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FLUID LIMITS FOR THE QUEUE LENGTH OF JOBS IN MULTISERVER OPEN QUEUEING NETWORKS

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Abstract. The object of this research in the queueing theory is a theorem about the Strong-Law-of-Large-Numbers (SLLN) under the conditions of heavy traffic in a multiserver open queueing network. SLLN is known as a fluid limit or fluid approximation. In this work, we prove that the long-term average rate of growth of the queue length process of a multiserver open queueing network under heavy traffic strongly converges to a particular vector of rates. SLLN is proved for the values of an important probabilistic characteristic of the multiserver open queueing network investigated as well as the queue length of jobs.

Keywords. Mathematical models of information systems, performance evaluation, queueing theory, multiserver open queueing network, heavy traffic, limit theorem, queue length of jobs.

Mathematics Subject Classification. 60K25, 60G70, 60F17.

1. Introduction

The paper is devoted to the analysis of queueing systems in the context of the network and communications theory. We investigate SLLN about the queue length of customers (jobs) in a multiserver open queueing network under the conditions of heavy traffic. Queueing networks have been extensively used for the analysis of manufacturing and transportation systems, and for computer and communications networks. Therefore, many approximation methods have emerged, and SLLN is among them.

The investigation of delays arising in communications and computer systems is a very complicated problem which has not yet been solved in the general case.

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A valuable progress in this area has been achieved for models based on Kleinrock's hypothesis on the independence of transmission times of messages at different nodes (see [3–5, 20, 22, 24, 39]). The most fruitful approach to the calculation of delays in communication systems is based on limit theorems for heavy traffic and light traffic regimes. Some results have been obtained both for systems with finite (see Kelly [21, 24]) and infinite (see [4, 9, 11, 12, 14, 15, 17, 18, 32, 33, 37–39]) waiting rooms. The theoretical base for heavy traffic limit theorems includes the weak convergence results for stochastic processes (see [1,2,30,35]), as well as the martingale approach to limit theorems (see [10,23,26]). The limit theorems usually state that, in the heavy traffic regime, properly normalized random variables (or random processes) describing queue lengths or waiting times converge in distribution to a normal random variable (or certain diffusion processes).

The first results for open queueing networks in heavy traffic were obtained by Iglehart and Whitt [17, 18]. They considered a single station, multiserver and acyclic networks of queues. The limit process for networks in heavy traffic was described as a complicated functional of multidimensional Brownian motion. Harrison [11] considered the heavy traffic approximation to the stationary distribution of the waiting times in single server queues in series. His limit process was also given as a complicated functional of Brownian motion. In reference [12], Harrison again considered the diffusion approximation to tandem queues, described the limit process and found analytical solutions in several special cases. Reiman [31] proved the heavy traffic limit theorems for the queue length process associated with open queueing networks. These theorems state that the limit process is a reflected Brownian motion on the nonnegative orthant with constant directions to each boundary hyperplane. Harrison and Reiman [14] considered the properties of distribution of the multidimensional reflected Brownian motion. Harrison and Williams [15] also analysed Brownian models of open queueing networks with homogeneous customer populations. Reiman [32] studied a multiclass feedback queue in heavy traffic. A network of priority queues with one bottleneck station in heavy traffic was considered by Reiman and Simon [33]. Note that the theory of heavy traffic analysis is rather well developed for systems that satisfy the Kleinrock hypothesis. Without this hypothesis the complexity of the problem dramatically increases.

Limit theorems (diffusion approximations) and SLLN for the queueing system under the conditions of heavy traffic are closely connected (they belong to the same field of research, *i.e.*, investigations on the theory of queueing systems in heavy traffic). Therefore, first we shall try to trace the development of research on the general theory of a queueing system in heavy traffic. There is a vast literature on the diffusion approximation. Readers are referred to [7,8,25,39] for a general survey of the subject. The present work extends the studies by Iglehart and Whitt [17,18] on a single station of multiserver queues, and by Reiman [31], Johnson [19], Chen and Mandelbaum [6] on networks of single server queues. Other closely related papers are by Harrison and Lemoine [13] on networks of infinite server queues, and Whitt [40] for a $GI/G/\infty$ queue.

