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ANALYSIS OF A $M^X/G(a,b)/1$ QUEUEING SYSTEM WITH VACATION INTERRUPTION

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Abstract. In this paper, a batch arrival general bulk service queueing system with interrupted vacation (secondary job) is considered. At a service completion epoch, if the server finds at least 'a' customers waiting for service say ξ, he serves a batch of min (ξ, b) customers, where $b \geq a$. On the other hand, if the queue length is at the most 'a-1', the server leaves for a secondary job (vacation) of random length. It is assumed that the secondary job is interrupted abruptly and the server resumes for primary service, if the queue size reaches 'a', during the secondary job period. On completion of the secondary job, the server remains in the system (dormant period) until the queue length reaches 'a'. For the proposed model, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained. Various performance measures are derived. A cost model for the queueing system is also developed. To optimize the cost, a numerical illustration is provided.

Keywords. Bulk arrival, single server, batch service, vacation, interruption.

Mathematics Subject Classification. 60K25, 60K20, 90B22, 68M20.

1. Introduction

The motivation of this paper comes from a real life situation that exists in an industry involving Sintering Process. Sintering is a method for making objects from powder, by heating the material in a sintering furnace below its melting

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point (solid state sintering), until its particles adhere to each other. Sintering is traditionally used for manufacturing ceramic objects, and has also found uses in such fields as powder metallurgy. Sintering is part of the firing process used in the manufacture of pottery, ceramic objects, etc. Sintering strengthens a powder mass and normally produces densification and, in powdered metals, recrystallization. After compaction, the components pass through a sintering furnace. This typically has two heating zones, the first removes the lubricant, and the second higher temperature zone allows diffusion and bonding between powder particles.

The components arrive for sintering process in batches. The sintering process is done in bulk by sintering furnace. The sintering furnace will be operated only if sufficient amount of components are available to start the sintering process. Once the process is started, the bulk operation has to continue successively for many batches; otherwise, the operating cost will increase. It is assumed that the processing time has a general distribution. After completing the sintering process, if the number of components to be processed is less than the batch quantity, say 'a', then the operator stops the sintering process and performs some other work (secondary job/vacation), like mixing the powders for the next process, cleaning the outer surface and checking dimensions of the components, etc. During the secondary job, if the required number of components reaches the required threshold value 'a', the operator returns (vacation interruption) and starts the sintering process. After completing a secondary job, the operator waits in the system till the required number of components arrives. This sintering process can be modeled as a bulk arrival bulk service queue with interrupted vacation.

In this paper, a single server bulk service queueing system with interrupted secondary job (*vacation*) is considered. In the literature, all vacation models with the bulk service consider that the server can start the service only when he completes the secondary job. But in emergency, the server has to terminate the secondary job and must give priority for primary job. Once the required level is reached to start the primary service, there is no point in continuing the secondary job. This model is proposed to overcome this difficulty and to make the system operate more efficiently. In this paper, the focus is on a vacation interruption policy which provides the solution to the situation mentioned above. Under the vacation interruption policy, the vacation is interrupted and the server resumes to a regular busy period. These vacation models have important applications in practice.

General single server vacation models have been well studied and surveyed by Doshi [6,7] and the monographs of Takagi [21] and Tian and Zhang [22]. Detailed analysis of some bulk queueing models can be seen in the studies of Chaudhry and Templeton [5] and Medhi [17]. Borthakur and Medhi [4] have studied a queueing system with arrival and services in batches of variable size. They have derived the queue length distribution for the $M^X/G(a,b)/1$ model without vacation concepts. Krishna *et al.* [14] have discussed a $M^X/G(a,b)/1$ model with N-policy, multiple vacations, and setup times. Arumuganathan and Jeyakumar [1] analyzed a bulk queue with multiple vacations, setup times with N-policy and closedown times. Lee *et al.* [15] analyzed an $M^x/G/1$queue with N-policy and multiple vacation

Balasubramanian *et al.* [2] discussed steady state analysis of a Non-Markovian bulk queueing system with overloading and multiple vacations. Haridass and Arumuganathan [8] discussed a batch arrival general bulk service queueing system with variant threshold policy for secondary jobs. Jau-Chuan Ke *et al.* [10] discussed an algorithmic analysis of the multi-server system with a modified Bernoulli vacation schedule. Ji Hong Li and Nai-Shuo Tian [13] discussed performance analysis of a GI/M/1 queue with single working vacation. *In all the aforesaid models with vacations, the server cannot come back (vacation interruption) to the normal working level (regular busy period), until the vacation period ends.*

For vacation interruption models, Li and Tian [16] analyzed the discrete-time GI/Geo/1 queue with working vacations and vacation interruption. Ji Hong Li *et al.* [12] analyzed GI/M/1 queue with working vacations and vacation interruption. Mian Zhang and Zhengting Hou [18] studied an M/G/1 queue with working vacations and vacation interruption. Ji-Hong Li and Nai-Shuo Tian [11] analyzed the M/M/1 queue with working vacations and vacation interruptions. Yutaka BABA [23] analyzed the M/PH/1 queue with working vacations and vacation interruption. Hongbo Zhang and Dinghua Shi [9] provided a study on the M/M/1 queue with Bernoulli-Schedule-Controlled vacation and vacation interruption. Mian Zhang and Zhengting Hou [19] analyzed an MAP/G/1 queue with working vacations and vacation interruption. All the above models consider single arrival and single service only. *This stimulates the authors to develop a single server bulk arrival bulk service queueing system with vacation interruption policy.*

The following points are addressed in this paper. Vacation interruption concept is introduced for a bulk service queueing vacation model. Probability generating function (PGF) of the steady state queue size distribution at an arbitrary time epoch is obtained. Various performance measures are derived. *A recursive approach is used to express the unknown function in the PGF of interrupted vacation queueing system in terms of known values, to break the barrier in solving vacation interruption model.* A cost model has been developed, and an important contribution of this is, the study of cost model for a practical situation and to optimize the cost.

The structure of the paper is as follows: Introduction and Literature survey is presented in Section 1. The mathematical model is developed with necessary balance equations in Section 2. In section 3, the queue size distribution is developed. Various performance measures are derived in Section 4. Particular case and some special cases are discussed in Section 5. The cost model for the proposed queueing system is given in Section 6 and the effects of various parameters on the system performance are analyzed numerically in Section 7. The conclusion and future work are presented in Section 8.

2. Model Description

In this section, the mathematical model for a batch arrival and bulk service queueing system with single vacation and vacation interruption is considered.

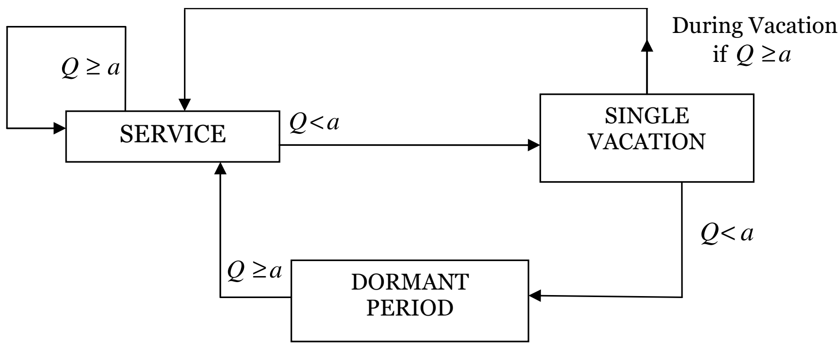

FIGURE 1. Schematic representation of the model: Q-queue length.

At a service completion epoch, if the server finds at least 'a' customers waiting for service say ξ, he serves a batch of min (ξ, b) customers, where $b \geq a$. On the other hand, if the queue length is at the most 'a-1', the server leaves for a secondary job (vacation) of random length. It is assumed that the secondary job is interrupted abruptly and the server resumes for primary service, if the queue size reaches 'a' during the secondary job period. On completion of the secondary job, the server remains in the system (dormant period), until the queue length reaches 'a'. For the proposed model, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained. Various performance measures are derived. *The effects of several parameters on the total average cost for the proposed queueing system and to optimize the cost, a numerical illustration is provided.* The above system is modeled using the supplementary variable technique, by considering remaining service time of the batch in service and remaining vacation time of the server as supplementary variables at an arbitrary time. One can refer the book by Nikolaos Limnios and Gheorghe Oprişan [20] for the general reference of supplementary variable technique. The figure above illustrates the schematic representation of the proposed model.

2.1. NOTATIONS

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate. g_k be the probability that 'k' customers arrive in a batch and $X(z)$ be its probability generating function (PGF). Let $S(x)(s(x))\{\tilde{S}(\theta)\}[S^0(x)]$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining service time] of service. Let $V(x)(v(x))\{\tilde{V}(\theta)\}[V^0(x)]$ be the cumulative distribution function (probability density function) {Laplace-Stieltjes transform} [remaining vacation time] of vacation. $N_q(t)$ denotes the number of customers waiting for service at time t, $N_s(t)$ denotes the number of customers under the service at time t.

$$C(t) = \begin{cases} 0, \text{ when the server is busy with service} \\ 1, \text{ when the server is on vacation} \\ 2, \text{ when the server is on dormant period.} \end{cases}$$

The state probabilities are defined as follows:

$P_{ij}(x,t)\mathrm{d}t = \Pr\left\{N_s(t) = i, N_q(t) = j, x \leq S^0(t) \leq x + \mathrm{d}t, C(t) = 1\right\}, a \leq i \leq b, j \geq 0$ is the joint probability that at time t the server is busy, the queue size is j, the number of customers under service is i and the remaining service time of a batch under service at an arbitrary time is between x and $x + \mathrm{d}t$.

