq 0\}$ for each $i \in \bar{I}^k$.

We denote $T(X^{k+1}) = T_1^{k+1}$ and $T(Y^{k+1}) = T_2^{k+1}$.

Remark 4.13. A SMFS of the problem $P_1^*(T_2^k)$, yields a pair of Stage-I and Stage-II times in which Stage-II time is strictly less than T_2^k. The existence of such an OFS of Stage-II problem is established in next section.

5. Theoretical justification of the proposed algorithm

Theorem 5.1. *At an optimal SMFS solution of the problem $P_1^*(T_2^k)$ yielding Stage-I time as T_1^{k+1}, an OFS of the corresponding Stage-II problem exists that yields a time T_2^{k+1} such that $T_2^{k+1} < T_2^k$.*

Proof. Let $Z^{k+1} = \{z_{ij}\}_{I^* \times J^*}$ be an optimal SMFS of $P_1^*(T_2^k)$. Let $X^{k+1} = \{x_{ij}\}_{I \times J}$ be the corresponding feasible solution of Stage-I problem and $Y^{k+1} = \{y_{ij}\}_{I \times J}$ be a solution of Stage-II corresponding to X^{k+1} obtained by using the strategy discussed in Section 4.2.

Claim 1 (Feasibility). Y^{k+1} is a feasible solution of Stage-II problem corresponding to the feasible solution X^{k+1} of Stage-I problem.

On contrary, suppose $Y^{k+1} = \{y_{ij}\}_{I \times J}$ is not a feasible solution of Stage-II problem.

By the construction of Y^{k+1}, $y_{ij} \geq 0\ \ \forall\ (i,j) \in \bar{S}^k$ and $y_{ij} = 0\ \ \forall\ (i,j) \in (I \times J) \backslash \bar{S}^k$ which implies that $y_{ij} \geq 0\ \forall\ (i,j) \in I \times J$.

Therefore $\sum_{i \in I} y_{ij} \geq 0\ \forall\ j \in J$.

Also for each $i \in \bar{I}^k$, $\sum_{j \in \bar{J}_i{}^k} y_{ij} \leq z_{i,n+1}$ and $y_{ij} = 0$ for $j \in J \backslash \bar{J}_i{}^k$.

$$\Rightarrow \qquad \sum_{j \in J} y_{ij} = \sum_{j \in \bar{J}_i{}^k} y_{ij} + \sum_{j \in J \backslash \bar{J}_i{}^k} y_{ij} \leq z_{i,n+1} + 0 = a_i - \sum_{j \in J} x_{ij} = \bar{a}_i$$

and for each $i \in I \backslash \bar{I}^k$, $\sum_{j \in J} y_{ij} = 0 \leq \bar{a}_i$.

Therefore $\sum_{j \in J} y_{ij} \leq \bar{a}_i\ \forall\ i \in I$.

So, if we assume that Y^{k+1} is not a feasible solution of Stage-II, then the only constraint that is violated is

$$\sum_{i \in I} \sum_{j \in J} y_{ij} = F_2.$$

Then by construction of Y^{k+1}, there is only one possibility that $\sum_{i \in I} \sum_{j \in J} y_{ij} < F_2$. This implies $\sum_{(i,j) \in \bar{S}^k} y_{ij} < F_2$, as $\bar{S}^k \subseteq I \times J$ and $y_{ij} = 0\ \forall\ (i,j) \notin \bar{S}^k$.

$$\Rightarrow \sum_{i \in \bar{I}^k} \sum_{j \in \bar{J}_i{}^k} y_{ij} < F_2. \tag{5.1}$$

In this situation, by construction of Y^{k+i},

$$\sum_{j\in \bar{J}_i^k} y_{ij} = z_{i,n+1} \ \forall\ i \in \bar{I}^k.$$

Summing over $i \in \bar{I}^k$

$$\sum_{i\in \bar{I}^k} \sum_{j\in \bar{J}_i^k} y_{ij} = \sum_{i\in \bar{I}^k} z_{i,n+1}. \tag{5.2}$$

From (5.2) and (5.3) we get, $\sum_{i\in \bar{I}^k} z_{i,n+1} < F_2$.

Also $z_{i,n+1} = 0 \ \forall\ i \in I^k \backslash \bar{I}^k$ (by definition of $\bar{I}^k$), which implies

$$\sum_{i\in I^k} z_{i,n+1} < F_2. \tag{5.3}$$

A contradiction to the assumption that Z^{k+1} is a Special M-feasible solution. Therefore, Y^{k+1} is a feasible solution of Stage-II corresponding to the feasible solution X^{k+1} of Stage-I problem.