The natural setting for functional limit theorems in this paper is the weak convergence of probability measures on the function space $D[0,1]$. Since an excellent treatment of this subject has been recently published by Billingsley [1], we shall only make a few remarks here about our terminology and notation. Stochastic processes characterizing the queueing system give rise to sequences of random functions in D, the space of all right-continuous functions on $[0,1]$ having left limits and endowed with Skorohod metric, d. In [1], this metric is denoted by d_0. With d, D is a complete, separable metric space. Let $\mathcal{D}$ be the class of Borel sets of D. Then, if P_n and P are probability measures on $\mathcal{D}$ which satisfy

$$\lim_{n\to\infty}\int_D f\mathrm{d}P_n=\int_D f\mathrm{d}P$$

for every bounded, continuous, real-valued function f on D, we say that P_n *weakly converges* to P, as $n\to\infty$, and write $P_n\Rightarrow P$. A *random function* X is a measurable mapping from some probability space $(\Omega,\mathcal{B},\mathcal{P})$ into D with the *distribution* $P=\mathcal{P}X^{-1}$ on $(D,\mathcal{D})$. We say that a sequence of random functions $\{X_n\}$ weakly converges to the random function X, and write $X_n\Rightarrow X$, if the distribution P_n of X_n converges to the distribution P of X. A sequence of random functions $\{X_n\}$ weakly converges to X in probability if X_n and X are defined on a common domain and for all $\varepsilon>0$, $P\{\mathrm{d}(X_n,X)\geq\varepsilon\}\to 0$. When X is a constant function (not random), the convergence in probability is equivalent to a weak convergence. In such cases, we write $\mathrm{d}(X_n,X)\Rightarrow 0$ or $X_n\Rightarrow X$. If X_n and Y_n have a common domain, we also write $\mathrm{d}(X_n,Y_n)\Rightarrow 0$, $P\{\mathrm{d}(X_n,Y_n)>\varepsilon\}\to 0$ as for all $\varepsilon>0$. We also use the uniform metric ρ which is defined by $\rho(x,y)=\sup_{0\leq t\leq 1}|x(t)-y(t)|$ for $x,y\in D$. Also, note that $\mathrm{d}(x,y)\leq\rho(x,y)$ for $x,y\in D$.

Next, we state two extremely useful theorems for obtaining weak convergence results in applications. The first one has come to be known as the "converging together theorem". For it we assume that X_n and Y_n are defined on a common domain and take values in a separable metric space (S,m). This result can be found in [1], Theorem 4.1.

Theorem 1.1.

$$\textit{If } X_n\Rightarrow X \textit{ and } \mathrm{d}(X_n,Y_n)\Rightarrow 0,\ \textit{then } Y_n\Rightarrow X. \tag{1.1}$$

Now, suppose h is a measurable mapping of S into S', a second metric space with Borel sets $\mathcal{B}$. Each probability measure P on $(S,\mathcal{B})$ induces a unique probability measure $Ph^{-1}(A)=P(h^{-1}A)$ on $(S',\mathcal{B}')$ for $A\in\mathcal{B}'$. Let D_h be a set of discontinuities of h. The next result, known as a continuous mapping theorem, is an analog of the Mann–Wald theorem for Euclidean spaces (*see* [1], *Thm.* 5.1). *Define $h\circ X=h(X), X\in D$.*

Theorem 1.2.

$$\textit{If } X_n\Rightarrow X \textit{ and } P\{X\in D_h\}=0,\ \textit{then } h\circ X_n\Rightarrow h\circ X. \tag{1.2}$$

In practice we use this result as follows. First we show $X_n \Rightarrow X$, often by just quoting the known results. Then, we find an appropriate mapping h which gives us the random elements we are really interested in, $h \circ X_n$, and finally, we apply (1.2).

So, we prove here SLLN for the queue length of jobs in a multiserver open queueing network under heavy traffic conditions. The main tool for the analysis of these queueing systems in heavy traffic is a functional limit theorem for a complex renewal process (the proof can be found in [1]).

2. The network model

Consider a network of j stations, indexed by $j = 1, 2, \ldots, J$, and the station j with c_j servers, indexed by $(j, 1), \ldots, (j, c_j)$. Description of the primitive data and construction of processes of interest are the focus of this section. No probability space will be mentioned in this section, and of course, one can always think that all the variables and processes are defined on the same probability space.

First, $\{u_j(e), e \geq 1\}, j = 1, 2, \ldots, J$, are J sequences of exogenous interarrival times, where $u_j(e) \geq 0$ is the interarrival time between the $(e-1)$-job and the eth job that arrive at the station j exogenously (from the outside of the network). Define $U_j(0) = 0$, $U_j(n) = \sum_{e=1}^{n} u_j(e)$, $n \geq 1$ and $A_j(t) = \sup\{n \geq 0 : U_j(n) \leq t\}$, where $A_j = \{A_j(t), t \geq 0\}$ is called an exogenous arrival process at the station j, *i.e.*, $A_j(t)$ counts the number of jobs that arrived at the station j from the outside of the network.