$Q_n(x,t)\mathrm{d}t = Pr\left\{N_q(t) = n, x \leq V^0(t) \leq x + \mathrm{d}t, C(t) = 1\right\}, 0 \leq n \leq a-1$ is the joint probability that at time t the server is on a vacation, the queue size is n and the remaining vacation time is between x and $x + \mathrm{d}t$.

$T_n(t) = Pr\left\{N_q(t) = n, C(t) = 2\right\}, 0 \leq n \leq a-1$ is the probability that at time t, the system size is n and the server is on dormant period.

The following system equations are obtained for the queueing system, using supplementary variable technique:

$$\begin{aligned} P_{i0}(x-\Delta t, t+\Delta t) &= P_{i0}(x,t)\left(1-\lambda\Delta t\right) + \sum_{m=a}^{b} P_{mi}(0,t)s(x)\Delta t \\ &+ \sum_{k=0}^{a-1} Q_k(x,t)\lambda g_{i-k}s(x)\Delta t + \sum_{m=0}^{a-1} T_m(t)\lambda g_{i-m}s(x)\Delta t \qquad a \leq i \leq b. \end{aligned}$$

The above equation gives the probability that, there are 'i' customers under service and no customer in the queue with the remaining service time $x - \Delta t$ at time $t + \Delta t$ from other states. Similarly, the other system equations as follows are got:

$$\begin{aligned} P_{ij}(x-\Delta t, t+\Delta t) &= P_{ij}(x,t)\left(1-\lambda\Delta t\right) + \sum_{k=1}^{j} P_{i,j-k}(x,t)\lambda g_k\Delta t \\ &\qquad a \leq i \leq b-1; j \geq 1 \\ P_{bj}(x-\Delta t, t+\Delta t) &= P_{bj}(x,t)\left(1-\lambda\Delta t\right) + \sum_{m=a}^{b} P_{m,b+j}(0,t)s(x)\Delta t \\ &+ \sum_{k=1}^{j} P_{b,j-k}(x,t)\lambda g_k\Delta t + \sum_{m=0}^{a-1} T_m(t)\lambda g_{b+j-m}s(x)\Delta t \\ &+ \sum_{k=0}^{a-1} Q_k(x,t)\lambda g_{b+j-k}s(x)\Delta t, j \geq 1 \end{aligned}$$

$$Q_0(x-\Delta t, t+\Delta t) = Q_0(x,t)\,(1-\lambda\Delta t) + \sum_{m=a}^{b} P_{m0}(0,t)v(x)\Delta t$$

$$\begin{aligned} Q_n(x-\Delta t, t+\Delta t) &= Q_n(x,t)\,(1-\lambda\Delta t) + \sum_{m=a}^{b} P_{mn}(0,t)v(x)\Delta t \\ &\quad + \sum_{k=1}^{n} Q_{n-k}(x,t)\lambda g_k \Delta t, \qquad 1 \le n \le a-1 \end{aligned}$$

$$\begin{aligned} T_0(t+\Delta t) &= T_0(t)\,(1-\lambda\Delta t) + Q_0(0,t)\Delta t \\ T_n(t+\Delta t) &= T_n(t)\,(1-\lambda\Delta t) + Q_n(0,t)\Delta t \\ &\quad + \sum_{k=1}^{n} T_{n-k}(t)\lambda g_k \Delta t, \qquad 1 \le n \le a-1. \end{aligned}$$

3. Steady-state analysis

In this section, the Probability Generating Function (PGF) of the queue size at an arbitrary time epoch is derived. The PGF will be useful in deriving the important performance measures.

3.1. Steady state queue size distribution

From the above set of system equations, the steady state queue size equations for the queueing model are obtained as follows:

$$0 = -\lambda T_0 + Q_0(0) \tag{3.1}$$

$$0 = -\lambda T_n + Q_n(0) + \sum_{k=1}^{n} T_{n-k}\lambda g_k, 1 \le n \le a-1 \tag{3.2}$$

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}P_{i0}(x) &= -\lambda P_{i0}(x) + \sum_{m=a}^{b} P_{mi}(0)s(x) + \sum_{k=0}^{a-1} Q_k(x)\lambda g_{i-k}s(x) \\ &\quad + \sum_{m=0}^{a-1} T_m \lambda g_{i-m}s(x) \qquad a \le i \le b \end{aligned} \tag{3.3}$$

$$-\frac{d}{dx}P_{ij}(x) = -\lambda P_{ij}(x) + \sum_{k=1}^{j} P_{i,j-k}(x)\lambda g_k \qquad a \le i \le b-1, j \ge 1 \tag{3.4}$$

$$\begin{aligned} -\frac{d}{dx}P_{bj}(x) &= -\lambda P_{bj}(x) + \sum_{m=a}^{b} P_{m,b+j}(0)s(x) + \sum_{k=1}^{j} P_{b,j-k}(x)\lambda g_k \\ &\quad + \sum_{m=0}^{a-1} T_m \lambda g_{b+j-m}s(x) + \sum_{k=0}^{a-1} Q_k(x)\lambda g_{b+j-k}s(x), \qquad j \ge 1 \end{aligned} \tag{3.5}$$

$$-\frac{d}{dx}Q_0(x) = -\lambda Q_0(x) + \sum_{m=a}^{b} P_{m0}(0)v(x) \tag{3.6}$$

$$-\frac{d}{dx}Q_n(x) = -\lambda Q_n(x) + \sum_{m=a}^{b} P_{mn}(0)v(x) + \sum_{k=1}^{n} Q_{n-k}(x)\lambda g_k, \qquad 1 \le n \le a-1. \tag{3.7}$$

The Laplace - Stieltjes transforms of $P_{in}(x)$ and $Q_{jn}(x)$ are defined as

$$\tilde{P}_{in}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} P_{in}(x)\mathrm{d}x \text{ and } \tilde{Q}_{jn}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} Q_{jn}(x)\mathrm{d}x.$$

Taking Laplace-Stieltjes transform on both sides of the equation (3.3) through (3.7), we have

$$\lambda T_0 = Q_0(0) \tag{3.8}$$

$$0 = \lambda T_n - Q_n(0) - \sum_{k=1}^{n} T_{n-k}\lambda g_k, \qquad 1 \le n \le a-1 \tag{3.9}$$

$$\theta\tilde{P}_{i0}(\theta) - P_{i0}(0) = \lambda\tilde{P}_{i0}(\theta) - \sum_{m=a}^{b} P_{mi}(0)\tilde{S}(\theta) - \sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{i-k}\tilde{S}(\theta) - \sum_{m=0}^{a-1} T_m\lambda g_{i-m}\tilde{S}(\theta) \quad a \le i \le b \tag{3.10}$$

$$\theta\tilde{P}_{ij}(\theta) - P_{ij}(0) = \lambda\tilde{P}_{ij}(\theta) - \sum_{k=1}^{j}\tilde{P}_{i,j-k}(\theta)\lambda g_k \;\; a \le i \le b-1, \quad j \ge 1 \tag{3.11}$$

$$\theta\tilde{P}_{bj}(\theta) - P_{bj}(0) = \lambda\tilde{P}_{bj}(\theta) - \sum_{m=a}^{b} P_{m,b+j}(0)\tilde{S}(\theta) - \sum_{k=1}^{j} P_{b,j-k}(\theta)\lambda g_k - \sum_{m=0}^{a-1} T_m\lambda g_{b+j-m}\tilde{S}(\theta) - \sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{b+j-k}\tilde{S}(\theta), \quad j \ge 1 \tag{3.12}$$

$$\theta\tilde{Q}_0(\theta) - Q_0(0) = \lambda\tilde{Q}_0(\theta) - \sum_{m=a}^{b} P_{m0}(0)\tilde{V}(\theta) \tag{3.13}$$

$$\theta\tilde{Q}_n(\theta) - Q_n(0) = \lambda\tilde{Q}_n(\theta) - \sum_{m=a}^{b} P_{mn}(0)\tilde{V}(\theta) - \sum_{k=1}^{n}\tilde{Q}_{n-k}(\theta)\lambda g_k, \quad 1 \le n \le a-1. \tag{3.14}$$

To obtain the probability generating function (PGF) of the queue size at an arbitrary time, the following probability generating functions are defined.

$$\tilde{P}_i(z,\theta)=\sum_{j=0}^{\infty}\tilde{P}_{ij}(\theta)z^j;\quad P_i(z,0)=\sum_{j=0}^{\infty}P_{ij}(0)z^j; a\le i\le b;$$

$$\tilde{Q}(z,\theta)=\sum_{n=0}^{a-1}\tilde{Q}_n(\theta)z^n;\quad Q(z,0)=\sum_{n=0}^{a-1}Q_n(0)z^n;\quad T(z)=\sum_{n=0}^{a-1}T_nz^n. \tag{3.15}$$

