RAIRO-Oper. Res. 48 (2014) 381–398
DOI: 10.1051/ro/2014013

RAIRO Operations Research
www.rairo-ro.org

PERFORMANCE ANALYSIS OF SINGLE SERVER NON-MARKOVIAN RETRIAL QUEUE WITH WORKING VACATION AND CONSTANT RETRIAL POLICY

V. Jailaxmi[1], R. Arumuganathan[1] and M. Senthil Kumar[2]

Abstract. This paper analyses an M/G/1 retrial queue with working vacation and constant retrial policy. As soon as the system becomes empty, the server begins a working vacation. The server works with different service rates rather than completely stopping service during a vacation. We construct the mathematical model and derive the steady-state queue distribution of number of customer in the retrial group. The effects of various performance measures are derived.

Keywords. Retrial queue, working vacation, constant retrial policy.

Mathematics Subject Classification. 60K25, 90B22.

1. Introduction

A retrial system consists of a primary service facility and an orbit. Customers arrive at the service facility at a Poisson rate from main pool. Upon arrival of a customer, if the server is busy or under repair or on vacation the arrival will join the retrial group in the orbit and attempt for service again at some time later. Such situations arise in many communication protocols, local area networks and daily life situations. In aviation, where a plane finds the runway occupied remakes its attempt of landing later and in this case the plane is said to be in orbit. In telephone

Received September 24, 2012. Accepted January 8, 2014.

[1] Department of Mathematics, PSG College of Technology, 641004 - Coimbatore, India. subaprasannaa@yahoo.co.in; ran_psgtech@yahoo.co.in

[2] Department of Applied Mathematics and Computational Sciences, PSG College of Technology, 641004 - Coimbatore, India. ms_kumar@yahoo.com

Article published by EDP Sciences

where a telephone subscriber who obtains a busy signal repeats the call until the required connection is made. The detailed overviews of the related references with retrial queues can be found in the book of Falin and Templetion [1] and the survey papers, of Artalejo [2, 3]. The single server retrial queue with priority calls have been studied by Choi *et al.* [4–6] for many applications in telecommunication and mobile communication. Artalejo and Gomez−Corral [7] have made a detailed study on retrial queueing systems.

Vacation models had been the subject of interest in queueing theory in recent years because of their applications in real life congestion situations such as manufacturing and production, computer and communication systems, service and distribution systems, etc. A comprehensive and excellent study on the vacation models can be found in Takagi [8]. For related literature of retrial queues with vacations, Li and Yang [9] developed an M/G/1 retrial system with server vacations and M independent identical input sources. Later Artalejo [10] analyzed an M/G/1 retrial queue with exhausted server vacations, that is the server takes a vacation only when there are no customers in the orbit. A literature survey on queueing systems with server vacations can be found in Doshi [11]. Doshi [12] discussed an M/G/1 system with variable vacations. Batch arrival Markovian single server queueing systems with multiple vacations were first studied by Baba [13]. Later Senthilkumar and Arumuganathan [14] have analyzed single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service. The variations and extensions of these vacation models can be referred to Lee *et al.* [15,16] and Krishna Reddy *et al.* [17]. Arumuganathan *et al.* [18] analyzed a steady state non-Markovian bulk queueing system with N-policy and different types of vacation. Haridass and Arumuganathan [19] analyzed a batch arrival, bulk service queueing system with interrupted vacation.

A queue with working vacation was first analyzed by Servi and Finn [20], they obtained the queue length distribution of M/M/1/Wv queue. They discussed a classical single server vacation model in which a single server works at a different rate rather than completely stopping during the vacation period. Further, they applied the model for the performance evaluation of Wavelength Division Multiplexing (WDM) optical systems. But they have assumed exponential service time, which may not be the case always. Subsequently, Kim *et al.* [21] have analyzed an M/G/1 queue with exponentially distributed working vacations and obtained the steady state queue length distribution through the decomposition approach. Later Wu and Takagi [22] extended Servi and Finns model to an M/G/1 working vacation in which, both regular service time and the service time in working vacation are assumed to be generally distributed. Li *et al.* [23] considered an M/G/1 queue with exponentially distributed working vacations, which is a special case of that in Wu and Takagi [22]. All the above contributors consider classical queueing model.

Tien Van Do [24] studied a Markovian retrial queue with working vacation. But in practice, there must be generally distributed service times which are motivated by the performance analysis of Media Access Control (MAC) function in wireless networks. Wireless MAC protocols often use collision avoidance techniques, in conjunction with a (physical or virtual) carrier sense mechanism. In carrier sense mechanism, when a node wishes to transmit a packet, it first waits until the channel is idle. Nodes hearing RTS(Request-to-Send) or CTS(Clear-to-Send) stay silent for the duration of the corresponding transmission. Once channel becomes idle, the node waits for a randomly chosen duration before attempting to transmit. This mechanism can be modeled as M/G/1 retrial queueing model with working vacation model by considering the orbit as pool of packets waiting for transmission once it senses the idle channel and RTS and CTS as working vacation times. So, in this paper we introduce an M/G/1 retrial queue with single working vacation and constant retrial policy. Analytical treatment of this model is obtained using supplementary variable technique. The probability generating function of number of customers in the retrial group is obtained.

2. The mathematical model

In this paper an M/G/1 retrial queue with working vacation and constant retrial policy is analyzed. The customers arrive according to Poisson process with rate λ. If the server is busy at the arrival time, the customers join the orbit to repeat their request later, whereas if the server is idle then the arriving customer begins its service immediately. The customers in the orbit try for service one by one with a constant retrial rate γ when the server is idle. The single server takes a working vacation at times when the customers being served depart from the system and no customers are in the orbit. The server works with different service rates rather than completely stopping service during a vacation. The service rate is μ_b when the server is not on vacation and μ_v during working vacation ($\mu_v < \mu_b$). Vacation durations are exponentially distributed with parameter η. After completing a vacation, the server stays idle in the system until a customer arrives from main pool or from orbit.

Let $S_v(x)$ $(s_v(x))$ $\{\tilde{S}_v(\theta)\}$ $[S_v^0(x)]$ be the cumulative distribution function (probability density function) {Laplace transform} [remaining service time] of service during working vacation. Let $S_b(x)$ $(s_b(x))$ $\{\tilde{S}_b(\theta)\}$ $[S_b^0(x)]$ be the cumulative distribution function (probability density function) {Laplace transform} [remaining service time] of service when the server is not on working vacation. $N(t)$ denotes the number of customers in the orbit at time t. The process considered here is a semi-Markov process which become Markov by including additional random variable as the remaining service time as given by Limnios and Oprisan [25].