Claim 2 (Optimality)**.** Y^{k+1} is an optimal feasible solution of Stage-II corresponding to the feasible solution X^{k+1} of Stage-I.

On contrary, let $Y' = \{y'_{ij}\}$ be any other feasible solution of Stage-II problem corresponding to the feasible solution X^{k+1} of Stage-I problem yielding time of Stage-II problem strictly less than that yielded by Y^{k+1}.

Let there be p elements in $\bar{S}^k$ arranged in ascending order of transportation time (as discussed in Sect. 4.2), *i.e.*,

$$t_{i_1j_1} \le t_{i_2j_2} \le t_{i_3j_3} \le \ldots \le t_{i_pj_p}.$$

We will prove the optimality of Y^{k+1} by using induction on the cardinality of the set $\bar{S}^k$.

Case I. Let $|\bar{S}^k| = 1$, *i.e.*, $\bar{S}^k = \{(i_1, j_1)\}$.
By construction of Y^{k+1} and its feasibility $y_{i_1j_1} = \min\{F_2, z_{i_1,n+1}, \bar{u}_{i_1j_1}\}$, therefore $y_{i_1j_1} = F_2$.
As Y' yields Stage-II time strictly less than that yielded by Y^{k+1}, therefore $y'_{i_1j_1}$ should be 0 which is absurd, as we assume Y' to be a feasible solution of Stage-II in which flow F_2 is transported.

Case II. Let $|\bar{S}^k| = 2$, *i.e.*, $\bar{S}^k = \{(i_1, j_1), (i_2, j_2)\}$.
The cells allowed for allocation are $(i_1, j_1), (i_2, j_2)$ and $\sum_{q=1,2} y_{i_qj_q} = F_2$.

If for Y^{k+1}, $y_{ij} > 0$ only for (i_1, j_1) and $y_{ij} = 0$ for (i_2, j_2), then it can be proved on the same lines as proved for Case I. But if $y_{ij} > 0$ for (i_1, j_1) and (i_2, j_2) both, then by construction of Y^{k+1}, $y_{i_1j_1} = \min\{F_2, z_{i_1,n+1}, \bar{u}_{i_1j_1}\}$

Clearly $y_{i_2j_2} > 0$, this implies that $y_{i_1j_1} \neq F_2$.

Following subcases arise:
Subcase (i)

$$\text{If} \ \ y_{i_1j_1} = \bar{u}_{i_1j_1} < F_2. \tag{5.4}$$

By assumption that Y' is a feasible solution yielding Stage-II time strictly less than that yielded by Y^{k+1},

$$F_2 = y'_{i_1j_1} \le \bar{u}_{i_1j_1} \tag{5.5}$$

Combining (5.4) and (5.5)

$$F_2 \le \bar{u}_{i_1j_1} < F_2$$

which is a contradiction.
Subcase (ii)

$$\text{If} \ \ y_{i_1j_1} = z_{i_1,n+1} < F_2 \tag{5.6}$$

By assumption that Y' yields Stage-II time strictly less than that yielded by Y^{k+1},

$$F_2 = y'_{i_1j_1} \leq z_{i_1,n+1}. \tag{5.7}$$

Combining (5.6) and (5.7)

$$F_2 \leq z_{i_1,n+1} < F_2.$$

which is a contradiction.

Case III. Let $|\bar{S}^k| = 3$.
The cells allowed for allocation are $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ and $\sum\limits_{q=1,2,3} y_{i_qj_q} = F_2$.
If for Y^{k+1}, $y_{ij} > 0$ only for (i_1, j_1) and $y_{ij} = 0$ for (i_2, j_2) and (i_3, j_3), then it can be proved on the same lines as proved for Case I.
If for Y^{k+1}, $y_{ij} > 0$ for (i_1, j_1), (i_2, j_2) and $y_{ij} = 0$ for (i_3, j_3), then it can be proved on the same lines as proved for Case II.
But if for $Y^{k+1}, y_{ij} > 0$ for (i_1, j_1), (i_2, j_2) and (i_3, j_3), then start with (i_1, j_1) and find

$$y_{i_1j_1} = \min\{F_2, z_{i_1,n+1}, \bar{u}_{i_1j_1}\}.$$

Either $y_{i_1j_1} = \bar{u}_{i_1j_1}$ or $y_{i_1j_1} = z_{i_1,n+1}$.