Theorem 3.1. *If p_n is the steady state probability of 'n' customers in the queue, then the probability generating function of the queue size at an arbitrary time epoch $P(z)$ is given by*

$$P(z)=\frac{\begin{array}{c}(-\lambda+\lambda\xi(z))\begin{pmatrix}\sum\limits_{i=a}^{b-1}g(\tilde{S},E1,E2,E3,z)-\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum\limits_{j=0}^{b-1}p_jz^j\\ +\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\lambda E4+h(\tilde{S},X(z),z)\\ +\lambda\sum\limits_{i=0}^{a-1}z^i f\left(\tilde{S},\tilde{Q},z\right)-\lambda\sum\limits_{i=0}^{a-1}z^{i-b}\psi\left(\tilde{S},\tilde{Q},z\right)\end{pmatrix}\\ +\left((-\lambda+\lambda X(z))\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\left(\tilde{V}(\lambda-\lambda\xi(z))-1\right)\sum\limits_{n=0}^{a-1}p_nz^n\right)\end{array}}{(-\lambda+\lambda\xi(z))(-\lambda+\lambda X(z))\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)} \tag{3.16}$$

where

$$\begin{aligned}g(\tilde{S},E1,E2,E3,z)&=\Big(\left(z^b-1\right)\tilde{S}(\lambda-\lambda X(z))\lambda E1+\left(\tilde{S}(\lambda-\lambda X(z))-1\right)z^bE3\\&\quad-\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\lambda E2\Big),\\ h(\tilde{S},X(z),z)&=\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)(-\lambda+\lambda X(z))\,T(z),\\ f(\tilde{S},\tilde{Q},z)&=\left(\tilde{Q}_i(\lambda-\lambda X(z))\tilde{S}(\lambda-\lambda X(z))-\tilde{Q}_i(0)\right)\left(X(z)-\sum_{j=1}^{b-i-1}g_jz^j\right),\\ \psi(\tilde{S},\tilde{Q},z)&=\left(\tilde{Q}_i(\lambda-\lambda X(z))-\tilde{Q}_i(0)\right)\tilde{S}(\lambda-\lambda X(z))\left(X(z)-\sum_{j=1}^{b-i-1}g_jz^j\right),\end{aligned}$$

$$E1=\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))g_{i-k},\quad E2=\sum_{k=0}^{a-1}\tilde{Q}_k(0)\,g_{i-k},$$

$$E3=p_i+\lambda\sum_{m=0}^{a-1}T_mg_{i-m}\ \textit{and}\ E4=T(z)X(z)-\sum_{m=0}^{a-1}\left(T_mz^m\sum_{j=1}^{b-m-1}g_jz^j\right).$$

Proof. Given in Appendix A.

The probability generating function $P(z)$ has to satisfy $P(1) = 1$. In order to satisfy this condition, applying L'Hospital's rule and evaluating $\lim_{z\to 1} P(z)$ and equating the expression to 1, it is derived that, $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration, where $\rho = \frac{\lambda E(X)E(S)}{b}$. □

3.2. Computational aspects

Equation (3.16) gives the probability generating function of the number of customers in the queue, which involves the unknowns T_i and $\tilde{Q}_i(\theta)$. Using the following theorems, T_i and $\tilde{Q}_i(\theta)$ are expressed in terms of p_i and the known function $\tilde{V}(\lambda)$ respectively. To find the unknown constants, Rouche's theorem of complex variables can be used. It follows that the expression $z^b - \tilde{S}(\lambda - \lambda X(z))$ has $b-1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator of (3.16) must vanish at these points, which gives 'b' equations and 'b' unknowns. These equations can be solved by suitable numerical techniques. MATLAB is used for programming.

Theorem 3.2. *The unknown constants q_n involved in T_n are expressed in terms of p_n as, $q_n = \sum_{i=0}^{n} p_{n-i}\,\beta_i$; $n = 0, 1, 2, \ldots, a-1$, where β_i is the probability that 'i' customers arrive during the vacation.*

Proof. From equation (A.4), we have $Q(z, 0) = \tilde{V}(\lambda - \lambda\xi(z)) \sum_{n=0}^{a-1} p_n z^n$

$$\sum_{n=0}^{a-1} q_n z^n = \left(\sum_{n=0}^{\infty} \beta_n z^n\right)\left(\sum_{n=0}^{a-1} p_n z^n\right)$$

$$\sum_{n=0}^{a-1} q_n z^n = \sum_{n=0}^{a-1}\left(\sum_{i=0}^{n} p_{n-i}\beta_i\right) z^n + \sum_{n=a}^{\infty}\left(\sum_{i=0}^{a-1} \beta_{n-i} p_i\right) z^n. \tag{3.17}$$

Equating the coefficient of z^n; $n = 0, 1, 2, 3, \ldots a-1$, on both sides of the equation (3.17), we get

$$q_n = \sum_{i=0}^{n} p_{n-i}\beta_i. \tag{3.18}$$

□

The unknown constants T_n involved in $P(z)$ are expressed in terms of p_n in the following theorem.

Theorem 3.3. *Let B_j be the collection of set of positive integers (not necessarily distinct) A, such that, sum of elements in A is j, then, $T_n = \frac{1}{\lambda}\left(\sum_{j=0}^{n} q_{n-j} \sum_{j=1}^{n(B_j)} \prod_{l\in A} g_l\right)$, $n = 0, 1, 2, 3, \ldots a-1$.*

Proof. From the equations (3.1) and (3.2), we have $\lambda T_0 = Q_0(0) = q_0$

$$\lambda T_n = Q_n(0) + \lambda \sum_{k=1}^{n} T_{n-k} g_k; \quad 1 \leq n \leq a-1.$$

When $n = 1$,

$$\begin{aligned}\lambda T_1 &= Q_1(0) + \lambda T_0 g_1 \\ &= q_1 + q_0 g_1.\end{aligned}$$

When $n = 2$,

$$\begin{aligned}\lambda T_2 &= Q_2(0) + \sum_{k=1}^{2} \lambda T_{2-k} g_k \\ &= Q_2(0) + \lambda T_1 g_1 + \lambda T_0 g_2 \\ &= q_2 + q_1 g_1 + q_0 (g_1^2 + g_2).\end{aligned}$$

When $n = 3$,

$$\begin{aligned}\lambda T_3 &= Q_2(0) + \sum_{k=1}^{3} \lambda T_{3-k} g_k \\ &= Q_3(0) + \lambda T_2 g_1 + \lambda T_1 g_2 + \lambda T_0 g_3 \\ &= q_3 + q_2 g_1 + q_1 (g_1^2 + g_2) + q_0 (g_1^3 + 2 g_1 g_2 + g_3)\end{aligned}$$

$$T_3 = \frac{1}{\lambda} \left(\sum_{j=0}^{3} q_{3-j} \sum_{j=1}^{n(B_j)} \prod_{l \in A} g_l \right)$$

where $B_1 = \{\{1\}\}$, $B_2 = \{\{1,1\},\{2\}\}$ and $B_3 = \{\{3\},\{1,1,1\},\{1,2\},\{2,1\}\}$

By induction, we get

$$T(z) = \sum_{n=0}^{a-1} T_n z^n = \frac{1}{\lambda} \left(\sum_{n=0}^{a-1} \left(\sum_{j=0}^{n} q_{n-j} \sum_{j=1}^{n(B_i)} \prod_{l \in A} g_l \right) z^n \right). \tag{3.19}$$

Therefore,

$$T_n = \frac{1}{\lambda} \left(\sum_{j=0}^{n} q_{n-j} \sum_{j=1}^{n(B_i} \prod_{l \in A} g_l \right). \tag{3.20}$$

□

An important theorem, which breaks the barrier in solving vacation interruption model, is proved. The following theorem gives a compact way of representing the unknown functions $\tilde{Q}_i(\theta)$ in terms of known values.

Theorem 3.4. *The Laplace-Stieltjes transform of the unknown function $\tilde{Q}_i(\theta)$; $i = 0, 1, 2, 3 \ldots a-1$ are expressed in terms of $\tilde{V}(\lambda)$ and higher derivatives $\tilde{V}^{(n)}(\lambda)$ as,*

$$\tilde{Q}_i(\theta) = \sum_{j=0}^{i} \sum_{A_j \in \phi} \left((-1)^{n(A_j)} \left(\prod_{i \in A_j} \lambda g_i \right) p_{i-j} k_{n(A_j)} \right)$$

where

$$k_l = \frac{\tilde{V}^{n(A_l)}(\lambda) + k_{l-1}}{\theta - \lambda};$$

$l = 1, 2, 3, \ldots i, k_0 = \frac{\tilde{V}(\lambda) - \tilde{V}(\theta)}{\theta - \lambda}$ *and* ϕ *is the collection of all possible distinct sets of positive integers* A_j *such that, sum of elements in* A_j *is* j.

Proof. Given in Appendix B. □

4. Performance measures

In a waiting line, it is customary to access the mean number of waiting units and mean waiting time. In this section, some useful performance measures of the proposed model like expected number of customers in the queue $E(Q)$, expected length of idle period $E(I)$, expected length of busy period $E(B)$ are derived, which are useful to find the total average cost of the system. Also, expected waiting time in the queue W_Q, probability that the server is on vacation $P(V)$ and probability that the server is busy $P(B)$, are derived.