The server state is denoted as

$$C(t) = \begin{cases} 0, & \textit{if the server is idle during working vacation} \\ 1, & \textit{if the server is idle and not on working vacation} \\ 2, & \textit{if the server is busy during working vacation} \\ 3, & \textit{if the server is busy and not on working vacation} \end{cases}$$

Now the system state probabilities are defined as follows:

1) $W_n(t) = Pr\{N(t) = n, C(t) = 0\}, n \geq 0$ is the probability that at time t the server is idle during vacation and the orbit size n.
2) $I_n(t) = Pr\{N(t) = n, C(t) = 1\}, n \geq 0$ is the probability that at time t the server is idle but not on working vacation and the orbit size is n.
3) $Q_n(x,t)\mathrm{d}t = Pr\{N(t) = n, C(t) = 2, x \leq S_v^0(t) \leq x + \mathrm{d}t\}, n \geq 0$ is the is the probability that at time t the server is busy during working vacation, the orbit size is n and the remaining service time of a customer during working vacation at an arbitrary time is between x and $x + \mathrm{d}t$.
4) $P_n(x,t)\mathrm{d}t = Pr\{N(t) = n, C(t) = 3, x \leq S_b^0(t) \leq x + \mathrm{d}t\}, n \geq 0$ is the joint probability that at time t the server is busy when it is not on working vacation, the orbit size is n and the remaining service time of a customer when the server is not on working vacation at an arbitrary time is between x and $x + \mathrm{d}t$.

3. Steady state queue size distribution

To derive the steady state queue size distribution the following equations are obtained, using supplementary variable technique,

$$\begin{aligned}
W_0(t+\Delta t) &= W_0(t)(1-\lambda\Delta t-\eta\Delta t)+Q_0(0,t)\Delta t+P_0(0,t)\Delta t\\
W_n(t+\Delta t) &= W_n(t)(1-\lambda\Delta t-\eta\Delta t-\gamma\Delta t)+Q_n(0,t)\Delta t\\
I_0(t+\Delta t) &= I_0(t)(1-\lambda\Delta t)+W_0(t)\eta\Delta t\\
I_n(t+\Delta t) &= I_n(t)(1-\lambda\Delta t-\gamma\Delta t)+W_n(t)\eta\Delta t+P_n(0,t)\Delta t\\
Q_n(x-\Delta t,t+\Delta t) &= Q_n(x,t)(1-\lambda\Delta t-\eta\Delta t)+\lambda W_n(t)s_v(x)\Delta t\\
&\quad +\gamma W_{n+1}(t)s_v(x)\Delta t+\lambda Q_{n-1}(x,t)\Delta t(1-\delta_{n,0})\\
P_n(x-\Delta t,t+\Delta t) &= P_n(x,t)(1-\lambda\Delta t)+\lambda I_n(t)s_b(x)\Delta t+\gamma I_{n+1}s_b(x)\Delta t\\
&\quad +\left[\int_0^\infty Q_n(y,t)\mathrm{d}y\right]\eta s_b(x)\Delta t+\lambda P_{n-1}(x,t)(1-\delta_{n,0})\Delta t
\end{aligned}$$

where $\delta_{n,0} = \begin{bmatrix} 0 \; if \; n \neq 0 \\ 1 \; if \; n = 0 \end{bmatrix}.$

In steady state, we can set $W_0 = \lim_{t\to\infty} W_0(t)$, $I_0 = \lim_{t\to\infty} I_0(t)$, $W_n = \lim_{t\to\infty} W_n(t)$, $I_n = \lim_{t\to\infty} I_n(t)$ and limiting densities $Q_n(x) = \lim_{t\to\infty} Q_n(x,t)$ for $x > 0$ and $P_n(x) = \lim_{t\to\infty} P_n(x,t)$ for $x > 0$.

Now the above equations under steady state conditions can be written as follows:

$$(\lambda+\eta)W_0 = Q_0(0)+P_0(0) \tag{3.1}$$

$$(\lambda+\gamma+\eta)W_n(0) = Q_n(0) \tag{3.2}$$

$$\lambda I_0 = \eta W_0 \tag{3.3}$$

$$(\lambda+\gamma)I_n(0) = W_n(0)\eta+P_n(0) \tag{3.4}$$

$$\frac{-\mathrm{d}}{\mathrm{d}x}Q_n(x) = -(\lambda+\eta)Q_n(x) + \lambda W_n(0)s_v(x) + \gamma W_{n+1}(0)s_v(x) + \lambda Q_{n-1}(x)(1-\delta_{n,0}) \tag{3.5}$$

$$\frac{-\mathrm{d}}{\mathrm{d}x}P_n(x) = -\lambda P_n(x) + \lambda I_n(0)s_b(x) + \gamma I_{n+1}(0)s_b(x) + \lambda P_{n-1}(x)(1-\delta_{n,0}) + \left[\int_0^\infty Q_n(y)\mathrm{d}y\right]\eta s_b(x). \tag{3.6}$$

Assume that
$Laplace\ transform(P_n(x)) = \tilde{P}_n(\theta) = \int_0^\infty e^{-\theta x}P_n(x)\mathrm{d}x$;
$Laplace\ transform(Q_n(x)) = \tilde{Q}_n(\theta) = \int_0^\infty e^{-\theta x}Q_n(x)\mathrm{d}x$.

Taking Laplace transform on steady state equations (5) and (6), we have

$$\theta\tilde{Q}_n(\theta) - Q_n(0) = (\lambda+\eta)\tilde{Q}_n(\theta) - \lambda W_n(0)\tilde{S}_v(\theta) - \gamma W_{n+1}(0) \times \tilde{S}_v(\theta) - \lambda\tilde{Q}_{n-1}(\theta)(1-\delta_{n,0}) \tag{3.7}$$

$$\theta\tilde{P}_n(\theta) - P_n(0) = \lambda\tilde{P}_n(\theta) - \lambda I_n(0)\tilde{S}_b(\theta) - \gamma I_{n+1}(0)\tilde{S}_b(\theta) - \lambda\tilde{P}_{n-1}(\theta)(1-\delta_{n,0}) - \tilde{Q}_n(0)\eta\tilde{S}_b(\theta). \tag{3.8}$$

The following generating functions are helpful in deriving the probability generating function of orbit size.

$$\begin{aligned} W(z,0) &= \sum_{n=0}^{\infty} W_n(0)z^n; I(z,0) = \sum_{n=0}^{\infty} I_n(0)z^n; \\ \tilde{Q}(z,\theta) &= \sum_{n=0}^{\infty} \tilde{Q}_n(\theta)z^n; Q(z,0) = \sum_{n=0}^{\infty} Q_n(0)z^n \\ \tilde{P}(z,\theta) &= \sum_{n=0}^{\infty} \tilde{P}_n(\theta)z^n; P(z,0) = \sum_{n=0}^{\infty} P_n(0)z^n \end{aligned} \tag{3.9}$$

where $|z| \leq 1$.