In any of the above case, go to (i_2, j_2) and find

$$y_{i_2j_2} = \min\{\hat{F}_2, \hat{z}_{i_2,n+1}, \bar{u}_{i_2j_2}\} = \min\{F_2 - y_{i_1j_1}, \hat{z}_{i_2,n+1}, \bar{u}_{i_2j_2}\}$$

Now either $y_{i_2j_2} = \bar{u}_{i_2j_2}$ or $y_{i_2j_2} = \hat{z}_{i_2,n+1}$ for the reason that $y_{i_2j_2}$ can not be equal to $F_2 - y_{i_1j_1}$ as $y_{ij} > 0$ for (i_1, j_1), (i_2, j_2) and (i_3, j_3).
Subcase (i) $y_{i_1j_1} = \bar{u}_{i_1j_1}$ and $y_{i_2j_2} = \bar{u}_{i_2j_2}$.
Clearly

$$\sum_{q=1}^{3} y_{i_qj_q} = F_2 \Rightarrow \quad \sum_{q=1}^{2} y_{i_qj_q} < F_2 \Rightarrow \quad \bar{u}_{i_1j_1} + \bar{u}_{i_2j_2} < F_2.$$

By assumption that Y' is a feasible solution and yields better time than Y^{k+1} we have

$$F_2 = \sum_{q=1}^{2} y'_{i_qj_q} \leq \bar{u}_{i_1j_1} + \bar{u}_{i_2j_2} < F_2.$$

which is a contradiction.
Subcase (ii) $y_{i_1j_1} = \bar{u}_{i_1j_1}$ and $y_{i_2j_2} = \hat{z}_{i_2,n+1}$.
Clearly

$$\sum_{q=1}^{3} y_{i_qj_q} = F_2 \Rightarrow \quad \sum_{q=1}^{2} y_{i_qj_q} < F_2.$$

$$\Rightarrow \quad \bar{u}_{i_1j_1} + \hat{z}_{i_2,n+1} < F_2 \tag{5.8}$$

There are two possibilities:

(i) If $i_1 = i_2$, then $\hat{z}_{i_2,n+1} = \hat{z}_{i_1,n+1} = z_{i_1,n+1} - \bar{u}_{i_1j_1}$
Inequality (5.8) implies that $\bar{u}_{i_1j_1} + z_{i_1,n+1} - \bar{u}_{i_1j_1} < F_2 \Rightarrow \quad z_{i_1,n+1} < F_2$.

Again, by assumption that Y' is a feasible solution and yields better time than Y^{k+1} we have

$$F_2 = \sum_{q=1}^{2} y'_{i_q j_q} \leq z_{i_1,n+1} < F_2$$

which is a contradiction.

(ii) If $i_1 \neq i_2$, then $\hat{z}_{i_2,n+1} = z_{i_2,n+1}$ (because nothing has been shipped from the source i_2 yet). Inequality (5.8) implies that $\bar{u}_{i_1 j_1} + z_{i_2,n+1} < F_2$. Again, by assumption that Y' is a feasible solution and yields better time than Y^{k+1} we have

$$F_2 = \sum_{q=1}^{2} y'_{i_q j_q} \leq \bar{u}_{i_1 j_1} + z_{i_2,n+1} < F_2$$

which is a contradiction.

Subcase (iii) $y_{i_1 j_1} = z_{i_1,n+1}$ and $y_{i_2 j_2} = \bar{u}_{i_2 j_2}$.

Here it is clear that $i_1 \neq i_2$.

Also

$$\sum_{q=1}^{3} y_{i_q j_q} = F_2 \Rightarrow \quad \sum_{q=1}^{2} y_{i_q j_q} < F_2 \Rightarrow \quad z_{i_1,n+1} + \bar{u}_{i_2 j_2} < F_2$$

Again, by assumption that Y' is a feasible solution and yields better time than Y^{k+1}, we have

$$F_2 = \sum_{q=1}^{2} y'_{i_q j_q} \leq z_{i_1,n+1} + \bar{u}_{i_2 j_2} < F_2$$

which is a contradiction.

Subcase (iv) $y_{i_1 j_1} = z_{i_1,n+1}$and $y_{i_2 j_2} = \hat{z}_{i_2,n+1}$.

Here also, $i_1 \neq i_2$.

Now

$$\sum_{q=1}^{3} y_{i_q j_q} = F_2 \Rightarrow \quad \sum_{q=1}^{2} y_{i_q j_q} < F_2 \Rightarrow \quad z_{i_1,n+1} + z_{i_2,n+1} < F_2.$$

By assumption that Y' is a feasible solution and yields better time than Y^{k+1} we have,

$$F_2 = \sum_{q=1}^{q=2} y'_{i_q j_q} \leq z_{i_1,n+1} + z_{i_2,n+1} < F_2$$

which is a contradiction.