4.1. Expected queue length

The expected queue length $E(Q)$ (*i.e.* mean number of customers waiting in the queue) at an arbitrary time epoch, is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$\lim_{z \to 1} P(z) = E(Q)$$

$$E(Q) = \frac{\begin{array}{l}\sum_{i=a}^{b-1} (4(T11)f1 - 2(f7)(f14)) - \sum_{j=0}^{b-1} p_j \,(4(T11)(f2) - 2(f7)(f9) \\ -3(f13)(S1)) + \sum_{n=0}^{a-1} (4(T11)\lambda(f5 - f6) - 2(f7)\lambda(f12) \\ -3(f13)\lambda(f16)) + 4(T11)f4 - 2(f7)(f11) - 3(f13)(f15)\end{array}}{24(T11)^2} + \frac{\sum_{n=0}^{a-1} p_n \,(f3 + f10)}{(T27)^2} \tag{4.1}$$

where

$$S1 = \lambda E(X)E(S), \quad S2 = \lambda X''(1)E(S) + \lambda^2 E^2(X)E(S^2),$$
$$S3 = \lambda^3 E^3(X)E(S^3), \quad S4 = \lambda^2 E(X)E(X^2)E(S^2),$$
$$S5 = \lambda E(S)E(X^3), \quad S6 = \lambda^2 E(X)E(X^2), \quad S7 = \lambda^2 E(S)E(S^3),$$
$$S8 = \lambda E(S)E(X^2), \quad V1 = \lambda E(X)E(V),$$
$$V2 = \lambda X''(1)E(V) + \lambda^2 E^2(X)E(V^2), \quad G1 = 1 - \sum_{j=1}^{b-i-1} g_j;$$
$$G2 = E(X) - \sum_{j=1}^{b-i-1} j\, g_j, \quad G3 = E(X^2) - \sum_{j=1}^{b-i-1} j\,(j-1)g_j,$$
$$G4 = T(1) - \sum_{m=0}^{a-1} \sum_{j=1}^{b-m-1} T_m g_j,$$
$$G5 = \sum_{m=0}^{a-1} \sum_{j=1}^{b-m-1} (m+j)(m+j-1)T_m g_j,$$
$$G6 = \sum_{m=0}^{a-1} \sum_{j=1}^{b-m-1} (m+j)T_m g_j,$$
$$G7 = E(X)T'(1), \quad G8 = E(X)T(1), \quad G9 = E(X^2)T'(1);$$
$$G10 = p_i + q_i + \lambda \sum_{m=0}^{a-1} T_m g_{i-m},$$
$$T1 = \left(\sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z)) g_{i-k} \right)_{z=1},$$
$$T2 = \left(\frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z))\, g_{i-k} \right) \right)_{z=1},$$
$$T3 = \left(\sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z)) \right)_{z=1},$$
$$T4 = \left(\frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z)) \right) \right)_{z=1},$$
$$T5 = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(\sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z)) \right) \right)_{z=1}, \quad T6 = \sum_{k=0}^{a-1} \tilde{Q}_k(0)\, g_{i-k};$$
$$T7 = \frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{k=0}^{a-1} \tilde{Q}_k(0) \right), \quad T8 = \sum_{k=0}^{a-1} \tilde{Q}_k(0),$$

$$
\begin{aligned}
T9 &= \left(\frac{d^3}{dz^3}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\right)\right)_{z=1},\\
T10 &= \left(\frac{d^2}{dz^2}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))g_{i-k}\right)\right)_{z=1},\\
T11 &= \lambda E(X)(b-S1)\\
T12 &= T2+(T1)(S1);\quad T13=T4+(T3)(S1);\\
T14 &= S3+3(S4)+S5;\quad T15=S3+3(S4)+S7;\\
T16 &= T10+(b-1)(T12);\quad T17=(S1)(T2)+(T1)(S2);\\
T18 &= 9(S4)+8(S5)+6(S8)-3(S3);\\
T19 &= 3i(i-1)(G1)+G3+(G2)[2+4i];\\
T20 &= (T3)(S2)+2(T4)(S1)+T5;\quad T21=2G1(2i+1)+G2;\\
T22 &= 2T4(G2+(S1)(G1))+(T5)(S1);\\
T23 &= G3+2(S1)(G2)+(S2)(G1);\\
T24 &= 2T2+(T1)((b-1)+2S1)-(b-1)(T6);\\
T25 &= (G1)(2i)+2(G2);\quad T26=(T8)(S2)+2(T3)(S1)+T5;\\
T27 &= (-\lambda+\lambda\xi(1)),\xi(1)=\sum_{k=1}^{a-1}g_k,\xi'(1)=\sum_{k=1}^{a-1}kg_k,\\
f1(\lambda,b,S,G,\tilde{Q}) &= 3\lambda b(T16+T17+(S1)(T2))+\lambda b(b-1)(b-2)(T1-T6)\\
&\quad+G10[b(b-1)(3S1+2S2)]+T14+b(S2)\\
&\quad-\lambda T6(S3-S5+S4-2(S6)),\\
f2(\lambda,X,S) &= 3j(S2+(j-1)S1)+T14;\\
f3(\lambda,\xi,\tilde{V}) &= \left[\tilde{V}(\lambda-\lambda\xi(1))-1\right](n(T27)-\lambda\xi'(1)),\\
f4(\lambda,b,S,G) &= 3\lambda S1[G9+2(G7)+T''(1)-G5]+3\lambda(S2)[G8+T'(1)-G6]\\
&\quad+\lambda(T15)(G4)+(G8)\lambda[b(b+1)-3]-\lambda(b-S1)[2(G7)+3(G9)],\\
f5(\lambda,S,G,\tilde{Q}) &= (T13)(T19)+(T20)(T21)+[(T3)(T14)\\
&\quad+3(T4)(S2)+3(T5)(S1)+T9](G1),\\
f6(\lambda,S,G,\tilde{Q}) &= 3(i-b)[(i-b-1)(G1)(T4)+T22]+(3T4)(T23)\\
&\quad+(3T5)[G2+(S1)(G1)]+(T9)(G1),\\
f7(\lambda,X,b,S) &= 3[\lambda X''(1)(b-S1)+\lambda E(X)b(b-1)-S3-S4],\\
f8(\lambda,b,S,G,\tilde{Q}) &= \lambda[b(T24)+(T6)(S2)]+(G10)[2bS1+S2],\\
f9(\lambda,X,S) &= 2jS1+S2,\\
f10(\lambda,\xi',\tilde{V}') &= (T27)\left[-\lambda\xi'(1)\tilde{V}'(\lambda-\lambda\xi(1))\right],
\end{aligned}
$$

$$
\begin{aligned}
f11(\lambda, S, G) &= 2\lambda S1[G8 - T'(1) - G6 + \lambda(S2)(G4)] + 2\lambda(b - S1)G8,\\
f12(\lambda, S, G, \tilde{Q}) &= [(S1)(T8) + T3]T25 + (G1)(T26) - 2(T4)(G1)[i - b + S1]\\
&\quad - 2(T4)(G2) - (T5)(G1),\\
f13(\lambda, b, X, S) &= 2\lambda b(b - 1)[2(b - 2)E(X) + 3X''(1)]\\
&\quad - \lambda[E(X)(T18)4bX'''(1) - 9S4],\\
f14(\lambda, b, S, \tilde{Q}) &= \lambda b(T1) + (G10)(S1) - \lambda(T6)[b - S1],\\
f15(\lambda, S, G) &= \lambda(S1)(G4) \text{ and } f16(S, \tilde{Q}) = (S1)(T8).
\end{aligned}
$$

4.2. Expected waiting time in the queue

The mean waiting time of the customers in the queue E(W) can be easily obtained using Little's formula

$$E(W) = \frac{E(Q)}{\lambda E(X)}\cdot \tag{4.2}$$

4.3. Expected length of idle period

The time period from the vacation initiation epoch to the busy period initiation epoch is called the idle time period. Let I be the random variable for 'idle period'.

π_j, $j = 0, 1, 2, \ldots a - 1$, is the probability that the system state (number of customers in the system) visits 'j' during an idle period.

Let $I_j = \begin{cases} 1 \text{ if the state 'j' is visited during an idle period} \\ 0 \; otherwise \end{cases}$

Conditioning on the queue size at service completion epoch, we have $\pi_0 = \alpha_0$

$$\pi_j = P(I_j = 1) = \alpha_j + \sum_{k=0}^{j-1} \alpha_k \, P(I^1_{j-k} = 1); j = 1, 2, 3, \ldots a - 1,$$

where $P(I^1_j = 1)$ is the probability that the system state becomes j during an idle period of $M^X/G(a, b)/1$ queueing system without vacation and α_j is the probability that 'j' customers in the queue at a service completion epoch.

$P(I^1_j = 1)$ is obtained as $P(I^1_j = 1) = \phi_j$; where $\phi_0 = 1$, $\phi_n = \sum\limits_{i=1}^{n} g_i \phi_{n-i}$.

Thus the expected length of the idle period is obtained as

$$E(I) = \frac{1}{\lambda} \sum_{j=0}^{a-1} \pi_j. \tag{4.3}$$

where $\frac{1}{\lambda}$ is the expected staying time in the state 'j' during an idle period.

4.4. Expected length of busy period

Let B be the busy period random variable. A random variable J is defined as, $J = 0$, if the server finds less than 'a' customers in the queue at a service completion epoch and $J = 1$, if the server finds 'a' or more customers in the queue at a service completion epoch. Then,

$$\begin{aligned} E(B) &= E(B/J=0)P(J=0) + E(B/J=1)P(J=1) \\ &= E(S)P(J=0) + (E(S)+E(B))P(J=1) \end{aligned}$$

and since $P(J=0)+P(J=1)=1$, solving for $E(B)$, we get

$$E(B) = \frac{E(S)}{P(J=0)} = \frac{E(S)}{\sum\limits_{i=0}^{a-1} p_i}. \tag{4.4}$$

4.5. Probability that the server is on vacation

Let $P(V)$ be the probability that the server is on vacation at time t.
From equation (A.7), we have

$$\tilde{Q}(z,0) = \frac{1}{(-\lambda+\lambda\xi(z))}\left(\tilde{V}(\lambda-\lambda\xi(z))-1\right)\sum_{n=0}^{a-1} p_n z^n.$$

Therefore,

$$P(V) = \lim_{z\to 1}\tilde{Q}(z,0) = \frac{\left(\tilde{V}(\lambda-\lambda\xi(1))-1\right)}{\lambda-\lambda\xi(1)}\sum_{n=0}^{a-1} p_n. \tag{4.5}$$