Multiplying equations (1) and (3) by z^0, equations (2), (4), (7) and (8) by z^n, taking summation from $n = 0$ to ∞ and using (9), we get,

$$(\lambda+\eta)W(z,0) + \gamma(W(z,0) - W_0) = Q(z,0) + P_0 \tag{3.10}$$

$$\lambda I(z,0) + \gamma(I(z,0) - I_0) = \eta W(z,0) + (P(z,0) - P_0) \tag{3.11}$$

$$(\theta - (\lambda+\eta) + \lambda z)\,\tilde{Q}(z,\theta) = Q(z,0) - \lambda W(z,0)\tilde{S}_v(\theta) - \left(\frac{\gamma}{z}\right)(W(z,0) - W_0)\tilde{S}_v(\theta) \tag{3.12}$$

$$(\theta - \lambda + \lambda z)\tilde{P}(z,\theta) = P(z,0) - \left(\lambda + \left(\frac{\gamma}{z}\right)\right)\tilde{S}_b(\theta)I(z,0) + \left(\frac{\gamma}{z}\right) \times \tilde{S}_b(\theta)I_0 - \tilde{Q}(z,0)\eta\tilde{S}_b(\theta). \tag{3.13}$$

Theorem 3.1. *The probability generating function $P(z)$ of number of customers in orbit is given by,*

$$P(z) = \frac{\begin{array}{l} W(z,0)\begin{bmatrix} (\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+(\lambda+\gamma/z)(\tilde{S}_v(\lambda+\eta-\lambda z)) \\ \eta(\tilde{S}_b(\lambda-\lambda z)-1)+(\lambda+\gamma/z)\tilde{S}_v(\lambda+\eta-\lambda z)(-\lambda+\lambda z) \end{bmatrix} \\ +I(z,0)\left[(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)\right. \\ \left.+(\lambda+(\gamma/z))(\tilde{S}_b(\lambda-\lambda z)-1)(\lambda z-(\lambda+\eta))\right] \\ +W_0\left[(\gamma/z)\eta(\tilde{S}_v(\lambda+\eta-\lambda z)-1)(1-\tilde{S}_b(\lambda-\lambda z))\right. \\ \left.+(\gamma/z)(\tilde{S}_v(\lambda+\eta-\lambda z)-1)(\lambda z-\lambda)\right] \\ +I_0\left[(\gamma/z)(1-\tilde{S}_b(\lambda-\lambda z))(\lambda z-(\lambda+\eta))\right] \end{array}}{(\lambda z-(\lambda+\eta))\,(-\lambda+\lambda z)}$$

where

$$W(z,0) = \frac{W_0[\gamma-(\gamma/z)\tilde{S}_v(\lambda+\eta-\lambda z)]+p_0}{(\lambda+\eta+\gamma)-(\lambda+(\gamma/z))\tilde{S}_v(\lambda+\eta-\lambda z)}$$

$$I(z,0) = \frac{\begin{array}{l} W(z,0)[\eta(\lambda z-\lambda-\eta)+(\lambda+\gamma/z)\tilde{S}_v(\lambda+\eta-\lambda z)\eta\tilde{S}_b(\lambda-\lambda z)] \\ +[\gamma I_0-p_0-(\gamma/z)\tilde{S}_b(\lambda-\lambda z)I_0](\lambda z-\lambda-\eta) \\ -(\gamma/z)W_0[\tilde{S}_v(\lambda+\eta-\lambda z)-1][\eta\tilde{S}_b(\lambda-\lambda z)] \end{array}}{(\lambda z-\lambda-\eta)[(\lambda+\gamma)-(\lambda+(\gamma/z))\tilde{S}_b(\lambda-\lambda z)]}.$$

Proof. The probability generating function $P(z)$ of number of customers in orbit at an arbitrary time instant can be expressed as follows:

$$P(z) = W(z,0)+I(z,0)+\tilde{P}(z,0)+\tilde{Q}(z,0). \tag{3.14}$$

Using equations (10), (11) and (12) we derive the expressions for $W(z,0), I(z,0), \tilde{P}(z,0), \tilde{Q}(z,0)$ as (complete derivation is given in appendix)

$$W(z,0) = \frac{W_0[\gamma-(\gamma/z)\tilde{S}_v(\lambda+\eta-\lambda z)]+p_0}{[(\lambda+\eta+\gamma)-(\lambda+(\gamma/z))\tilde{S}_v(\lambda+\eta-\lambda z)]} \tag{3.15}$$

$$I(z,0) = \frac{\begin{array}{l} W(z,0)\left[\eta(\lambda z-\lambda-\eta)+(\lambda+\gamma/z)\tilde{S}_v(\lambda+\eta-\lambda z)\eta\times\tilde{S}_b(\lambda-\lambda z)\right] \\ +\left[\gamma I_0-p_0-(\gamma/z)\tilde{S}_b(\lambda-\lambda z)I_0\right](\lambda z-\lambda-\eta)-(\gamma/z) \\ \times W_0\left[\tilde{S}_v(\lambda+\eta-\lambda z)-1\right]\left[\eta\tilde{S}_b(\lambda-\lambda z)\right] \end{array}}{(\lambda z-\lambda-\eta)\left[(\lambda+\gamma)-(\lambda+(\gamma/z))\tilde{S}_b(\lambda-\lambda z)\right]} \tag{3.16}$$

$$\tilde{P}(z,0) = \frac{\begin{array}{l}[\lambda W(z,0) + (\gamma/z)(W(z,0) - W_0)][\tilde{S}_v(\lambda+\eta-\lambda z) - 1] \\ \eta[\tilde{S}_b(\lambda - \lambda z) - 1] \\ +(\lambda + (\gamma/z))I(z,0)[\tilde{S}_b(\lambda - \lambda z) - 1][\lambda z - \lambda - \eta] \\ -(\gamma/z)(\tilde{S}_b(\lambda - \lambda z) - 1)I_0[\lambda z - (\lambda+\eta)]\end{array}}{[\lambda z - (\lambda+\eta)][-\lambda + \lambda z]} \quad (3.17)$$

$$\tilde{Q}(z,0) = \frac{[\lambda W(z,0) + (\gamma/z)(W(z,0) - W_0)][\tilde{S}_v(\lambda+\eta-\lambda z) - 1]}{[\lambda z - (\lambda+\eta)]}. \quad (3.18)$$

Substituting the equations (15)–(18) in (14) we get

$$P(z) = \frac{\begin{array}{l} W(z,0)\begin{bmatrix}(\lambda z - (\lambda+\eta))(-\lambda+\lambda z) + (\lambda+\gamma/z)(\tilde{S}_v(\lambda+\eta-\lambda z)) \\ \eta(\tilde{S}_b(\lambda-\lambda z) - 1) + (\lambda+\gamma/z)\tilde{S}_v(\lambda+\eta-\lambda z)(-\lambda+\lambda z)\end{bmatrix} \\ +I(z,0)\big[(\lambda z - (\lambda+\eta))(-\lambda+\lambda z) \\ +(\lambda + (\gamma/z))(\tilde{S}_b(\lambda - \lambda z) - 1)(\lambda z - (\lambda+\eta))\big] \\ +W_0\Big[(\gamma/z)\eta(\tilde{S}_v(\lambda+\eta-\lambda z) - 1)(1 - \tilde{S}_b(\lambda - \lambda z)) \\ +(\gamma/z)(\tilde{S}_v(\lambda+\eta-\lambda z) - 1)(\lambda z - \lambda)\Big] \\ +I_0\Big[(\gamma/z)(1 - \tilde{S}_b(\lambda - \lambda z))(\lambda z - (\lambda+\eta))\Big]\end{array}}{(\lambda z - (\lambda+\eta))\,(-\lambda+\lambda z)}. \quad (3.19)$$