Proceeding this way for $|\bar{S}^k| = 4, 5, \ldots, p$, we get a similar type of contradiction in each case.

Thus, there does not exist a solution like Y' giving Stage-II time strictly less than that given by Y^{k+1}. Thus Y^{k+1} is an OFS of Stage-II problem corresponding to a feasible solution X^{k+1} of Stage-I. Also by construction of Y^{k+1}, it is clear that $T_2^{k+1} < T_2^k$. □

Theorem 5.2. *Let an OFS of the problem $P_1^*(T_2^k)$ is a Special M-feasible solution. Then $T_1^{k+1} \geq T_1^k$.*

Proof. On contrary, suppose $T_1^{k+1} < T_1^k$.

Since the pair (T_1^k, T_2^k) is obtained from an optimal SMFS of the problem $P_1^*(T_2^{k-1})$, so T_1^k is the time of Stage-I for the problem $P_1^*(T_2^{k-1})$ and T_2^k is the corresponding optimal time for Stage-II. If $T_1^{k+1} < T_1^k$, this means that the cell $(i,j) \in I \times J$ with $t_{ij} = T_1^{k+1}$ is not blocked in the problem $P_1^*(T_2^{k-1})$ and $t_{i,n+1} = 0$ for $i \in I$ for which $\min_{j\in J} t_{ij} < T_2^{k-1}$, *i.e.*, $\min_{j\in J} t_{ij} \leq T_2^k$ (as $T_2^k < T_2^{k-1}$).

Also $T_2^k < T_2^{k-1}$ implies that $J_i^k \subset J_i^{k-1}$ for each $i \in I$.

Thus any feasible solution $Z^{k+1} = \{z_{ij}\}_{I^*\times J^*}$ of the problem $P_1^*(T_2^k)$ satisfying

$$\sum_{j\in J_i^k} z_{ij} + z_{i,n+1} \leq \sum_{j\in J_i^k} u_{ij} \qquad \forall\, i \in I^k$$

must also satisfy

$$\sum_{j\in J_i^{k-1}} z_{ij} + z_{i,n+1} \leq \sum_{j\in J_i^{k-1}} u_{ij} \qquad \forall\, i \in I^{k-1}$$

as $I^k \subset I^{k-1}$. Therefore, an optimal SMFS, Z^{k+1} of the problem $P_1^*(T_2^k)$ giving time of transportation of Stage-I as T_1^{k+1} is a feasible solution of the problem $P_1^*(T_2^{k-1})$ as it satisfies the partial sum constraint for $J_i^{k-1} = \{j \in J | \min_{j\in J} t_{ij} < T_2^{k-1}\}$ also. Thus, Z^{k+1} is a SMFS of the problem $P_1^*(T_2^{k-1})$ with associated time as T_1^{k+1} which is strictly less than T_1^k. This is a contradiction to the fact that T_1^k is the optimal time of the problem $P_1^*(T_2^{k-1})$. Hence $T_1^{k+1} \geq T_1^k$. □

Theorem 5.3. *If $\hat{T}_1 + \hat{T}_2 = \min_{k\geq 0}[T_1^k + T_2^k]$, where T_1^k and T_2^k are the times of transportation of Stage-I and Stage-II respectively, corresponding to an optimal SMFS of the problem (P_1^*), for $k = 0$ and $P_1^*(T_2^{k-1})$, for $k \geq 1$, then $\hat{T}_1 + \hat{T}_2$ is the optimal value of objective function of the problem (P_3).*

Proof. Let if possible $\hat{T}_1 + \hat{T}_2$ not be the optimal value of the objective function of the problem (P_3). Suppose there exists a feasible solution $(\tilde{X}, \tilde{Y})$ of (P_3) that provides a pair of Stage-I and Stage-II times as $(\tilde{T}_1, \tilde{T}_2)$ such that $\tilde{T}_1 + \tilde{T}_2 < \hat{T}_1 + \hat{T}_2$, *i.e.*,

$$\tilde{T}_1 + \tilde{T}_2 < \min_{k\geq 0}\left[T_1^k + T_2^k\right]. \tag{5.9}$$