Second, $\{v_{jk_j}(e), e \geq 1\}$, $j = 1, 2, \ldots, J$, $k_j = 1, 2, \ldots, c_j$, are $c_1 + \ldots + c_J$ sequences of service times, where $v_{jk_j}(e) \geq 0$ is the service time for the e-th job served by the server k_j at the station j. Define $V_{jk_j}(0) = 0$, $V_{jk_j}(n) = \sum_{e=1}^{n} v_{jk_j}(e)$, $n \geq 1$ and $x_{jk_j}(t) = \sup\{n \geq 0 : V_{jk_j}(n) \leq t\}$, where $x_{jk_j} = \{x_{jk_j}(t), t \geq 0\}$ is called a service process of the server k_j at the station j, *i.e.*, $x_{jk_j}(t)$ counts the number of services completed by the server k_j at the station j during the server's busy time. We write $\mu_{jk_j} = \left(E\left[v_{jk_j}(e)\right]\right)^{-1} > 0$, $\sigma_{jk_j} = D\left(v_{jk_j}(e)\right) > 0$ and $\lambda_j = \left(E\left[u_j(e)\right]\right)^{-1} > 0$, $a_j = D\left(u_j(e)\right) > 0$, $j = 1, 2, \ldots, k$; with all of these terms assumed finite.

In addition, let $\tilde{\tau}_j(t)$ be the total number of jobs routed to the jth station of the network in the interval $[0, t]$, $\tau_j(t)$ be the total number of jobs after service departure from the jth station of the network in the interval $[0, t]$, $\tilde{\tau}_{jk_j}(t)$ be the total number of jobs routed to the k_j server at the jth station of the network in the interval $[0, t]$, let $\tau_{jk_j}(t)$ be the total number of customers after service departure from the k_j server at the jth station of the network in the interval $[0, t]$, and $\tau_{ijk_i}(t)$ be the total number of jobs after service departure from the k_i server at the ith station of the network and routed to the k_j server of the jth station of the network in the interval $[0, t]$. Let p_{ij} be a probability of the job after service at the ith station of the network routed to the jth station of the network. Denote $p_{ijk_i}^t = \dfrac{\tau_{ijk_i}(t)}{\tau_{ik_i}(t)}$ as part of the total number of jobs which, after service at the k_i

server of the ith station of the network, are routed to the jth station of the network in the interval $[0,t]$, $i,j=1,2,\ldots,J$, $k_i=1,\ldots,c_i$ and $t>0$. Also, assume that arrival and service times are independent identically distributed random variables.

The processes of primary interest are the queue length process $Q=(Q_j)$ with $Q_j=\{Q_j(t),\, t\geq 0\}$, where $Q_j(t)$ indicates the number of jobs at the station j at time t. Now we introduce the following processes $Q_{jk_j}=\{Q_{jk_j}(t),\, t\geq 0\}$, where $Q_{jk_j}(t)$ indicates the number of jobs waiting to be served by the server k_j of the station j at time t; clearly, we have $Q_j(t)=\sum_{k_i=1}^{c_j} Q_{jk_i}(t)$, $j=1,2,\ldots,J$.

The dynamics of the queueing system (to be specified) depends on the service discipline at each service station. To be more precise, the "first come, first served" (FCFS) service discipline is assumed for all J stations. When a job arrives at a station and finds more than one server available, it will join one of the servers with the smallest index. We assume that the service station is work conserving; namely, not all servers at a station can be idle when there are customers waiting for service at that station. In particular, we assume that a station must works at its full capacity when the number of jobs waiting is equal to or exceeds the number of servers at that station. Suppose that the queue of jobs in each station of the open queueing network is unlimited.

Let us denote

$$\beta_j=\sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\mu_{ik_i}\cdot p_{ij}+\lambda_j-\sum_{k_j=1}^{c_j}\mu_{jk_j},\ \hat{\sigma}_j^2=\sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\mu_{ik_i}^3\cdot\sigma_{ik_i}\cdot p_{ij}^2$$
$$+\lambda_j^3\cdot a_j+\sum_{k_j=1}^{c_j}\mu_{jk_j}^3\cdot\sigma_{jk_j}>0,\ j=1,2,\ldots,J.$$

Also, we define

$$\mu_{jk_j}=\big(Ev_{jk_j}(e)\big)^{-1}>0,\ \sigma_{jk_j}=Dv_{jk_j}(e)>0,\ \lambda_j=\big(Eu_j(e)\big)^{-1}>0,$$
$$a_j=Du_j(e)>0,\ j=1,2,\ldots,J,\ k_j=1,2,\ldots,c_j.$$

In this work, we also admit that the following "overload conditions" are fulfilled

$$\sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\mu_{ik_i,n}\cdot p_{ij}+\lambda_j>\sum_{k_i=1}^{c_j}\mu_{ik_i,n},\ j=1,2,\ldots,J. \tag{2.1}$$

Note that conditions (2.1) quarantee that, with probability one, there exists a queue length of jobs which is constantly growing.