□

3.1. Stability condition

On using the condition $\lim_{z\to 1} P(z) = 1$ in equation (19) we derive the steady state condition as $\gamma > \frac{\lambda^2}{\mu_b - \lambda}$ (which coincides with the stability condition given by Tien Van Do [24])

4. Performance characteristics

In this section, some useful performance measures of the proposed model such as probability that the server is idle during working vacation and not working vacation, the probability that the server is busy during working vacation and when not in working vacation are derived. And also the mean orbit size L_v during working vacation, the mean orbit size L_b when the server is not on working vacation are studied.

(i) *The probability that the server is idle during working vacation*:
Using equation (15) and applying limit $z \to 1$ we get P_{iv} probability that the server is idle during working vacation as,

$$P_{\text{iv}} = \frac{W_0\gamma(1 - \tilde{S}_v(\eta) + p_0}{(\lambda+\gamma)(1 - \tilde{S}_v(\eta)) + \eta}\cdot \tag{4.1}$$

(ii) *The probability that the server is idle but not on working vacation*:
Using equation (16) and applying limit $z \to 1$, we get P_{inv} probability that the server is idle and not on working vacation as

$$P_{\text{inv}} = \frac{\begin{array}{l} W_0\gamma\eta\left[(\lambda S_{v1} + \eta S_{b1})(1 - \tilde{S}_v(\eta)) - \eta\right] \\ +I_0\gamma\eta\left[(\lambda+\gamma)\left(1 - \tilde{S}_v(\eta)\right) + \eta][S_{b1} - 1\right] \\ +p_0\left[(\lambda+\gamma)(\eta S_{b1}(\tilde{S}_v(\eta) - 1) + \lambda\left(\tilde{S}_v(\eta)\right)\right] + \gamma(\eta - \lambda) \end{array}}{\eta\gamma[(\lambda+\gamma)(1 - \tilde{S}_v(\eta)) + \eta][S_{b1} - 1]}\cdot \tag{4.2}$$

(iii) *The probability that the server is busy during working vacation*:
Using equation (18) and applying limit $z \to 1$ we get P_{bv}, the probability that the server is busy during working vacation as

$$P_{\text{bv}} = \frac{\left[1 - \tilde{S}_v(\eta)\right][(\lambda+\gamma)p_0 - \eta\gamma W_0]}{\eta\left[(\lambda+\gamma)(1 - \tilde{S}_v(\eta)) + \eta\right]}\cdot \tag{4.3}$$

(iv) *The probability that the server is busy but not on working vacation*:
Using equation (17) and applying limit $z \to 1$ we get P_{bnv}, the probability that the server is busy when not on working vacation as

$$P_{\text{bnv}} = \{[(\lambda+\gamma)(1 - \tilde{S}_v(\eta)) + \eta_0][1 - \tilde{S}_v(\eta)] + (\lambda+\gamma)P_{\text{inv}} - \gamma\}(S_{b1}/\lambda) \tag{4.4}$$

where $S_{b1} = \lambda E(S_b); S_{v1} = \lambda\int_0^\infty t\mathrm{e}^{-\eta t}s_v(t)\mathrm{d}t$.

(v) *The mean orbit size L_Q:*
Let L_v and L_b denote the mean orbit size during working vacation and regular busy period respectively. Since the expressions are too large the numerical values of L_v and L_b are calculated using Mathematica.

$$L_v = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}[W(z,0] + \tilde{Q}(z,0)]$$
$$L_b = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}[I(z,0) + \tilde{P}(z,0)]\cdot$$

Hence the mean orbit size is given by $L_Q = L_v + L_b$.

(vi) *Mean waiting time in the retrial queue*: $W = \frac{L_Q}{\lambda}$.

5. Special cases

Further, by specifying service time random variables as Exponential, Erlang and Hyper Exponential distribution, some special cases of this model are discussed below:

Case I. Single server retrial queue with working vacation (Exponential service time both for the service during working vacation and for the service when the server is not on working vacation) and constant retrial policy.

If the service times are assumed to be Exponential with probability density $s_i(x) = u_i e^{-u_i x}$ where $u_i (i = 1, 2)$ is the parameter and $u_i > 0, x \geq 0$ then

$$\tilde{S}_b(\lambda - \lambda z) = \left(\frac{u_1}{u_1 + \lambda - \lambda z}\right)$$
$$\tilde{S}_v(\lambda + \eta - \lambda z) = \left(\frac{u_2}{u_2 + \lambda + \eta - \lambda z}\right).$$

Substituting in (19), the *PGF* of the retrial queue size distribution for single server retrial queue with working vacation and constant retrial policy is given

$$P(z) = \frac{\begin{gathered} W(z,0)\left[\begin{gathered}(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+[\lambda+\gamma/z]\left(\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}-1\right)\right) \\ \eta\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)-1)+(\lambda+\gamma/z)\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}-1\right)(-\lambda+\lambda z)\end{gathered}\right] \\ +I(z,0)\left[(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+(\lambda\right. \\ \left.+(\gamma/z))\left(\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)-1\right)(\lambda z-(\lambda+\eta))\right] \\ +W_0\left[(\gamma/z)\eta\left(\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}\right)-1\right)\left(1-\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)\right)\right. \\ \left.+(\gamma/z)\left(\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}\right)-1\right)(\lambda z-\lambda)\right] \\ +I_0\left[(\gamma/z)\left(1-\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)\right)(\lambda z-(\lambda+\eta))\right]\end{gathered}}{(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)} \tag{5.1}$$

where

$$W(z,0) = \frac{W_0\left[\gamma - (\gamma/z)\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}\right)\right] + p_0}{(\lambda+\eta+\gamma) - (\lambda + (\gamma/z))\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}\right)}$$

$$I(z,0)=\frac{\begin{array}{l}W(z,0)\left[\eta(\lambda z-\lambda-\eta)+(\lambda+\gamma/z)\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}-1\right)\eta\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)\right]\\+[\gamma I_0-p_0-(\gamma/z)\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)I_0](\lambda z-\lambda-\eta)-(\gamma/z)\\\times W_0[\left(\frac{u_2}{u_2+\lambda+\eta-\lambda z}\right)-1]\left[\eta\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)\right]\end{array}}{(\lambda z-\lambda-\eta)\left[(\lambda+\gamma)-(\lambda+(\gamma/z))\left(\frac{u_1}{u_1+\lambda-\lambda z}\right)\right]}.$$

Case II. Single server retrial queue with working vacation (Erlang service time both for the service during working vacation and for the service when the server is not on working vacation) and constant retrial policy.