As $(\tilde{X}, \tilde{Y})$ is a feasible solution of (P_3), so $\tilde{X}$ is the feasible solution of Stage-I problem (P_1) and $\tilde{Y}$ be the corresponding OFS of Stage-II problem (P_2). Corresponding to $(\tilde{X}, \tilde{Y})$, define a solution $\tilde{Z} = \{z_{ij}\}_{I^*\times J^*}$ of the problem (P_1^*) (a balanced transportation problem equivalent to Stage-I problem (P_1)) as

$$\begin{array}{c} z_{ij} = x_{ij} \quad \forall \ \ (i,j) \in I \times J \\ z_{i,n+1} = a_i - \sum_{j\in J} x_{ij} \quad \forall\ i \in I \\ z_{m+1,j} = \sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij} \quad \forall\ j\ \in J. \\ z_{m+1,n+1} = 0 \end{array} \tag{5.10}$$

Now, firstly we prove that $\tilde{Z} = \{z_{ij}\}_{I^*\times J^*}$ defined by equation (5.10) is a feasible solution of the problem (P_1^*). For all $i \in I$,

$$\sum_{j\in J^*} z_{ij} = \sum_{j\in J} z_{ij} + z_{i,n+1} = \sum_{j\in J} x_{ij} + a_i - \sum_{j\in J} x_{ij} = a_i = a_i^*.$$

For $i = m+1$, $\displaystyle\sum_{j\in J^*} z_{m+1,j} = \sum_{j\in J} z_{m+1,j} + z_{m+1,n+1} = \sum_{j\in J} z_{m+1,j} + 0$

$$= \sum_{j\in J}\left(\sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij}\right)$$

$$= \sum_{j\in J}\sum_{i\in I} u_{ij} - \sum_{j\in J}\sum_{i\in I} x_{ij}$$

$$= \sum_{j\in J}\sum_{i\in I} u_{ij} - F_1 = a^*_{m+1}.$$

Similarly, for all $j \in J$,

$$\sum_{i\in I^*} z_{ij} = \sum_{i\in I} z_{ij} + z_{m+1,j} = \sum_{i\in I} x_{ij} + \sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij} = \sum_{i\in I} u_{ij} = b^*_j.$$

For $j = n+1$, $\displaystyle\sum_{i\in I^*} z_{i,n+1} = \sum_{i\in I} z_{i,n+1} + z_{m+1,n+1} = \sum_{i\in I}(a_i - \sum_{j\in J} x_{ij}) + 0 = \sum_{i\in I} a_i - F_1 = b^*_{n+1}$. Also for $(i,j) \in I \times J,\ 0 \le z_{ij} \le u^*_{ij}$ (as $z_{ij} = x_{ij}$ and $u^*_{ij} = u_{ij}\ \ \forall\,(i,j) \in I \times J$).

For $i = m+1$, $z_{m+1,j} = \displaystyle\sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij} \le \sum_{i\in I} u_{ij} - b_j = u^*_{m+1,j}\ \forall\ j \in J$.

Also, $z_{m+1,j} = \displaystyle\sum_{i\in I} u_{ij} - \sum_{i\in I} x_{ij} \ge 0$ and $z_{i,n+1} = a_i - \displaystyle\sum_{j\in J} x_{ij} \le a_i = u^*_{i,n+1}$.

Also, $z_{i,n+1} = a_i - \displaystyle\sum_{j\in J} x_{ij} \ge 0$ (as $\tilde{X} = \{x_{ij}\}$ is a feasible solution of (P_1)).

Therefore, $\tilde{Z} = \{z_{ij}\}_{I^*\times J^*}$ is a feasible solution of the problem (P_1^*).

Now clearly, $\tilde{T}_2 \ge T_2^L$ because T_2^L is the minimum possible time of Stage-II. Also $\tilde{T}_2 \le T_2^0$, for if $\tilde{T}_2 > T_2^0$, then the inequality (5.9) implies $\tilde{T}_1 < T_0^1$ which contradicts the optimality of T_1^0 for the problem (P_1^*). Therefore

$$T_2^L \le \tilde{T}_2 \le T_2^0. \tag{5.11}$$

Now if (T_1^r, T_2^r) is a pair of times of Stage-I and Stage-II such that either the optimal solution of $P_1^*(T_2^r)$ is not SMFS or $T_2^r = T_2^L$, then $T_2^r \le \tilde{T}_2 \le T_2^0$.

If $\tilde{T}_2 \ge T_2^r$ then by inequality (5.9),

$$\tilde{T}_1 < T_1^r. \tag{5.12}$$

Consider the problem $P_1^*(T_2^{r-1}),\ \ r \ge 1$ withT_1^r as the Stage-I time given by optimal SMFS of this problem. As proved earlier, $\tilde{Z}$ is a feasible solution of (P_1^*) but it may or may not be a feasible solution of $P_1^*(T_2^{r-1})$.