In addition, we assume throughout that

$$\max_{1\leq j\leq J}\max_{1\leq k_j\leq c_j}\sup_{e\geq 1}E\left(v_{jk_j}(e)\right)^{2+\gamma}<\infty\ \text{ for some } \gamma>0, \tag{2.2}$$
$$\max_{1\leq j\leq J}\max_{1\leq k_j\leq c_j}\sup_{e\geq 1}E(u_j(e))^{2+\gamma}<\infty\ \text{ for some } \gamma>0. \tag{2.3}$$

Conditions (2.2) and (2.3) imply the Lindeberg conditions for the respective sequences, and are easier to verify in practice (usually $\gamma=1$ work).

3. Main results

At first we prove the key lemma.

Lemma 3.1. *If* $Q_j(0) = 0,\ j = 1, 2, \ldots, J,$ *then*

$$|Q_j(t) - \hat{x}_j(t)| \le w(t) + \gamma(t),$$

where

$$\hat{x}_j(t) = \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} x_{ik_i}(t) \cdot p_{ij} + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t),\ w(t) = \sum_{j=1}^{J} \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} x_{ik_i}(t) \cdot |p_{ijk_i}^{t} - p_{ij}|,\ \gamma(t) = \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} \sup_{0 \le s \le t} (x_{ik_i}(s) - \tau_{ik_i}(s)).$$

Proof. By definition of the queue of jobs at the stations of the network, we get that, for $j = 1, 2, \ldots, J,\ k_j = 1, 2, \ldots, c_j$

$$\begin{aligned}
Q_j(t) &= \tilde{\tau}_j(t) - \tau_j(t) = \sum_{k_i=1}^{c_j} Q_{ik_i}(t) = \sum_{k_i=1}^{c_j} \tilde{\tau}_{ik_i}(t) - \sum_{k_i=1}^{c_j} \tau_{ik_i}(t) \\
&= \sum_{k_i=1}^{c_j} \tilde{\tau}_{ik_i}(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{k_i=1}^{c_j} x_{ik_i}(t) - \sum_{k_i=1}^{c_j} \tau_{ik_i}(t) \\
&\le \sum_{k_i=1}^{c_j} \tilde{\tau}_{ik_i}(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{k_i=1}^{c_j} \sup_{0 \le s \le t} (x_{ik_i}(s) - \tau_{ik_i}(s)) \\
&= \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} \tau_{ijk_i}(t) + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{k_i=1}^{c_j} \sup_{0 \le s \le t} (x_{ik_i}(s) - \tau_{ik_i}(s)) \\
&\le \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} \tau_{ik_i}(t) \cdot \frac{\tau_{ijk_i}(t)}{\tau_{ik_i}(t)} + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) + \sum_{k_i=1}^{c_j} \sup_{0 \le s \le t} (x_{ik_i}(s) - \tau_{ik_i}(s)) \\
&\le \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} x_{ik_i}(t) \cdot p_{ijk_i}^{t} + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) + \sup_{0 \le s \le t} (x_{jk_j}(s) - \tau_{jk_j}(s)) \\
&= \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} x_{ik_i}(t) \cdot (p_{ijk_i}^{t} - p_{ij} + p_{ij}) + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) \\
&\le \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} x_{ik_i}(t) \cdot p_{ij} + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{i=1}^{J} \sum_{k_i=1}^{c_i} x_{ik_i}(t) \cdot |p_{ijk_i}^{t} - p_{ij}| \\
&\quad + \sum_{k_i=1}^{c_j} \sup_{0 \le s \le t} (x_{ik_i}(s) - \tau_{ik_i}(s)) = \hat{x}_j(t) + w(t) + \gamma(t),
\end{aligned}$$

$j = 1, 2, \ldots, J$ and $t > 0$.