If the service times are assumed to be Erlang with probability density function $s_i(x)=\frac{(k_iu_i)^{k_i}x^{k_i-1}e^{-k_iu_ix}}{(k_i-1)!}; i=1,2; u_i>0; x>0$ and k_i is the positive integer and u_i is the parameter then

$$\tilde{S}_b(\lambda-\lambda z)=\left(\frac{u_1k_1}{u_1k_1+\lambda-\lambda z}\right)^{k_1}$$
$$\tilde{S}_v(\lambda+\eta-\lambda z)=\left(\frac{u_2k_2}{u_2k_2+\lambda+\eta-\lambda z}\right)^{k_2}.$$

Substituting in (19), the PGF of the retrial queue size distribution for single server retrial queue with working vacation and constant retrial policy is given by

$$P(z)=\frac{\begin{array}{l}W(z,0)\left[\begin{array}{l}(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+[\lambda+\gamma/z]\left(\left(\frac{u_2k_2}{u_2k_2+\lambda+\eta-\lambda z}\right)^{k_2}-1\right)\\\eta(\left(\frac{u_1k_1}{u_1k_1+\lambda-\lambda z}\right)^{k_1}-1)+(\lambda+\gamma/z)\\\qquad\times\left[\left(\frac{u_2k_2}{u_2k_2+\lambda+\eta-\lambda z}\right)^{k_2}-1\right](-\lambda+\lambda z)\end{array}\right]\\+I(z,0)\left[(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)+(\lambda+(\gamma/z))\right.\\\left.\times\left(\left(\frac{u_1k_1}{u_1k_1+\lambda-\lambda z}\right)^{k_1}-1\right)(\lambda z-(\lambda+\eta))\right]\\+W_0\left[(\gamma/z)\eta\left(\left(\frac{u_2k_2}{u_2k_2+\lambda+\eta-\lambda z}\right)^{k_2}-1\right)\left(1-\left(\frac{u_1k_1}{u_1k_1+\lambda-\lambda z}\right)^{k_1}\right)\right.\\\left.+(\gamma/z)\left(\left(\frac{u_2k_2}{u_2k_2+\lambda+\eta-\lambda z}\right)^{k_2}-1\right)(\lambda z-\lambda)\right]\\+I_0\left[(\gamma/z)\left(1-\left(\frac{u_1k_1}{u_1k_1+\lambda-\lambda z}\right)^{k_1}\right)(\lambda z-(\lambda+\eta))\right]\end{array}}{(\lambda z-(\lambda+\eta))(-\lambda+\lambda z)}. \quad (5.2)$$

Case III. Single server retrial queue with working vacation (Hyper Exponential service time both for the service during working vacation and for the service when the server is not on working vacation) and constant retrial policy.

If the service times are assumed to be Hyper Exponential with probability density function $s(x) = cu\mathrm{e}^{-ux} + (1-c)w\mathrm{e}^{-wx}$ where $x > 0, u > 0, w > 0; 0 \leq c \leq 1$, then

$$\tilde{S}_b(\lambda - \lambda z) = \left(\frac{u_1 c}{u_1 + \lambda - \lambda z}\right) + \left(\frac{w_1(1-c)}{w_1 + (\lambda - \lambda z)}\right)$$

$$\tilde{S}_v(\lambda + \eta - \lambda z) = \left(\frac{u_2 c}{u_2 + \lambda + \eta - \lambda z}\right) + \left(\frac{w_2(1-c)}{w_2 + (\lambda + \eta - \lambda z)}\right).$$

Substituting in (19), the PGF of the retrial queue size distribution for single server retrial queue with working vacation and constant retrial policy is obtained.

6. Numerical results

In this section to justify the theoretical results obtained, we present some numerical results. To study the effect of arrival rate λ and retrial rate γ on the mean orbit size L_Q and the mean waiting time W the following notations are used and some assumptions are made:

(i) Average arrival rate $\lambda = 0.3$.
(ii) Service rate during working vacation μ_v.
(iii) Regular service rate (when the server is not on working vacation)μ_b.
(iv) Vacation duration is exponential with parameter η.
(v) Retrial rate γ.

Table 1, Figures 1 and 2 represent the effect of retrial rate γ on the mean orbit size and the mean waiting time W. The service times are considered as exponential, Erlang-2 and Hyper exponential with parameters $\lambda = 0.3$, $\mu_v = 0.2$, $\mu_b = 1$, and $\eta = 2$.

It is observed that:

* Mean orbit size is decreasing when retrial rate increases.
* Mean waiting is decreasing when retrial rate increases.

Table 2, Figures 3 and 4 represent the effect of arrival rate λ on the mean orbit size and the mean waiting time W. The service times are considered as Exponential, Erlang-2 and Hyper exponential with parameters $\gamma = 0.6$, $\mu_v = 0.2$, $\mu_b = 1$, and $\eta = 2$. It is observed that:

* Mean orbit size is increasing when arrival rate increases.
* Mean waiting time is increasing when arrival rate increases.

Table 3, Figures 5 and 6 represent the effect of service rate μ_b on the mean orbit size and the mean waiting time W. The service times are considered as Exponential, Erlang-2 and Hyper exponential with parameters $\lambda = 0.3$, $\gamma = 0.6$,$\mu_v = 0.2$ and $\eta = 2$. It is observed that:

* Mean orbit size is decreasing when service rate μ_b increases.
* Mean waiting time is decreasing when service rate μ_b increases.

Table 4 represents the effect of retrial rate γ on the probability that the server is busy during working vacation with $\lambda = 0.1$, $\mu_v = 0.2$, $\mu_b = 0.9$ and $\eta = 2$ when the service time distribution follow Exponential, Erlang-2 and Hyper exponential respectively. It is observed that probability that the server is busy during working vacation is increasing when retrial rate increases.

Table 5 represents the effect of retrial rate γ on the probability that the server is busy and not on working vacation with $\lambda = 0.1$, $\mu_v = 0.2$, $\mu_b = 0.9$ and $\eta = 2$ when the service time distribution follow Exponential, Erlang-2 and Hyper exponential respectively. It is observed that probability that the server is busy and not on working vacation is increasing when retrial rate increases.

Table 6 represents the effect of retrial rate γ on the probability that the server is idle during working vacation with $\lambda = 0.1$, $\mu_v = 0.2$, $\mu_b = 0.9$ and $\eta = 2$ when the service time distribution follow Exponential, Erlang-2 and Hyper exponential respectively. It is observed that probability that the server is not occupied during working vacation is decreasing when retrial rate increases.