Case (i): $\tilde{Z}$ is a feasible solution of $P_1^*(T_2^{r-1}),\ r \ge 1$.
Subcase (i) Suppose it is also a SMFS of $P_1^*(T_2^{r-1})$ which implies that it is an MFS with respect to (P_1). It means that the solution $\tilde{Z}$ is giving Stage-I time equal to $\tilde{T}_1$, such that $\tilde{T}_1 < T_1^r$ which is a contradiction to the optimality of T_1^r for the problem $P_1^*(T_2^{r-1})$.
Subcase (ii) Suppose $\tilde{Z}$ is not a SMFS of $P_1^*(T_2^{r-1})$, then either $\tilde{Z}$ is not MFS with respect to (P_1) or $\sum_{i\in \bar{I}^{r-1}} z_{i,n+1} < F_2$.

(a) If $\tilde{Z}$ is not MFS with respect to (P_1). It means $z_{ij} > 0$ for some $(i,j) \in I \times J$ with $t_{ij} = M$.
This means $z_{ij} > 0$ for some $(i,j) \notin L$ (because $t_{ij} = M$ only for $(i,j) \notin L$ where $L = \{(i,j) \in I \times J | t_{ij} + T_2^L < \min_{h=0,1,\ldots,r-1}[T_1^h + T_2^h]\}$). But $\tilde{Z}$ yields a solution giving Stage-I time $\tilde{T_1}$, therefore, for $(i,j) \in I \times J$ with $t_{ij} = \tilde{T_1},\ z_{ij} > 0$ and eventually that (i,j) does not belong to L.
$\Rightarrow \quad \tilde{T_1} + T_2^L \geq \min_{h=0,1,\ldots,r-1}[T_1^h + T_2^h]$.
$\Rightarrow \quad \tilde{T_1} + \tilde{T_2} \geq \tilde{T_1} + T_2^L \geq \min_{h=0,1,\ldots,r-1}[T_1^h + T_2^h]$ (as $\tilde{T_2} \geq T_2^L$).
which is a contradiction as to (5.9).

(b) If $\sum_{\mathbf{i} \in \bar{\mathbf{I}}^{\mathbf{r}-\mathbf{1}}} \mathbf{z_{i,n+1}} < \mathbf{F_2}$, then F_2 cannot be transported in time less than T_2^{r-1}, therefore for $\tilde{Z}$, $\tilde{T_2} \geq T_2^{r-1}$. From (5.9), $\tilde{T_1} < T_1^{r-1}$.

So, we consider the problem $P_1^*(T_2^{r-2})$ and proceed as before and continue this process until we reach at a step with $\tilde{T_2} \geq T_2^0$ which gives that $\tilde{T_1} < T_1^0$. A contradiction to optimality of T_1^0.

Case (ii): Let $\tilde{Z}$ not be a feasible solution of $P_1^*(T_2^{r-1})$, i.e, the partial sum constraints are not satisfied. This implies $\displaystyle\sum_{j \in J_i^{r-1}} z_{ij} + z_{i,n+1} > \sum_{j \in J_i^{r-1}} u_{ij}$ for some $i \in \bar{I}^{r-1}$.

It means in Stage-II, F_2 has to be sent at a route with time greater than or equal to T_2^{r-1} but then $\tilde{T_1} < T_1^{r-1}$
So, we consider the problem $P_1^*(T_2^{r-2})$ and proceed as before and continue this process until we reach at a step with $\tilde{T_2} \geq T_2^0$ which gives that $\tilde{T_1} < T_1^0$. A contradiction to optimality of T_1^0.
Hence there does not exist any solution of (P_3) giving a pair of the type $(\tilde{T_1}, \tilde{T_2})$ satisfying (5.9). □

5.1. Computing T_2^L

Find $\min_{I \times J} t_{ij} = t_{r_1 s_1}$.
If $F_2 \leq u_{r_1 s_1}$, then $T_2^L = t_{r_1 s_1}$ else, find $\min_{I \times J \setminus (r_1 s_1)} t_{ij} = t_{r_2 s_2}$.
If $F_2 \leq u_{r_1 s_1} + u_{r_2 s_2}$, then $T_2^L = t_{r_2 s_2}$ else, find $\min_{I \times J \setminus (r_1 s_1),(r_2 s_2)} t_{ij} = t_{r_3 s_3}$.
Continuing this way, we get $T_2^L = t_{r_{k+1} s_{k+1}}$ where $\min_{I \times J \setminus \{(r_1 s_1),(r_2 s_2)\ldots(r_k s_k)\}} t_{ij} = t_{r_{k+1} s_{k+1}}$ if $\sum_{l=1}^{l=k} u_{r_l s_l} < F_2$ and $\sum_{l=1}^{l=k+1} u_{r_l s_l} \geq F_2$.