Hence it follows that

$$Q_j(t) \leq \hat{x}_j(t) + w(t) + \gamma(t),\ j = 1, 2, \ldots, J \text{ and } t > 0. \tag{3.1}$$

Besides, note that

$$\begin{aligned}
Q_j(t) &\geq \tilde{\tau}_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t) = \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \tau_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t)\\
&= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} (x_{ik_i}(t) + \tau_{ik_i}(t) - x_{ik_i}(t))\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_i=1}^{c_j} x_{ik_i}(t)\\
&= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} (\tau_{ik_i}(t) - x_{ik_i}(t))\cdot p_{ijk_i}^t + A_j(t)\\
&\quad - \sum_{k_i=1}^{c_j} x_{ik_i}(t) = \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} (x_{ik_i}(t) - \tau_{ik_i}(t))\cdot p_{ijk_i}^t\\
&\quad + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t) \geq \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t)\\
&\quad - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} (x_{ik_i}(t) - \tau_{ik_i}(t)) \geq \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t)\\
&\quad - \sup_{0\leq s\leq t}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} (x_{ik_i}(s) - \tau_{ik_i}(s)) \geq \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ijk_i}^t + A_j(t)\\
&\quad - \sum_{k_j=1}^{c_j} x_{jk_j}(t) - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \sup_{0\leq s\leq t} (x_{ik_i}(s) - \tau_{ik_i}(s))\\
&= \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot (p_{ijk_i}^t - p_{ij} + p_{ij}) + A_j(t) - \sum_{k_j=1}^{c_j} x_{jk_j}(t)\\
&\quad - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \sup_{0\leq s\leq t} (x_{ik_i}(s) - \tau_{ik_i}(s)) = \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot p_{ij} + A_j(t)\\
&\quad - \sum_{k_i=1}^{c_j} x_{ik_i}(t) + \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot (p_{ijk_i}^t - p_{ij})\\
&\quad - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \sup_{0\leq s\leq t} (x_{ik_i}(s) - \tau_{ik_i}(s)) \geq \hat{x}_j(t)\\
&\quad - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} x_{ik_i}(t)\cdot |p_{ijk_i}^t - p_{ij}| - \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \sup_{0\leq s\leq t} (x_{ik_i}(s) - \tau_{ik_i}(s))\\
&= \hat{x}_j(t) - w(t) - \gamma(t),\ j = 1, 2, \ldots, J \text{ and } t > 0.
\end{aligned} \tag{3.2}$$

Hence it follows that

$$Q_j(t) \geq \hat{x}_j(t) - w(t) - \gamma(t), \tag{3.3}$$

$j = 1, 2, \ldots, J$ and $t > 0$.

By combining (3.1) and (3.3), we can write

$$|Q_j(t) - \hat{x}_j(t)| \leq w(t) + \gamma(t), \tag{3.4}$$

$j = 1, 2, \ldots, J$ and $t > 0$. The proof of the lemma is complete. □

Now we use the formulation of Lemmas 3.2 and 3.3 the proof of which is presented in [28].

Lemma 3.2. *If conditions* (1) *are fulfilled, then for every* $\varepsilon > 0$

$$P\left(\overline{\lim_{t\to\infty}} \frac{\sup\limits_{0\leq s\leq t}(x_{jk_j}(s) - \tau_{jk_j}(s))}{t} > \varepsilon\right) = 0,$$

$j = 1, 2, \ldots, J,\ k_j = 1, 2, \ldots, c_j$.

Lemma 3.3. *If conditions* (1) *are fulfilled, then*

$$p_{ijk_i}^t \Rightarrow p_{ij},\ i, j = 1, 2, \ldots, J,\ k_i = 1, 2, \ldots, c_i.$$

Finally, applying the results of Lemmas 3.1–3.3, we prove the following theorem about SLLN for the queue length of jobs in multiserver open queueing networks.

Theorem 3.4. *If conditions* (2.1)–(2.3) *are fulfilled, then*

$$\left(\frac{Q_1(t)}{t}; \frac{Q_2(t)}{t} \ldots; \frac{Q_J(t)}{t}\right) \Rightarrow (\beta_1; \beta_2; \ldots; \beta_J).$$

Proof. At first we can find that for $\varepsilon > 0$

$$P\left(\left|\frac{Q_j(t)}{t} - \beta_j\right| > \varepsilon\right) \leq P\left(\sup_{0\leq s\leq t}\left|\frac{Q_j(s)}{s} - \frac{\hat{x}_j(s)}{s}\right| > \frac{\varepsilon}{2}\right) + P\left(\sup_{0\leq s\leq t}\left|\frac{\hat{x}_j(s)}{s} - \beta_j\right| > \frac{\varepsilon}{2}\right),\ j = 1, 2, \ldots, J \text{ and } t > 0. \tag{3.5}$$