Table 7 represents the effect of retrial rate γ on the probability that the server is idle during working vacation with $\lambda = 0.1$, $\mu_v = 0.2$, $\mu_b = 0.9$ and $\eta = 2$ when the service time distribution follow Exponential, Erlang-2 and Hyper exponential respectively. It is observed that probability that the server is not occupied and not on working vacation is decreasing when retrial rate increases.

7. Conclusion

In this paper a single server retrial queue with general retrial time, single working vacation and constant retrial policy is analyzed under the condition of stability. Some system performance measures are computed in steady state. Numerical illustrations are also presented. For future research one can consider the same model when the vacation time follows a phase-type distribution.

TABLE 1. Retrial rate γ versus Mean Orbit Size L_Q and mean waiting time W.

γ	Exponential		Erlang-2		Hyper-exponential	
	L_Q	W	L_Q	W	L_Q	W
0.5	0.5972	1.9907	0.5321	1.7738	0.5919	1.9730
0.6	0.4828	1.6093	0.4249	1.4164	0.4796	1.5986
0.7	0.4058	1.3525	0.3525	1.1749	0.4043	1.3476
0.8	0.3495	1.165	0.2994	0.9981	0.3495	1.1651
0.9	0.3061	1.0201	0.2583	0.8609	0.3074	1.0245
1	0.271	0.9032	0.2249	0.7499	0.2735	0.9116
1.1	0.2418	0.8059	0.1971	0.6571	0.2454	0.8179
1.2	0.2168	0.7227	0.1733	0.5775	0.2214	0.7381
1.3	0.195	0.6501	0.1524	0.5079	0.2006	0.6686
1.4	0.1757	0.5857	0.1338	0.4461	0.1822	0.6072
1.5	0.1583	0.5278	0.1171	0.3902	0.1656	0.5521

TABLE 2. Arrival rate λ versus mean orbit Size L_Q and mean waiting time W.

λ	Exponential		Erlang-2		Hyper-exponential	
	L_Q	W	L_Q	W	L_Q	W
0.1	0.0165	0.165	0.0112	0.1121	0.0149	0.1487
0.15	0.0685	0.4567	0.0566	0.3771	0.0421	0.2809
0.2	0.1664	0.8321	0.1429	0.7148	0.0909	0.4543
0.25	0.3362	1.3448	0.2929	1.1714	0.1767	0.7067
0.3	0.6309	2.0133	0.5522	1.8407	0.3369	1.1229
0.35	1.1784	3.3669	1.0312	2.9464	0.8166	2.3333
0.4	2.3793	5.9482	2.0759	5.1899	1.519	3.7975
0.45	6.4691	14.3757	5.6154	12.4786	4.8432	10.7627

TABLE 3. Service rate μ_b versus mean orbit Size L_Q and mean waiting time W.

μ_b	Exponential		Erlang-2		Hyper-exponential	
	L_Q	W	L_Q	W	L_Q	W
1.1	0.3842	1.2806	0.3397	1.1324	0.3310	1.1032
1.2	0.3145	1.0483	0.2791	0.9304	0.2397	0.7991
1.3	0.2629	0.8765	0.2340	0.7801	0.1803	0.6011
1.4	0.2235	0.7449	0.1993	0.6644	0.1392	0.4641
1.5	0.1924	0.6413	0.1719	0.5729	0.1094	0.3648
1.6	0.1674	0.5579	0.1496	0.4988	0.0871	0.2901
1.7	0.1468	0.4895	0.1313	0.4378	0.0697	0.2322
1.8	0.1297	0.4324	0.1160	0.3867	0.0559	0.1864
1.9	0.1153	0.3842	0.1030	0.3434	0.0448	0.1493
2.0	0.1029	0.3430	0.0919	0.3063	0.0356	0.1187

TABLE 4. Retrial rate γ versus the probability that the server is busy during working vacation P_{bv}.

γ	P_{bv}		
	Exponential	Erlang-2	Hyper exponential
0.2	0.0393	0.0392	0.0393
0.3	0.0400	0.0399	0.0399
0.4	0.0403	0.0403	0.0403
0.5	0.0404	0.0404	0.0404
0.6	0.0404	0.0404	0.0404
0.7	0.0404	0.0404	0.0404
0.8	0.0404	0.0404	0.0404
0.9	0.0404	0.0404	0.0404

TABLE 5. Retrial rate γ versus the probability that the server is busy during but not on working vacation P_{bnv}.

γ	P_{bnv}		
	Exponential	Erlang-2	Hyper exponential
0.2	0.0095	0.0098	0.0097
0.3	0.0144	0.0149	0.0148
0.4	0.0187	0.0193	0.0192
0.5	0.0224	0.0229	0.0228
0.6	0.0253	0.0258	0.0257
0.7	0.0275	0.0280	0.0279
0.8	0.0291	0.0296	0.0295
0.9	0.0303	0.0306	0.0306

TABLE 6. Retrial rate γ versus the probability that the server is idle during working vacation P_{iv}.

γ	P_{iv}		
	Exponential	Erlang-2	Hyper exponential
0.2	0.0032	0.0034	0.0033
0.3	0.0031	0.0033	0.0033
0.4	0.0030	0.0032	0.0032
0.5	0.0030	0.0032	0.0032
0.6	0.0029	0.0031	0.0030
0.7	0.0029	0.0031	0.0030
0.8	0.0028	0.0030	0.0029
0.9	0.0028	0.0029	0.0029

TABLE 7. Retrial rate γ versus the probability that the server is idle but not on working vacation P_{inv}.

γ	P_{inv}		
	Exponential	Erlang-2	Hyper exponential
0.2	0.0711	0.1030	0.0961
0.3	0.0609	0.1030	0.0944
0.4	0.0499	0.1030	0.0917
0.5	0.0389	0.1020	0.0888
0.6	0.0279	0.1007	0.0857
0.7	0.0168	0.0996	0.0826
0.8	0.0057	0.0983	0.0793
0.9	0.0056	0.0968	0.0758

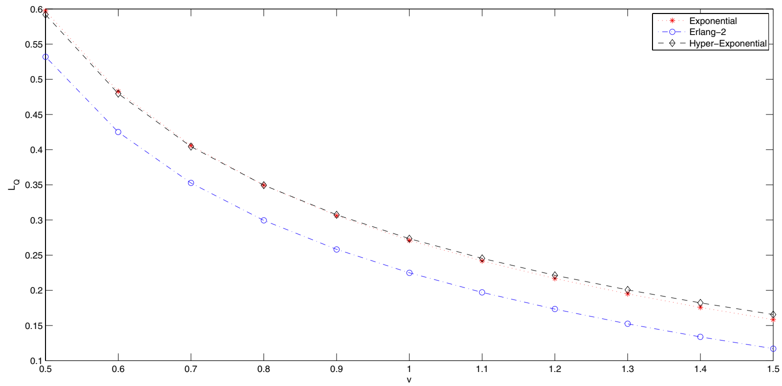

FIGURE 1. Retrial rate γ verses mean orbit size L_Q.