4.6. Probability that the server is busy

Let $P(B)$ be the probability that the server is in the busy period at time t.
From equations (A.8) and (A.9), we have

$$\begin{aligned} P(B) &= \lim_{z\to 1}\left(\sum_{i=a}^{b}\tilde{P}_i(z,0)\right) \\ &= \lim_{z\to 1}\left(\sum_{i=a}^{b-1}\tilde{P}_i(z,0)+\tilde{P}_b(z,0)\right) \\ &= \frac{\left(\begin{array}{l} \sum\limits_{i=a}^{b-1}\left(f_1(X,S,\tilde{Q})-f_2(X,S,\tilde{Q})\right)-\sum\limits_{i=0}^{b-1}\left((p_j)\left(2j\ S1+S2\right)\right) \\ +f_3(X,S,\tilde{Q})+\lambda\sum\limits_{i=0}^{a-1}\left(f_4(X,S,\tilde{Q})-f_5(X,\tilde{Q})\right) \end{array}\right)}{2\lambda E(X)\left(b-S1\right)} \end{aligned} \tag{4.6}$$

where

$$f_1(X,S,\tilde{Q}) = 2\lambda b(T2+S1.T1)+\lambda b(b-1).T1+\left(p_i+\lambda\sum_{m=0}^{a-1}T_m g_{i-m}\right)2bS1.S2,$$

$$f_2(X,S,\tilde{Q}) = \lambda T6\left(b(b-1)-S2\right),$$

$$f_3(X,S,\tilde{Q}) = 2\lambda S1\left(T(1)E(X)+T'(1)-\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}(m+j)T_m g_j\right)$$
$$+\lambda S2\left(T(1)-\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}T_m g_j\right),$$

$$f_4(X,S,\tilde{Q}) = 2\left(T8.S1+T4\right)\left(iG1+G2\right)+G1.T8.S2 \text{ and}$$

$$f_5(X,\tilde{Q}) = 2T4\left((i-b)G1+G2\right).$$

5. Special cases

The model so developed is general in nature as the service time and vacation time are arbitrary. But for practical purposes, service time and vacation time with particular distribution is required. In this section, some special cases of the proposed model by specifying vacation time random variable as Hyper Exponential distribution and bulk service time random variable as Exponential distribution, are discussed.

Case (i): ***Single server batch arrival queue with Hyper exponential vacation time***

Now, the case of hyper-exponential vacation time random variable is considered. The probability density function of hyper- exponential vacation time is given as follows:

$v(x) = cue^{-ux}+(1-c)we^{-wx}$, where u and w are the parameters.

Then, $\tilde{V}(\lambda-\lambda\xi(z)) = \left(\frac{uc}{u+\lambda(1-\xi(z))}\right)+\left(\frac{w(1-c)}{w+\lambda(1-\xi(z))}\right)$

Hence, the PGF of the queue size distribution of this model can be obtained by,

$$P(z) = \frac{\left(\begin{array}{l}\sum\limits_{i=a}^{b-1}\left(\left(z^b-1\right)\tilde{S}(\lambda-\lambda X(z))\lambda E1+\left(\tilde{S}(\lambda-\lambda X(z))-1\right)z^bE3\right.\\ \left.-\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\lambda E2\right)-\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum\limits_{j=0}^{b-1}(p_j)z^j\\ +\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\lambda E4+\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)(-\lambda+\lambda X(z))T(z)\\ +\lambda\sum\limits_{i=0}^{a-1}z^i f\left(\tilde{S},\tilde{Q},z\right)-\lambda\sum\limits_{i=0}^{a-1}z^{i-b}\psi\left(\tilde{S},\tilde{Q},z\right)\end{array}\right)}{(-\lambda+\lambda X(z))\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)}$$
$$+\frac{\left(\left(\frac{uc}{u+\lambda(1-\xi(z))}\right)+\left(\frac{w(1-c)}{w+\lambda(1-\xi(z))}\right)-1\right)}{(-\lambda+\lambda\xi(z))}\sum_{n=0}^{a-1}p_n.$$

Case (ii): ***Single server batch arrival queue with exponential bulk service time***

Now, the case of exponential service time random variable is considered. The probability density function of exponential service time is given as follows:

$$s(x) = \mu\, e^{-\mu x}, \text{ where } \gamma \text{ is the parameter. Then,}$$

$$\tilde{S}(\lambda - \lambda X(z)) = \left(\frac{\mu}{\mu + \lambda(1 - X(z))}\right).$$

Hence, the PGF of the queue size distribution of this model can be obtained by,

$$P(z) = \frac{\left(\begin{array}{l} \sum\limits_{i=a}^{b-1}\left(\left(z^b-1\right)\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)\lambda E1 + \left(\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)-1\right)z^b E3 \right.\\ \left. -\left(z^b - \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)\right)\lambda E2\right) - \left(\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)-1\right)\sum\limits_{j=0}^{b-1}(p_j)z^j \\ +\left(\left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)-1\right)\lambda E4 \\ +\left(z^b - \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)\right)(-\lambda + \lambda \mathrm{X(z)})\,T(z) + \lambda\sum\limits_{i=0}^{a-1} z^i f\left(\tilde{S},\tilde{Q},z\right) \\ -\lambda\sum\limits_{i=0}^{a-1} z^{i-b}\psi\left(\tilde{S},\tilde{Q},z\right) \end{array}\right)}{(\text{-}\lambda + \lambda \mathrm{X(z)})\left(z^b - \left(\frac{\mu}{\mu+\lambda(1-X(z))}\right)\right)} + \frac{\left(\tilde{V}(\lambda - \lambda\xi(z)) - 1\right)\sum\limits_{n=0}^{a-1} p_n z^n}{(-\lambda + \lambda\xi(z))}$$

where

$$f(\tilde{S},\tilde{Q},z) = \left(\tilde{Q}_i(\lambda - \lambda X(z))\left(\frac{\mu}{\mu + \lambda(1 - X(z))}\right) - \tilde{Q}_i(0)\right)\left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)$$

and

$$\psi(\tilde{S},\tilde{Q},z) = \left(\tilde{Q}_i(\lambda - \lambda X(z)) - \tilde{Q}_i(0)\right)\left(\frac{\mu}{\mu + \lambda(1 - X(z))}\right)\left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right).$$

Case(iii): If the server doesn't avail any vacation ($i.e.$, $\tilde{V}(\lambda - \lambda\xi(z)) = 1$), then the equation (3.16) reduces to

$$P(z) = \frac{\begin{pmatrix} \sum\limits_{i=a}^{b-1} \left(\tilde{S}(\lambda{-}\lambda X(z)) - 1\right) \left(z^b - z^i\right) p_i \\ \lambda \sum\limits_{i=a}^{b-1} \left(\left(\tilde{S}(\lambda{-}\lambda X(z)) - 1\right) z^b \left(\sum\limits_{m=0}^{a-1} T_m g_{i-m}\right)\right) \\ - \left(\tilde{S}(\lambda{-}\lambda X(z)){-}1\right) \sum\limits_{j=0}^{a-1} p_j z^j + \left(z^b{-}\tilde{S}(\lambda{-}\lambda X(z))\right) (-\lambda{+}\lambda X(z))\, T(z) \\ + \left(\tilde{S}(\lambda{-}\lambda X(z)) - 1\right) \lambda \left(T(z)(X(z) - \sum\limits_{m=0}^{a-1} T_m z^m \sum\limits_{j=1}^{b-m-1} g_j z^j\right) \end{pmatrix}}{(-\lambda + \lambda X(z)) \left(z^b - \tilde{S}(\lambda{-}\lambda X(z))\right)}. \tag{5.1}$$

Equation (5.1) is the PGF of queue size distribution of $M^X/G(a,b)/1$ queueing system without vacation and it coincides with the result of $M^X/G(a,b)/1$ queueing system without modified vacation and constant arrival rate of Balasubramanian and Arumuganathan [3].

6. Cost model

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost, set up cost and reward cost (if any). It is quite natural that the management of the system desires to minimize the total average cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

C_{h} : holding cost per customer
C_{o} : operating cost per unit time
C_{s} : startup cost per cycle
C_r : reward cost per cycle due vacation.

Since the length of the cycle is the sum of the idle period and busy period, from equations (4.3) and (4.4), the expected length of cycle, $E(T_c)$ is obtained as

$$E(T_c) = E(\text{length of the Idle Period}) + E(\text{length of the Busy Period})$$

$$E(T_c) = \frac{1}{\lambda}\sum_{j=0}^{a-1}\pi_j + \frac{E(S)}{\sum\limits_{i=0}^{a-1} p_i}.$$

Now, the total average cost per unit time is obtained as

Total average cost = Start-up cost per cycle + holding cost of number of customers in the queue per unit time + Operating cost per unit time * ρ – reward due to vacation per cycle.

$$TAC = [C_s - C_r E(I)]\,\frac{1}{E(T_c)} + C_h E(Q) + C_o \rho \tag{6.1}$$

where $\rho = \frac{\lambda E(X)E(S)}{b}$.

It is difficult to have a direct analytical result for the optimal value a^* (minimum batch size in $M^X/G(a,b)/1$ queueing system), to minimize the total average cost. The simple direct search method to find optimal policy for a threshold value a^* to minimize the total average cost, is defined.

Step 1 : Fix the value of maximum batch size 'b'
Step 2 : Select the value of 'a' which will satisfy the following relation

$$\text{TAC}\,(\text{a}^*) \le TAC(a), 1 \le a \le b$$

Step 3 : The value a^* is optimum, since it gives minimum total average cost.

By using the above procedure to find optimal value of 'a' which minimizes the total average cost function, some numerical example to illustrate the above solution is presented in the next section.