Denote $c = 2 \cdot J^2 \cdot (c_1 + c_2 + \cdots + c_J)$, $Wj(t) = Q_j(t) - \hat{x}_j(t)$, $j = 1, 2, \ldots, J$ and $t > 0$. Let us estimate the first term in inequality (3.6):

$$
\begin{aligned}
P\left(\frac{|W_j(t)|}{t} > \varepsilon\right) &\leq P\left(\frac{\sup\limits_{0\leq s\leq t}\left\{\sum\limits_{j=1}^{J}\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i} x_{ik_i}(s)\cdot|p^s_{ijk_i} - p_{ij}|\right\}}{t} > \frac{\varepsilon}{2}\right) \\
&\quad + P\left(\frac{\sup\limits_{0\leq s\leq t}\left\{\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i}\sup\limits_{0\leq l\leq s}(x_{ik_i}(s) - \tau_{ik_i}(s)\right\}}{t} > \frac{\varepsilon}{2}\right) \\
&\leq P\left(\frac{\sum\limits_{j=1}^{J}\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i}\sup\limits_{0\leq s\leq t}\left\{x_{ik_i}(s)\cdot|p^s_{ijk_i} - p_{ij}|\right\}}{t} > \frac{\varepsilon}{2}\right) \\
&\quad + P\left(\frac{\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i}\sup\limits_{0\leq s\leq t}\sup\limits_{0\leq l\leq s}\{x_{ik_i}(l) - \tau_{ik_i}(l)\}}{t} > \frac{\varepsilon}{2}\right) \\
&\leq P\left(\frac{\sum\limits_{j=1}^{J}\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i}\sup\limits_{0\leq s\leq t}\left\{x_{ik_i}(s)\cdot|p^s_{ijk_i} - p_{ij}|\right\}}{t} > \frac{\varepsilon}{2}\right) \\
&\quad + P\left(\frac{\sum\limits_{i=1}^{J}\sum\limits_{k_i=1}^{c_i}\sup\limits_{0\leq s\leq t}(x_{ik_i}(s) - \tau_{ik_i}(s))}{t} > \frac{\varepsilon}{2}\right) \\
&\leq \sum_{j=1}^{J}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} P\left(\frac{\sup\limits_{0\leq s\leq t}\left\{x_{ik_i}(s)\cdot|p^s_{ijk_i} - p_{ij}|\right\}}{t} > \frac{\varepsilon}{c}\right) \\
&\quad + \sum_{i=1}^{J}\sum_{k_i=1}^{c_i} P\left(\frac{\sup\limits_{0\leq s\leq t}(x_{ik_i}(s) - \tau_{ik_i}(s))}{t} > \frac{\varepsilon}{c}\right),
\end{aligned}
$$

$$j = 1, 2, \ldots, J, \quad t > 0. \tag{3.6}$$

Consequently, it follows from (3.6) that for $\varepsilon > 0$

$$P\left(\frac{|W_j(t)|}{t} > \varepsilon\right) \le \sum_{j=1}^{J}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} P\left(\frac{\sup\limits_{0\le s\le t}\left\{x_{ik_i}(s)\cdot|p^s_{ijk_i}-p_{ij}|\right\}}{t} > \frac{\varepsilon}{c}\right)$$
$$+\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} P\left(\frac{\sup\limits_{0\le s\le t}\left(x_{ik_i}(s)-\tau_{ik_i}(s)\right)}{t} > \frac{\varepsilon}{c}\right),$$
$$j=1,2,\dots,J, \quad t>0. \tag{3.7}$$

Suppose $\mu = \max_{1\le j\le J}\sup_{1\le k_j\le c_j}\mu_{ik_i} < \infty$. Let us estimate the first term in (3.7). We have for $\varepsilon > 0$ that

$$P\left(\frac{\sup\limits_{0\le s\le t}\left\{x_{ik_i}(s)\cdot|p^s_{ijk_i}-p_{ij}|\right\}}{t} > \varepsilon\right) \le P\left(\frac{x_{ik_i}(t)\cdot\sup\limits_{0\le s\le t}|p^s_{ijk_i}-p_{ij}|}{t} > \varepsilon\right)$$
$$\le P\left(\left\{\frac{x_{ik_i}(t)\cdot\sup\limits_{0\le s\le t}|p^s_{ijk_i}-p_{ij}|}{t} > \varepsilon\right\}\cap\left\{x_{ik_i}(t)\le\varepsilon\cdot t+\mu_{ik_i}\cdot t\right\}\right)$$
$$+P\left(\frac{x_{ik_i}(t)}{t}-\mu_{ik_i} > \varepsilon\right) \le P\left((\varepsilon\cdot t+\mu_{ik_i}\cdot t)\cdot\sup\limits_{0\le s\le t}|p^s_{ijk_i}-p_{ij}| > \varepsilon\cdot t\right)$$
$$+P\left(\frac{x_{ik_i}(t)}{t}-\mu_{ik_i} > \varepsilon\right) \le P\left(\sup\limits_{0\le s\le t}|p^s_{ijk_i}-p_{ij}| > \frac{\varepsilon}{\varepsilon+\mu_{ik_i}}\right)$$
$$+P\left(\sup\limits_{0\le s\le t}\left|\frac{x_{ik_i}(s)}{s}-\mu_{ik_i}\right| > \varepsilon\right) \le P\left(\sup\limits_{0\le s\le t}|p^s_{ijk_i}-p_{ij}| > \frac{\varepsilon}{2\cdot\mu_{ik_i}}\right)$$
$$+P\left(\sup\limits_{0\le s\le t}\left|\frac{x_{ik_i}(s)}{s}-\mu\right| > \varepsilon\right),\ i=1,2,\dots,J, \quad k_i=1,2,\dots,c_i. \tag{3.8}$$