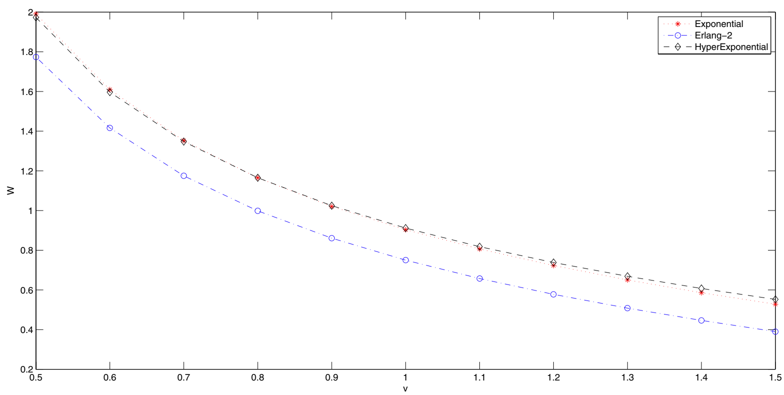

FIGURE 2. Retrial rate γ verses mean waiting time W.

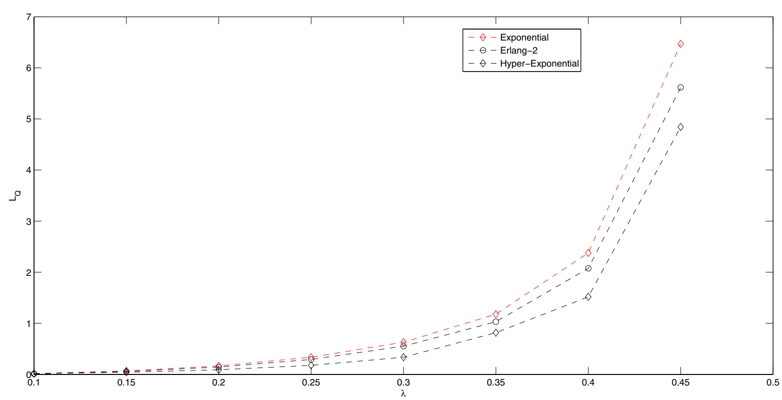

FIGURE 3. Arrival rate λ verses mean orbit size L_Q.

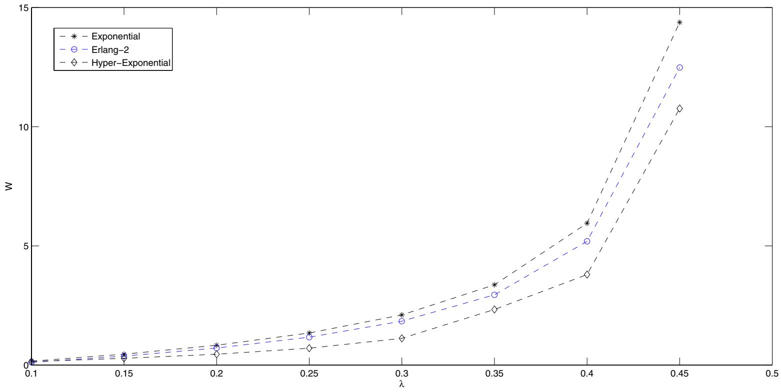

FIGURE 4. Arrival rate λ verses mean waiting time W.

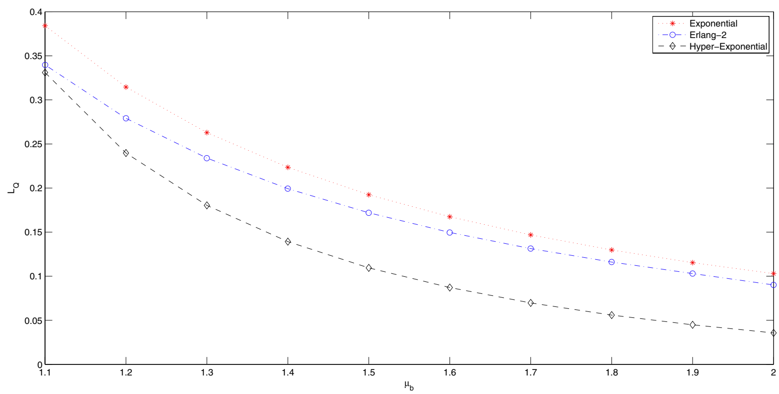

FIGURE 5. Service rate μ_b verses mean orbit size L_Q.

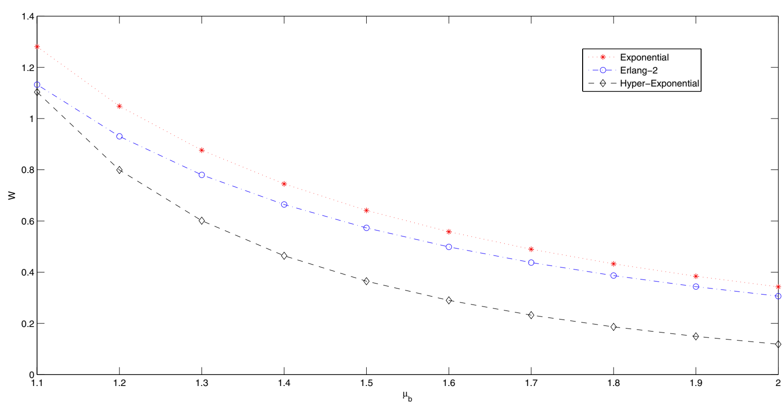

FIGURE 6. Service rate μ_b verses mean waiting time W.

Appendix A.

The derivation of equations (15), (16), (17) and (18)

Substituting $\theta = (\lambda + \eta - \lambda z)$ in equation (12) we have

$$Q(z,0) = \lambda W(z,0)\tilde{S}_v(\lambda + \eta - \lambda z) + (\gamma/z)(W(z,0) - W_0)\tilde{S}_v(\lambda + \eta - \lambda z). \quad \text{(A.1)}$$

Substituting for $Q(z,0)$ from equation (26) into equation (12) we have

$$\tilde{Q}(z,0) = \frac{[\lambda W(z,0) + (\gamma/z)(W(z,0) - W_0)][\tilde{S}_v(\lambda + \eta - \lambda z) - 1]}{\lambda z - (\lambda + \eta)} \quad \text{(A.2)}$$

which is equation(18).

Substituting for $Q(z,0)$ from equation (26) into equation (10), we get

$$W(z,0) = \frac{W_0[\gamma - (\gamma/z)\tilde{S}_v(\lambda + \eta - \lambda z)] + P_0}{[(\lambda + \eta + \gamma) - (\lambda + (\gamma/z))\tilde{S}_v(\lambda + \eta - \lambda z)]} \quad \text{(A.3)}$$

which is equation (15).