6. Algorithm

Initial step: Obtain an OFS of (P_1^*) and note the Stage-I and Stage-II times as T_1^0 and T_2^0 respectively. If $T_2^0 = T_2^L$, then stop and go to terminal step, else go to general step.

General step: Let the pairs in hand be (T_1^g, T_2^g) for $g = 0, 1, \ldots, k-1$.

Construct the problem $P_1^*(T_2^{k-1})$ and find its OFS. If this is not a SMFS, then stop and go to terminal step, otherwise read the time T_1^k of Stage-I and T_2^k of Stage-II.

If $T_2^k = T_2^L$, stop and go to terminal step, else repeat the general step for next higher values of k.

Terminal step: Declare $\min_{g \geq 0}[T_1^g + T_2^g]$ as the optimal value of the objective function of (P_3).

7. Numerical illustration

Consider the following 6×5 transportation problem given Table 1 having six bins and five machines. Table 2 gives the maximum capacity (the maximum number of units of the semi-finished jobs allowed for shipping) of each (i,j)th link. In this problem, $\sum_i a_i = 400$ and $\sum_j b_j = 300$ and the number of jobs to be sent in first stage is $F_1 = 350$ and in second stage is $F_2 = 40$.

Here $T_2^L = 13$.

TABLE 1. Time of transportation along various links.

	M_1	M_2	M_3	M_4	M_5	a_i
B_1	82	28	96	80	68	80
B_2	91	55	49	96	76	70
B_3	13	96	81	66	75	50
B_4	92	97	15	04	40	90
B_5	64	16	43	85	66	60
B_6	10	98	92	94	18	50
b_j	50	40	75	75	60	

TABLE 2. Capacity/upperbound on the bin-machine links.

20	25	25	30	20
15	20	25	25	20
35	25	25	15	20
25	20	25	20	20
20	25	20	25	15
15	20	25	30	20

7.1. Initial step:

An OFS of the problem (P_1^*) gives $T_1^0 = 85$ and the corresponding $T_2^0 = 92$ (given in the Tab. 3). So the current value of $T_1^0 + T_2^0$ is equal to 177. Since $T_2^0 > T_2^L$, go to general step of the algorithm.

7.2. General step:

Iteration 1: Construct the problem $P_1^*(T_2^0)$, its SMFS yields Stage-I time $T_1^1 = 92$ and the corresponding Stage-II time $T_2^1 = 76$.

The current value of $T_1^1 + T_2^1$ becomes 168. As $T_2^1 > T_2^L$, solve $P_1^*(T_2^1)$.

Iteration 2: $P_1^*(T_2^1)$ yields the pair $(T_1^2, T_2^2) = (92, 16)$.

Iteration 3: $P_1^*(T_2^2)$ yields the pair $(T_1^3, T_2^3) = (92, 13)$.

Here $T_2^3 = T_2^L$, therefore, Stage-II time can not be decreased further. So the algorithm terminates.

7.3. Terminal step:

The optimal value of the objective function of the problem (P_3) is given by $\min_{h=0,1,2,3}[T_1^h + T_2^h] = 105$. An optimal feasible solution of the problem $P_1^*(T_2^2)$ is shown in Table 4.

In Table 4, the entry at the center of each cell gives the time of transportation. Note that the entries in boldface represent basic cells and the entries of the form $\overline{a}$ represent the non basic cells which are at their upper bounds. Feasible solutions of Stage-I and Stage-II problems corresponding to the OFS of the problem $P_1^*(T_2^2)$ are depicted in Tables 5 and 6 respectively.

TABLE 3. (OFS) of the problem P_1^*.

	M_1	M_2	M_3	M_4	M_5	M_6
B_1	82	$\overline{25}$ 28	96	$\overline{30}$ 80	68 **5**	0
B_2	91	$\overline{20}$ 55	$\overline{25}$ 49	96	76 **15**	0 **10**
B_3	13 **30**	96	81 **5**	$\overline{15}$ 66	75	0
B_4	92	97	$\overline{25}$ 15	$\overline{20}$ 4	$\overline{20}$ 40	0 **25**
B_5	64 **5**	16 **5**	$\overline{20}$ 43	85 **10**	$\overline{15}$ 66	0
B_6	$\overline{15}$ 10	98	92	94	$\overline{20}$ 18	$\overline{15}$ 0
B_7	$\overline{80}$ 0	0 **80**	$\overline{70}$ 0	$\overline{70}$ 0	0 **20**	M

TABLE 4. (OFS) of the problem $P_1^*(T_2^2)$.