Thus, we achieve for $\varepsilon > 0$ that

$$P\left(\frac{\sup\limits_{0\le s\le t}\left\{x_{ik_i}(s)\cdot|p^s_{ijk_i}-p_{ij}|\right\}}{t} > \varepsilon\right) \le P\left(\sup\limits_{0\le s\le t}|p^s_{ijk_i}-p_{ij}| > \frac{\varepsilon}{2\cdot\mu}\right)$$
$$+P\left(\sup\limits_{0\le s\le t}\left|\frac{x_{ik_i}(s)}{s}-\mu_{ik_i}\right| > \varepsilon\right), \tag{3.9}$$

$i=1,2,\dots,J, \ \ k_i=1,2,\dots,c_i.$

Finally, we obtain that for $\varepsilon > 0$ (see (3.8)–(3.10))

$$\begin{aligned}
\lim_{t\to\infty} P\left(\frac{|W_j(t)|}{t} > \varepsilon\right) &\leq \sum_{j=1}^{J}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \lim_{t\to\infty} P\left(\sup_{0\leq s\leq t} |p^s_{ijk_i} - p_{ij}| > \frac{\varepsilon}{c\cdot\mu}\right) \\
&\quad + \sum_{j=1}^{J}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} \lim_{t\to\infty} P\left(\sup_{0\leq s\leq t} |\frac{x_{ik_i}(s)}{s} - \mu_{ik_i}| > \frac{\varepsilon}{c}\right) \\
&\quad + \sum_{j=1}^{J}\sum_{k_i=1}^{c_i} \lim_{t\to\infty} P\left(\frac{\sup_{0\leq s\leq t} (x_{ik_i}(s) - \tau_{ik_i}(s))}{t} > \frac{\varepsilon}{c}\right) \\
&\leq \sum_{j=1}^{J}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} P\left(\lim_{t\to\infty}\sup_{0\leq s\leq t} |p^s_{ijk_i} - p_{ij}| > \varepsilon\right) \\
&\quad + \sum_{j=1}^{J}\sum_{i=1}^{J}\sum_{k_i=1}^{c_i} P\left(\lim_{t\to\infty}\sup_{0\leq s\leq t} |\frac{x_{ik_i}(s)}{s} - \mu_{ik_i}| > \varepsilon\right) \\
&\quad + \sum_{j=1}^{J}\sum_{k_i=1}^{c_i} P\left(\frac{\lim_{t\to\infty}\sup_{0\leq s\leq t} (x_{ik_i}(s) - \tau_{ik_i}(s))}{t} > \varepsilon\right), \\
&j = 1, 2, \ldots, J. \qquad (3.10)
\end{aligned}$$

Let us prove that the first term in (3.10) converges to zero. Thus, we get (see the lemma of [34])

$$\begin{aligned}
P\left(\lim_{t\to\infty}\sup_{0\leq s\leq t} |p^s_{ijk_i} - p_{ij}| > \varepsilon\right) &\leq P\left(\lim_{t\to\infty}\lim_{\delta\downarrow 0}\sup_{0\leq s\leq t} |p^s_{ijk_i} - p_{ij}| > \delta\right) \\
&= \lim_{\delta\downarrow 0} P\left(\lim_{t\to\infty}\sup_{0\leq s\leq t} |p^s_{ijk_i} - p_{ij}| > \delta\right) = 0, \\
&i, j = 1, 2, \ldots, J,\ k_i = 1, 2, \ldots, c_i. \qquad (3.11)
\end{aligned}$$

Using the limit theorem for a renewal process, we see that the second term in (3.10) converges to zero (see [1]). Hence we get that the third term in (3.10) also converges to zero (see Lem. 3.2).