Substituting for $\theta = \lambda - \lambda z$ in the equation (13) we obtain

$$P(z,0) = (\lambda + (\gamma/z))\tilde{S}_b(\lambda - \lambda z)I(z,0) - (\gamma/z)\tilde{S}_b(\lambda - \lambda z)I_0 + \tilde{Q}(z,0)\eta\tilde{S}_b(\lambda - \lambda z). \quad \text{(A.4)}$$

Substituting for $\tilde{Q}(z,0)$ from equation (27) into equation (29) we get,

$$P(z,0) = \frac{\begin{pmatrix} \lambda + (\gamma/z))(\tilde{S}_b(\lambda - \lambda z)I(z,0) - \\ (\gamma/z)\tilde{S}_b(\lambda - \lambda z)I_0 + [\lambda W(z,0) + (\gamma/z)(W(z,0) - W_0)] \\ \times[\tilde{S}_v(\lambda + \eta - \lambda z) - 1]\eta\tilde{S}_b(\lambda - \lambda z) \end{pmatrix}}{\lambda z - (\lambda + \eta)}. \quad \text{(A.5)}$$

Substituting for $P(z,0)$ from equation (30) into equation (13) we obtain

$$\tilde{P}(z,0) = \frac{\begin{array}{l} [\lambda W(z,0) + (\gamma/z)(W(z,0) - W_0)] \\ \times[\tilde{S}_v(\lambda + \eta - \lambda z) - 1]\eta[\tilde{S}_b(\lambda - \lambda z) - 1] \\ +(\lambda + (\gamma/z))I(z,0)[\tilde{S}_b(\lambda - \lambda z) - 1][\lambda z - \lambda - \eta] - (\gamma/z) \\ \times(\tilde{S}_b(\lambda - \lambda z) - 1)I_0[\lambda z - (\lambda + \eta)] \end{array}}{[\lambda z - (\lambda + \eta)][-\lambda + \lambda z]} \quad \text{(A.6)}$$

which is equation (17).

Substituting for $P(z,0)$ from equation (30) into equation (11) we get,

$$I(z,0) = \frac{\begin{array}{l} W(z,0)[\eta(\lambda z - \lambda - \eta) + (\lambda + \gamma/z)\tilde{S}_v(\lambda + \eta - \lambda z)\eta\tilde{S}_b(\lambda - \lambda z)] \\ +[\gamma I_0 - p_0 - (\gamma/z)\tilde{S}_b(\lambda - \lambda z)I_0](\lambda z - \lambda - \eta) \\ -(\gamma/z)W_0[\tilde{S}_v(\lambda + \eta - \lambda z) - 1][\eta\tilde{S}_b(\lambda - \lambda z)] \end{array}}{(\lambda z - \lambda - \eta)[(\lambda + \gamma) - (\lambda + (\gamma/z))\tilde{S}_b(\lambda - \lambda z)]} \quad \text{(A.7)}$$

which is equation (16).

References

[1] G.I. Falin and J.K.C. Templeton, Retrial queues, Chapman and Hall, London (1997).
[2] J.R. Artalejo, Accessible bibliography on retrail queues. *Math. Comput. Model.* **30** (1999) 1−6.
[3] J.R. Artalejo, A classified bibliography of research on retrial queues: Progress in 1990, *Top* **7** (1999) 187−211.
[4] B.D. Choi and K.K. Park, The M/G/1 Retrial queue with Bernoulli schedule. *Queueing Syst.* **7** (1990) 219−227.
[5] B.D. Choi, K.B. Choi and Y.W. Lee, M/G/1 Retrial queueing system with two types of calls and finite capacity. *Queueing Syst.* **19** (1995) 215−229.
[6] B.D. Choi and Y. Chang, Single server retrial queues with priority calls. *Math. Comput. Model.* **30** (1999) 7−32.
[7] J.R. Artalejo and Gomez−Corral, *Retrial queueing systems. A Comput. Approach.* Springer-Verlag, Berlin (2008).
[8] H. Takagi, *Vacation and priority systems, Part I, Queueing analysis. A foundation of performance evaluation*, Vol. 1, North-Holland, Amsterdam (1991).
[9] H. Li and T. Yang, A single server retrial queue with server vacation and a finite number of input sources. *Eur. J. Oper. Res.* **85** (1995) 149−160.
[10] J.R. Artalejo, Analysis of an M/G/1 queue with constant repeated attempts and server vacations. *Comput. Oper. Res.* **24** (1997) 493−504.
[11] B.T. Doshi, Queueing systems with vacations a survey. *Queueing Syst.* **1** (1986) 29−66.
[12] B.T. Doshi, An M/G/1 queue with variable vacation. *Proc. Int. Conf. Performance Model.*, Sophia Antipolis, France (1985).
[13] Y. Baba, On the M^X/G/1 queue with vacation time. *Oper. Res. Lett.* **5** (1986) 93−98.
[14] M. Senthilkumar and R. Arumuganathan, On the single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service. *Quality Technology and Quantitative Management* **5** (2008) 145−160.
[15] H.W. Lee, S.S. Lee, J.O. Park and K.C. Chae, Analysis of M^X/G/1 queue with N-policy and multiple vacations. *J. Appl. Prob.* **31** (1994) 467−496.
[16] S.S. Lee, H.W. Lee and K.C. Chae, Batch arrival queue with N-policy and single vacation. *Comput. Oper. Res.* **22** (1995) 173−189.
[17] G.V. Krishna Reddy, R. Nadarajan and R. Arumuganathan, Analysis of a bulk queue with N- policy multiple vacations and setup times. *Comput. Oper. Res.* **25** (1998) 957−967.
[18] R. Arumuganathan, T. Judeth Malliga and A. Rathinasamy. Steady state analysis of non-Markorian bulk queueing system with N-Policy and different types of vacations. *Int. J. Modern Math.* **3** (2008) 47−66.
[19] M. Haridass and R. Arumuganathan, Analysis of a M^X/G/1 queueing system with vacation interruption. *RAIRO-Oper. Res.* **46** (2012) 304−334.
[20] L.D. Servi and S.G. Finn, M/M/1 queues with working vacation (M/M/1/Wv). *Performance Evaluation* **50** (2002) 41–52.
[21] J. Kim D. Choi and K. Chae, Analysis of queue length distribution of the M/G/1 queue with working vacations, *Int. Conf. Statistics and related fields*, Hawaii (2003).
[22] D. Wu. and H. Takagi, M/G/1 queue with multiple working vacations. *Performance Evaluations* **63** (2006) 654−681.
[23] J.L. Li, N. Tian and Z.G. Zhang, Analysis of the M/G/1 queue with exponentially distributed working vacations a matrix analytic approach. *Queueing Syst.* **61** (2009) 139−166.
[24] Do. Tien Van, M/M/1 retrial queue working vacation. *Acta Inf.* **47** (2009) 67−75.
[25] N. Limnios and Gh. Oprisan, *Semi-Markov Process and Reliability-Statistics for Industry and Technology*, Birkhauser Boston, Springer (2001).