7. Numerical illustration

In this section, a numerical example is analyzed to illustrate how the management of a sintering processing system can effectively use the results obtained in Sections 3 and 4 to take decision regarding effectively utilizing the idle time and to identify the threshold value to minimize the total average cost with the following assumptions:

Service time distribution is k-Erlang with	$k = 2$
Batch size distribution of the arrival is geometric with mean	2
Vacation period is exponential with parameter	γ
Minimum service capacity	a
Maximum service capacity	b

7.1. Effects of various parameters on performance measures

In this section, the effects of various parameters such as arrival rates, service rates, threshold value 'a', and the total average cost are discussed numerically. These results are presented in Tables 1–3 and represented in Figures 2–3. All numerical results are obtained using Mat Lab 7.1 software.

7.1.1. Effects of arrival rates on the performance measures

The effect of performance measures for various arrival rates are presented in Table 1. From the table, it is clear that, if the arrival rate increases, the expected queue length, the expected busy period and the expected waiting time increases whereas the expected idle period decreases.

TABLE 1. Effect of performance measures of the system. (For $\mu = 2.0$; $a = 3$; $b = 10$; $\gamma = 8$).

λ	$E(Q)$	$E(B)$	$E(I)$	$E(W)$	TAC
3.0	0.9649	1.2411	0.4707	0.1608	3.2189
3.2	1.0087	1.3030	0.4330	0.1681	3.3309
3.4	1.0474	1.3825	0.3997	0.1746	3.4209
3.6	1.0819	1.4812	0.3699	0.1803	3.4914
3.8	1.1089	1.6055	0.3438	0.1848	3.5398
4.0	1.1245	1.7586	0.3204	0.1874	3.5701

TABLE 2. Arrival rate (Vs) Servers state. (For $a = 2$, $b = 10$, $\mu = 2.0$).

λ	$P(B)$	$P(V)$	$P(D)$
1.0	0.2517	0.0563	0.6920
1.5	0.3525	0.0666	0.5809
2.0	0.4254	0.0681	0.5065
2.5	0.4715	0.0639	0.4645
3.0	0.4942	0.0568	0.4490
3.5	0.4975	0.0485	0.4339

TABLE 3. Effect of threshold value 'a' on the total average cost. (For $\mu = 1.0$; $\lambda = 0.5$; $b = 10$; $\gamma = 6$).

a	EQ	EB	EI	TAC
1	0.3196	3.8041	1.6000	0.8390
2	0.4666	4.2897	2.5600	0.6809
3	0.5309	6.8133	3.5360	0.5809
4	0.9822	6.5855	4.5220	0.6084
5	1.0171	12.4155	5.5130	0.6141
6	1.1445	17.0667	6.5070	0.6374
7	1.3262	22.7955	7.5034	0.6829
8	1.8138	20.0865	8.5010	0.7610
9	2.8266	14.6888	9.4993	0.9380

7.1.2. Effects of arrival rates on the server states

The effect of server's states for various arrival rates are presented in Table 2. From the table, it is observed that, as the arrival rate increases, the probability of busy period increases whereas the probability of vacation period and the probability of dormant period decrease.

7.2. OPTIMAL COST

In this section, a numerical example is analyzed to illustrate how the management of a sintering processing system can effectively use the results obtained in

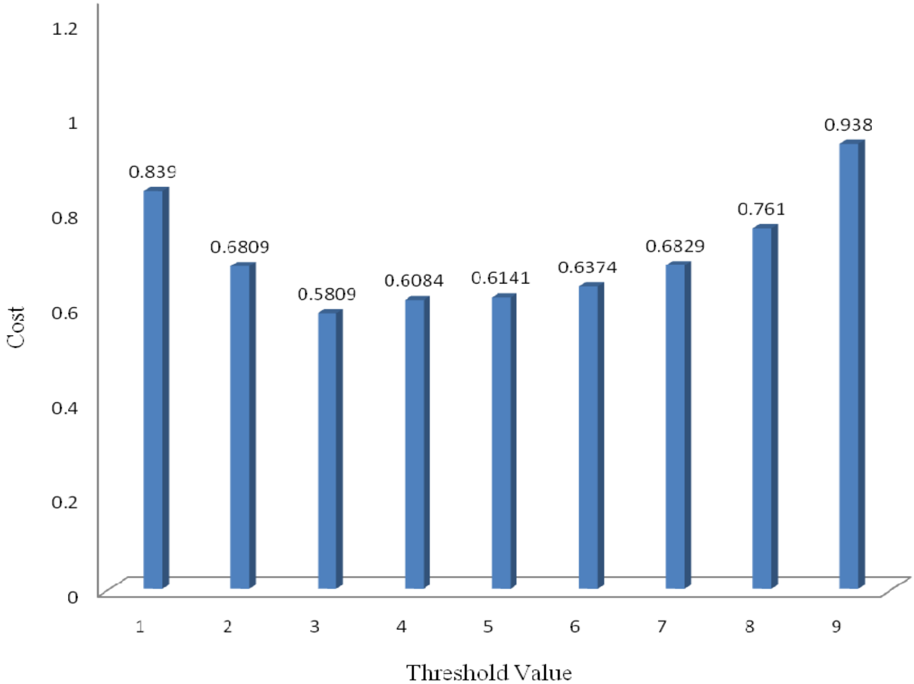

FIGURE 2. Threshold value (Vs) total average cost. (For $\mu = 1.0$; $\lambda = 0.5$; $b = 10$; $\gamma = 6$).

Sections 3 and 4, to make the decision regarding the threshold value to minimize the total average cost. The maximum capacity of the sintering process in bulk is 10 (*i.e.* $b = 10$). If the management of the sintering process allows the operator to start the process even for a single piece (*i.e.* $a = 1$) without waiting for further arrival, clearly, the operating cost will increase. On the other hand, if they start the process until all 10 pieces arrive, the holding cost/vacation cost may increase; hence, there must be some value between 1 and 10 that will optimize the cost. An optimal policy regarding the threshold value 'a' which will minimize the total average cost is wished to be obtained. The total average costs are obtained numerically with the following assumptions:

$$\begin{aligned} \text{Operating cost per unit time} &= Rs.5.00 \\ \text{Holding cost per customer} &= Rs.0.25 \\ \text{Start up cost} &= Rs.3.00 \\ \text{Reward cost due to vacation} &= Rs.1.00. \end{aligned}$$

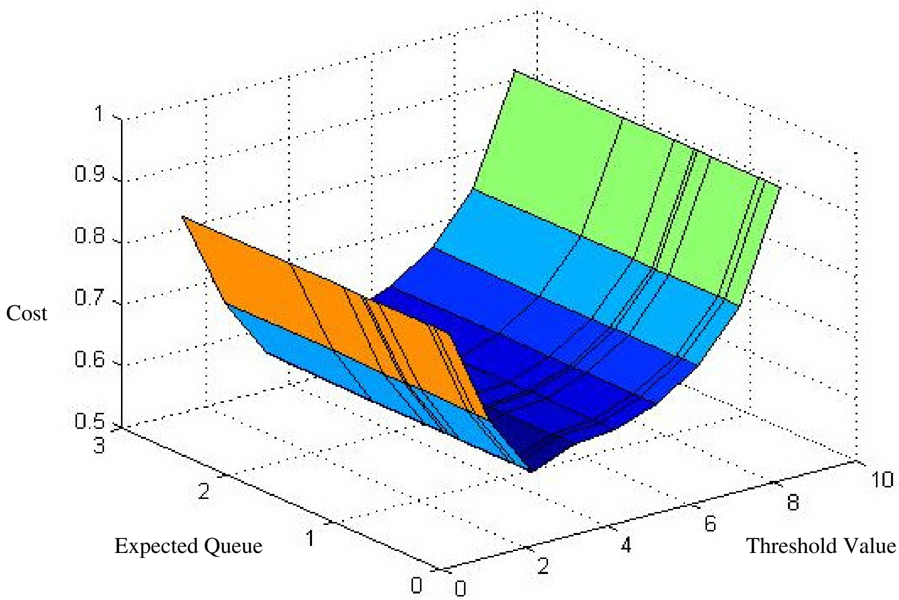

FIGURE 3. Threshold value (Vs) total average cost.(For $\mu = 1.0$; $\lambda = 0.5$; $b = 10$; $\gamma = 6$)

OPTIMAL VALUE OF 'a'

The effects of the threshold value 'a' on the total average cost with $b = 10$ are reported in Table 3 and Figures 2 and 3. From the table and the figures, one can observe that, to minimize the overall cost of the sintering process center, which is capable of processing 10 pieces at a time, the management has to fix the threshold value 'a' as 3, *i.e.*, the operator can start the service process only when 3 pieces have been accumulated.

Similarly, the management has to fix the threshold value 'a' to minimize the total average cost for various arrival rates, service rates and vacation rates.

8. CONCLUSION

In this paper, a "$M^X/G(a,b)/1$ queueing system with interrupted vacation" is analyzed. *The model so considered is unique in the sense that, vacation interruption concept is introduced for a bulk service queueing vacation model.* Probability generating function (PGF) of the steady state queue size distribution at an arbitrary time epoch is obtained. Expressions for various performances are derived. Some special cases of PGF of the queue size are discussed. The effect of various parameters on the system performance measures are also illustrated numerically with the cost model. *A recursive approach is used to express the unknown function* *in the PGF of interrupted vacation queueing system in terms of known values to break the barrier in solving vacation interruption model.* The results so obtained in this paper can be used for managerial decision to optimize the overall cost and search for the best operating policy in a waiting line system. The theoretical development of the model is justified with numerical results which are consistent with the fact that the total average cost decreases based on the threshold value 'a'.

In the direction of future research, the model can be extended with service interruptions, start up and close down concepts. An attempt may be made to derive the busy period distributions and idle period distributions. A discrete time model can also be developed.