Thus, we prove that (see (3.10) and (3.11))

$$\sup_{0\leq s\leq t} \frac{|W_j(s)|}{s} \Rightarrow 0,\ j = 1, 2, \ldots, J. \qquad (3.12)$$

Note that (see, for example, [4])

$$\frac{\hat{x}_j(t)-\beta_j\cdot t}{t}=\sum_{i=1}^{J}\sum_{k_i=1}^{c_i}\frac{(x_{ik_i}(t)-\mu_{ik_i}\cdot t)\cdot p_{ij}}{t}$$
$$+\frac{(A_j(t)-\lambda_j\cdot t)-\left\{\sum\limits_{k_i=1}^{c_i}(x_{ik_i}(t)-\mu_{ik_i}\cdot t)\right\}}{t}\Rightarrow 0,\ j=1,2,\ldots,J. \tag{3.13}$$

Thus, using the convergence together theorem (see Thm. 2.1), (3.12) and (3.13), we derive that

$$\frac{Q_j(t)-\beta_j\cdot t}{t}\Rightarrow 0,\ j=1,2,\ldots,J. \tag{3.14}$$

As a result, we complete the proof of the theorem. □

Let us denote $V(t)=\sum_{j=1}^{J}Q_j(t),\ t>0$. Finally, we prove the following limit theorem on the total queue length of jobs in open multiserver queueing networks.

Theorem 3.5. *If conditions* (2.1)–(2.3) *are fulfilled, then*

$$\sup_{0\le s\le t}\left|\frac{V(s)}{s}-\beta\right|\Rightarrow 0.$$

Proof. It suffices to note that $|V(t)-\sum_{j=1}^{J}\hat{x}_j(t)|\le\sum_{j=1}^{J}|W_j(t)|$ and apply Theorem 3.1. The proof of Theorem 3.2 is complete. □

4. Applications of the main results

First of all we present a theorem about the diffusion limit of the queue length of jobs in a multiserver open queueing network.

Theorem 4.1. *If conditions* (2.1)–(2.3) *are satisfied, then*

$$\left(\frac{Q_1(nt)-\beta_1\cdot nt}{\sqrt{n}};\frac{Q_2(nt)-\beta_2\cdot nt}{\sqrt{n}};\ldots;\frac{Q_J(nt)-\beta_J\cdot nt}{\sqrt{n}}\right)$$

$\Rightarrow(\hat{\sigma}_1\cdot z_1(t);\hat{\sigma}_2\cdot z_2(t);\ldots;\hat{\sigma}_J\cdot z_J(t))$, *where* $z_j(t),\ \ j=1,2,\ldots,J,\ \ 0\le t\le 1$ *are independent standard Wiener processes.*

Proof. The proof is based on Lemma 3.1, the convergence together theorem (see Thm. 2.1), and the limit theorem for the renewal process (see, for example, [1]). The proof is now complete. □

Next, we present the law of the iterated logarithm for the queue length of jobs in a multiserver open queueing network under heavy traffic conditions.

Theorem 4.2. *If conditions* (2.1)–(2.3) *are satisfied, then*

$$P\left(\overline{\lim_{t\to\infty}}\frac{Q_j(t)-\beta_j\cdot t}{\hat{\sigma}_j\cdot a(t)}=1\right)=P\left(\underline{\lim_{t\to\infty}}\frac{Q_j(t)-\beta_j\cdot t}{\widehat{\sigma}_j\cdot a(t)}=-1\right)=1,$$

$j=1,2,\ldots,J.$

Proof. This proof is also based on Lemma 3.1, the convergence together theorem (see Thm. 2.1), and the law of the iterated logarithm for the renewal process (see [16]). The proof is now complete. □

5. Concluding remarks and future research

1. If the conditions of the theorem on SLLN are fulfilled (*i.e.*, conditions (2.1) are satisfied), the network is occupied at first (see Corollary 4.1) and if conditions (2.1) are satisfied later on, the network becomes uncontrollable after a certain time (as $t\geq\max\limits_{1\leq j\leq J}\max\limits_{1\leq k_j\leq c_j}\frac{m_{jk_j}}{\beta_{jk_j}}$) (see Cor. 4.2).
2. Conditions (2.1) are fundamental, – the behaviour of the whole network and its evolution is not clear, if conditions (2.1) are not satisfied. Therefore, this fact is the object of a further research and discussion.
3. The theorems of this paper are proved for a class of multiserver open queueing network in heavy traffic with the service principle "first come, first served", endless waiting time of a customer in each node of the queueing system, and the times between the arrival of customers at the multiserver open queueing networks are independent identically distributed random variables. However, analogous theorems can be applied to a wider class of multiserver open queueing networks in heavy traffic: when the arrival and service of customers in a queue is by groups, when interarrival times of customers at a multiserver open queueing network are weakly dependent random variables, *etc.*

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