	M_1	M_2	M_3	M_4	M_5	M_6
B_1	$\overline{20}$ 82	$\overline{25}$ 28	M	$\overline{30}$ 80	68 **5**	M
B_2	91 **5**	$\overline{20}$ 55	$\overline{25}$ 49	M	$\overline{20}$ 76	M
B_3	13	M	81 **0**	$\overline{15}$ 66	75	0 **35**
B_4	92 **25**	M	$\overline{25}$ 15	$\overline{20}$ 4	$\overline{20}$ 40	0
B_5	64 **5**	$\overline{25}$ 16	$\overline{20}$ 43	85 **10**	66	M
B_6	10	M	92 **15**	94	$\overline{20}$ 18	$\overline{15}$ 0
B_7	0 **75**	0 **80**	0 **70**	$\overline{70}$ 0	0 **50**	M

TABLE 5. OFS of Stage-I of the problem $P_1^*(T_2^2)$.

	M_1	M_2	M_3	M_4	M_5
B_1	$\overline{20}$ 82	$\overline{25}$ 28	96	$\overline{30}$ 80	68 **5**
B_2	91 **5**	$\overline{20}$ 55	$\overline{25}$ 49	96	$\overline{20}$ 76
B_3	13	96	81 **0**	$\overline{15}$ 66	75
B_4	92 **25**	97	$\overline{25}$ 15	$\overline{20}$ 4	$\overline{20}$ 40
B_5	64 **5**	$\overline{25}$ 16	$\overline{20}$ 43	85 **10**	66
B_6	10	98	92 **15**	94	$\overline{20}$ 18

TABLE 6. OFS of Stage-II of the problem $P_1^*(T_2^2)$.

	M_1	M_2	M_3	M_4	M_5
B_1	82	28	96	80	68
B_2	91	55	49	96	76
B_3	$\overline{25}$ 13	96	81	66	75
B_4	92	97	15	4	40
B_5	64	16	43	85	66
B_6	$\overline{15}$ 10	98	92	94	18

8. Concluding remarks

(1) The problem considered in this paper is related to a steel industry manufacturing a product whose manufacturing process involves the transportation of semi-finished jobs from various bins to the various identical manufacturing facilities for the final processing and finishing. It is assumed that vehicles are available for the transportation and due to some constraints, a capacity is introduced on each bin-machine link. Also, depending upon various factors, the number of products equal to the market demand cannot be sent in one go. So the transportation has to take place in two stages. It is further assumed that the number of products equal to the market demand, when manufactured, are sent to the market in a truck consuming a fixed time of transportation. If a machine receives k number of jobs for processing from some bins, it takes k times the per unit processing time which would be a fixed time. So only the minimization of the time of transportation of semi-finished jobs from bins to the manufacturing units is considered to be of prime importance. So our objective is to find out that how the semi-finished jobs (equal to the market demand) must be transported on various bin-machine links in Stage-I and Stage-II so that total time of transportation is minimum.

(2) A polynomial time iterative algorithm is proposed to find an optimal solution to this capacitated two-stage time minimization transportation problem that generates a sequence of pairs of transportation times of Stage-I and Stage-II starting with minimum time of Stage-I. The generated pairs are such that at each iteration, Stage-II time decreases strictly and Stage-I transportation time may increase. The algorithm terminates as soon as the minimum time of Stage-II is reached or the problem is not Special M-feasible at some iteration. In the proposed algorithm, we are solving a balanced capacitated CMTP and its restricted versions at each iteration.
Various polynomial time solution techniques are available to solve capacitated CMTP [28] and any of the above mentioned techniques can be applied to solve this problem. Capacitated CMTP can be solved by converting it into an equivalent network problem with lower and upper arc capacities [1,3,23].

(3) The algorithm terminates in a finite number of steps. The maximum number of iterations that may be required is $s-r+1$ where s is the total number of distinct time entries and $r \in \{1, 2, \ldots, s\}$ such that $T_1^1 = t^r$.

(4) In the current problem, it is also assumed that total number of jobs to be manufactured is less than or equal to $2F_1$. If it is greater than $2F_1$, it would have become a multi-level problem which is a topic of further research.

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