Appendix A

Proof of Theorem 3.1

Using PGF (3.15) and taking Z-transforms on equations (3.10)–(3.14), we get the following equations:

$$(\theta - \lambda + \lambda\xi(z))\tilde{Q}(z,\theta) = Q(z,0) - \tilde{V}(\theta)\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}(0)z^n,$$

$$\text{where } \xi(z) = \sum_{k=1}^{a-1} g_k z^k \tag{A.1}$$

$$(\theta - \lambda + \lambda X(z))\tilde{P}_i(z,\theta) = P_i(z,0) - \tilde{S}(\theta)\left(\sum_{m=a}^{b} P_{mi}(0) + \sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\,\lambda\, g_{i-k} + \sum_{m=0}^{a-1} T_m\,\lambda\, g_{i-m}\right) \qquad a \le i \le b-1 \tag{A.2}$$

$$z^b(\theta - \lambda + \lambda X(z))\tilde{P}_b(z,\theta) = P_b(z,0)\left(z^b - \tilde{S}(\theta)\right) - \tilde{S}(\theta)\left(\begin{array}{l} \sum\limits_{m=a}^{b-1} P_m(z,0) - \sum\limits_{m=a}^{b}\sum\limits_{j=0}^{b-1} P_{mj}(0)\,z^j \\ +\lambda\left(T(z)X(z) - \sum\limits_{m=0}^{a-1}\left(T_m z^m \sum\limits_{j=1}^{b-m-1} g_j z^j\right)\right) \\ +\lambda\left(X(z)\sum\limits_{i=0}^{a-1}\tilde{Q}_i(\theta)z^i - \sum\limits_{i=0}^{a-1}\left(\tilde{Q}_i(\theta)z^i \sum\limits_{j=1}^{b-i-1} g_j z^j\right)\right) \end{array}\right). \tag{A.3}$$

Substituting $\theta = \lambda - \lambda\xi(z)$ in equation (A.1), we get

$$Q(z,0) = \tilde{V}(\lambda - \lambda\xi(z))\sum_{n=0}^{a-1}\sum_{m=a}^{b} P_{mn}(0)z^n. \tag{A.4}$$

Substituting $\theta = \lambda - \lambda X(z)$ in equations (A.2) and (A.3), we get

$$P_i(z,0) = \tilde{S}(\lambda - \lambda X(z)) \left(\sum_{m=a}^{b} P_{mi}(0) + \sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z))\, \lambda\, g_{i-k} + \sum_{m=0}^{a-1} T_m\, \lambda g_{i-m} \right), a \leq i \leq b-1 \quad \text{(A.5)}$$

$$P_b(z,0) = \frac{1}{z^b - \tilde{S}(\lambda - \lambda X(z))} \left(\tilde{S}(\lambda - \lambda X(z))\, f(z) \right) \quad \text{(A.6)}$$

where

$$\begin{aligned} f(z) = {} & \sum_{m=a}^{b-1} P_m(z,0) - \sum_{m=a}^{b} \sum_{j=0}^{b-1} P_{mj}(0) z^j \\ & + \lambda \left(T(z)X(z) - \sum_{m=0}^{a-1} \left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j \right) \right) \\ & + \lambda \left(X(z) \sum_{i=0}^{a-1} \tilde{Q}_i(\lambda - \lambda X(z)) z^i - \sum_{i=0}^{a-1} \left(\tilde{Q}_i(\lambda - \lambda X(z)) z^i \sum_{j=1}^{b-i-1} g_j z^j \right) \right). \end{aligned}$$

From equations (A.1) and (A.4), we have

$$\tilde{Q}(z,\theta) = \frac{1}{(\theta - \lambda + \lambda \xi(z))} \left(\tilde{V}(\lambda - \lambda \xi(z)) - \tilde{V}(\theta) \right) \sum_{n=0}^{a-1} \sum_{m=a}^{b} P_{mn}(0) z^n. \quad \text{(A.7)}$$

From equations (A.2) and (A.5), we have

$$\tilde{P}_i(z,\theta) = \frac{\left(\begin{array}{l} \left[\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta) \right] \left(\sum\limits_{m=a}^{b} P_{mi}(0) + \sum\limits_{m=0}^{a-1} T_m\, \lambda\, g_{i-m} \right) \\ + \tilde{S}(\lambda - \lambda X(z)) \sum\limits_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z))\, \lambda\, g_{i-k} - \tilde{S}(\theta) \sum\limits_{k=0}^{a-1} \tilde{Q}_k(\theta)\, \lambda\, g_{i-k} \end{array} \right)}{(\theta - \lambda + \lambda X(z))},$$

$$a \leq i \leq b-1. \quad \text{(A.8)}$$

From equations (A.3) and (A.6), we have

$$\tilde{P}_b(z,\theta) = \frac{\begin{pmatrix} \left(\tilde{S}(\lambda-\lambda X(z)) - \tilde{S}(\theta)\right) g(z) \\ +\lambda \sum_{i=0}^{a-1} z^i \left(\left(\tilde{Q}_i(\lambda-\lambda X(z))\tilde{S}(\lambda-\lambda X(z)) - \tilde{Q}_i(\theta)\tilde{S}(\theta)\right)\right. \\ \left.\times \left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)\right) \\ -\lambda \sum_{i=0}^{a-1} z^{i-b} \left(\left(\tilde{Q}_i(\lambda-\lambda X(z)) - \tilde{Q}_i(\theta)\right)\tilde{S}(\theta)\tilde{S}(\lambda-\lambda X(z))\right. \\ \left.\times \left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)\right) \end{pmatrix}}{(\theta-\lambda+\lambda X(z))\left(z^b - \tilde{S}(\lambda-\lambda X(z))\right)} \tag{A.9}$$

where

$$g(z) = \sum_{m=a}^{b-1} P_m(z,0) - \sum_{m=a}^{b}\sum_{j=0}^{b-1} P_{mj}(0)\, z^j + \lambda \left(T(z)X(z) - \sum_{m=0}^{a-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)\right).$$

Let

$$p_i = \sum_{m=a}^{b} P_{m\,i}(0), q_i = Q_i(0).$$

Let $P(z)$ be the probability generating function of the queue size at an arbitrary time epoch. Then $P(z) = \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \tilde{Q}(z,0) + T(z)$.

Using equations (A.7), (A.8) and (A.9), we get

$$P(z) = \frac{\begin{matrix} (-\lambda+\lambda\xi(z)) \begin{pmatrix} \sum_{i=a}^{b-1} g(\tilde{S},E1,E2,E3,z) - \left(\tilde{S}(\lambda-\lambda X(z)) - 1\right)\sum_{j=0}^{b-1} p_j z^j \\ + \left(\tilde{S}(\lambda-\lambda X(z)) - 1\right)\lambda E4 + h(\tilde{S}, X(z), z) \\ +\lambda \sum_{i=0}^{a-1} z^i f\left(\tilde{S},\tilde{Q},z\right) - \lambda \sum_{i=0}^{a-1} z^{i-b}\psi\left(\tilde{S},\tilde{Q},z\right) \end{pmatrix} \\ + \left((-\lambda+\lambda X(z))\left(z^b - \tilde{S}(\lambda-\lambda X(z))\right)\left(\tilde{V}(\lambda-\lambda\xi(z)) - 1\right)\sum_{n=0}^{a-1} p_n z^n\right) \end{matrix}}{(-\lambda+\lambda\xi(z))\,(-\lambda+\lambda X(z))\left(z^b - \tilde{S}(\lambda-\lambda X(z))\right)} \tag{A.10}$$

where

$$g(\tilde{S}, E1, E2, E3, z) = \Big(\left(z^b - 1\right)\tilde{S}(\lambda - \lambda X(z))\lambda E1 + \left(\tilde{S}(\lambda - \lambda X(z)) - 1\right) z^b E3 - \left(z^b - \tilde{S}(\lambda - \lambda X(z))\right)\lambda E2\Big),$$

$$h(\tilde{S}, X(z), z) = \left(z^b - \tilde{S}(\lambda - \lambda X(z))\right)(-\lambda + \lambda X(z))\, T(z),$$

$$f(\tilde{S}, \tilde{Q}, z) = \left(\tilde{Q}_i(\lambda - \lambda X(z))\tilde{S}(\lambda - \lambda X(z)) - \tilde{Q}_i(0)\right) \times \left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right),$$

$$\psi(\tilde{S}, \tilde{Q}, z) = \left(\tilde{Q}_i(\lambda - \lambda X(z)) - \tilde{Q}_i(0)\right)\tilde{S}(\lambda - \lambda X(z)) \times \left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right),$$

$$E1 = \sum_{k=0}^{a-1} \tilde{Q}_k(\lambda - \lambda X(z))\, g_{i-k}, \quad E2 = \sum_{k=0}^{a-1} \tilde{Q}_k(0)\, g_{i-k},$$

$$E3 = p_i + \lambda \sum_{m=0}^{a-1} T_m g_{i-m} \text{ and}$$

$$E4 = T(z)X(z) - \sum_{m=0}^{a-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)$$

Appendix B

Proof of Theorem 3.4

Substituting $\theta = \lambda$ in equation (3.13), we get $\tilde{Q}_0(0) = \tilde{V}(\lambda)p_0$ therefore

$$\tilde{Q}_0(\theta) = \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda} p_0$$

$$= k_0 p_0 \text{ where } k_0 = \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda}. \qquad \text{(B.1)}$$

Substituting $n = 1$ in equation (3.14), we get

$$(\theta - \lambda)\tilde{Q}_1(\theta) = Q_1(0) - p_1\tilde{V}(\theta) - \sum_{k=1}^{1} \tilde{Q}_{1-k}(\theta)\lambda\, g_k \qquad \text{(B.2)}$$

when $\theta = \lambda$, the above equation reduces to

$$Q_1(0) = p_1\tilde{V}(\lambda) + \lambda g_1\tilde{Q}_0(\lambda) \tag{B.3}$$

Substituting (B.3) in (B.2), we get

$$\tilde{Q}_1(\theta) = \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda}p_1 + \frac{\lambda g_1}{\theta - \lambda}\left(\tilde{Q}_0(\lambda) - \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda}p_0\right)$$
$$= k_0p_1 - \lambda g_1\left(\frac{\tilde{V}^1(\lambda) + k_0}{\theta - \lambda}\right)p_0$$
$$= k_0p_1 - \lambda g_1 k_1 p_0 \text{ where } k_1 = \frac{\tilde{V}^1(\lambda) + k_0}{\theta - \lambda}.$$

Substituting $n = 2$ in equation (3.14), we get

$$(\theta - \lambda)\tilde{Q}_2(\theta) = Q_2(0) - p_2\tilde{V}(\theta) - \sum_{k=1}^{2}\tilde{Q}_{2-k}(\theta)\lambda g_k) \tag{B.4}$$

when $\theta = \lambda$, the above equation reduces to

$$Q_2(0) = p_2\tilde{V}(\lambda) + \lambda\, g_1\tilde{Q}_1(\lambda) + \lambda\, g_2\tilde{Q}_0(\lambda). \tag{B.5}$$

Substituting (B.5) in (B.4), we get

$$\tilde{Q}_2(\theta) = \left(\frac{\tilde{V}(\lambda) - \tilde{V}(\theta)}{\theta - \lambda}\right)p_2 + \lambda g_2 p_0\left(\frac{\tilde{V}^1(\lambda) + \frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta - \lambda}\right)$$
$$- \lambda g_1 p_1\left(\frac{\tilde{V}^1(\lambda) + \frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta - \lambda}\right)$$
$$+ (\lambda g_1)(\lambda g_1)p_0\left(\frac{\tilde{V}^2(\lambda) + \frac{\tilde{V}^1(\lambda)+\frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta-\lambda}}{\theta - \lambda}\right)$$
$$= k_0p_2 - \lambda g_2 k_1 p_0 - \lambda g_1 k_1 p_1 + (\lambda g_1)(\lambda g_1)k_2 p_0 \text{ where } k_2 = \frac{\tilde{V}^2(\lambda) + k_1}{\theta - \lambda}.$$

Substituting $n = 3$ in equation (3.14), we get

$$(\theta - \lambda)\tilde{Q}_3(\theta) = Q_3(0) - p_3\tilde{V}(\theta) - \sum_{k=1}^{3}\tilde{Q}_{3-k}(\theta)\lambda\, g_k \tag{B.6}$$

when $\theta = \lambda$, the above equation reduces to

$$Q_3(0) = p_3\tilde{V}(\lambda) + \lambda g_1\tilde{Q}_2(\lambda) + \lambda\, g_2\tilde{Q}_1(\lambda) + \lambda g_3\tilde{Q}_0(\lambda). \tag{B.7}$$

Substituting (B.7) in (B.6), we get

$$\tilde{Q}_3(\theta) = \left(\frac{\tilde{V}(\lambda) - \tilde{V}(\theta)}{\theta - \lambda}\right) p_3 - \lambda g_1 p_2 \left(\frac{\tilde{V}^1(\lambda) + \frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$- \lambda g_2 p_1 \left(\frac{\tilde{V}^1(\lambda) + \frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$+ (\lambda g_1)(\lambda g_1) p_1 \left(\frac{\tilde{V}^2(\lambda) + \frac{\tilde{V}^1(\lambda)+\frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$- \lambda g_3 p_0 \left(\frac{\tilde{V}^1(\lambda) + \frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$+ (\lambda g_1)(\lambda g_2) p_0 \left(\frac{\tilde{V}^2(\lambda) + \frac{\tilde{V}^1(\lambda)+\frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$+ (\lambda g_1)(\lambda g_2) p_0 \left(\frac{\tilde{V}^2(\lambda) + \frac{\tilde{V}^1(\lambda)+\frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$- (\lambda g_1)(\lambda g_1)(\lambda g_1) p_0 \left(\frac{\tilde{V}^3(\lambda) + \frac{\tilde{V}^2(\lambda)+\frac{\tilde{V}^1(\lambda)+\frac{\tilde{V}(\lambda)-\tilde{V}(\theta)}{\theta-\lambda}}{\theta-\lambda}}{\theta-\lambda}}{\theta - \lambda}\right)$$

$$\tilde{Q}_3(\theta) = k_0\, p_3 - (\lambda g_1)\, k_1 p_2 - (\lambda g_2)\, k_1 p_1 + (\lambda g_1)\,(\lambda g_1)\, k_2 p_1$$

$$\left\{\begin{matrix} k & = 0 \\ A_0 & = \varphi \end{matrix}\right\} \quad \left\{\begin{matrix} k & = 1 \\ A_1 & = \{1\} \end{matrix}\right\} \quad \left\{\begin{matrix} k & = 2 \\ A_2 & = \{2\} \end{matrix}\right\} \quad \left\{\begin{matrix} k & = 2 \\ A_2 & = \{1,1\} \end{matrix}\right\}$$

$$-\lambda g_3 k_1\, p_0 + (\lambda g_2)\,(\lambda g_1)\, k_2 p_0 + (\lambda g_1)\,(\lambda g_2)\, k_2 p_0 - (\lambda g_1)\,(\lambda g_1)\,(\lambda g_1)\, k_3 p_0$$

$$\left\{\begin{matrix} k & = 3 \\ A_3 & = \{3\} \end{matrix}\right\} \quad \left\{\begin{matrix} k & = 3 \\ A_3 & = \{2,1\} \end{matrix}\right\} \quad \left\{\begin{matrix} k & = 3 \\ A_3 & = \{1,2\} \end{matrix}\right\} \quad \left\{\begin{matrix} k & = 3 \\ A_3 & = \{1,1,1\} \end{matrix}\right\}$$

where

$$k_3 = \frac{\tilde{V}^3(\lambda) + k_2}{\theta - \lambda}. \qquad \text{(B.8)}$$

Using appropriate notations, $\tilde{Q}_3(\theta)$ is expressed in a compact form as

$$\tilde{Q}_3(\theta) = \sum_{j=0}^{3} \sum_{A_j \in \phi} \left((-1)^{n(A_j)} \left(\prod_{i \in A_j} \lambda g_i \right) p_{3-j}\ k_{n(A_j)} \right).$$

Generalizing by recursive approach, we get

$$\tilde{Q}_i(\theta) = \sum_{j=0}^{i} \sum_{A_j \in \phi} \left((-1)^{n(A_j)} \left(\prod_{i \in A_j} \lambda g_i \right) p_{i-j}\ k_{n(A_j)} \right); \quad \mathrm{i} = 0, 1, 2, 3, \ldots a-1$$

where $k_l = \frac{\tilde{V}^{n(A_l)}(\lambda) + k_{l-1}}{\theta - \lambda}$; $l = 1, 2, 3,$
$ldotsi$ and $k_0 = \frac{\tilde{V}(\lambda) - \tilde{V}(\theta)}{\theta - \lambda}$.

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References

[1] R. Arumuganathan and S. Jeyakumar, Steady state analysis of a bulk queue with multiple vacations, setup times with N-policy and closedown times. *Appl. Math. Modell.* **29** (2005) 972–986.

[2] M. Balasubramanian, R. Arumuganathan and A. Senthil Vadivu, Steady state analysis of a non-Markovian bulk queueing system with overloading and multiple vacations. *Int. J. Oper. Res.* **9** (2010) 82–103.

[3] M. Balasubramanian and R. Arumuganathan, Steady state analysis of a bulk arrival general bulk service queueing system with modifed M-vacation policy and variant arrival rate. *Int. J. Oper. Res.* **11** (2011) 383–407.

[4] A. Borthakur and J. Medhi, A queueing system with arrival and services in batches of variable size. *Cahiers du. C.E.R.O.* **16** (1974) 117–126.

[5] M.L. Chaudhry and J.G.C. Templeton, *A first course in bulk queues.* New York, John Wiley and Sons (1983).

[6] B.T. Doshi, Single server queues with vacations: a survey, *Queueing Systems.* **I** (1986) 29–66.

[7] B.T. Doshi, *Single server queues with vacation, Stochastic Analysis of the Computer and Communication Systems*, edited by H. Takagi. North-Holland/Elsevier, Amsterdam (1990) 217–264.

[8] M. Haridass and R. Arumuganathan, Analysis of a batch arrival general bulk service queueing system with variant threshold policy for secondary jobs. *Int. J. Math. Oper. Res.* **3** (2011) 56–77.

[9] H. Zhang and D. Shi, The M/M/1 queue with Bernoulli-Schedule-Controlled vacation and vacation interruption. *Int. J. Inf. Manag. Sci.* **20** (2009) 579–587.

[10] Jau-Chuan Ke, Chia-Huang Wu and Wen Lea Pearn, Algorithmic analysis of the multi-server system with a modified Bernoulli vacation schedule. *Appl. Math. Model.* **35** (2011) 2196–2208.

[11] Ji-Hong Li and Nai-Shuo Tian, The M/M/1 queue with working vacations and vacation interruptions. *J. Syst. Sci. Eng.* **16** (2007) 121–127.

[12] Ji-Hong Li, Nai-Shuo Tian and Zhan-You Ma, Performance analysis of GI/M/1 queue with working vacations and vacation interruption. *Appl. Math. Model.* **32** (2008) 2715–2730.

[13] Ji-Hong Li and Nai-Shuo Tian, Performance analysis of a GI/M/1 queue with single working vacation. *Appl. Math. Comput.* **217** (2001) 4960–4971.

[14] G.V Krishna Reddy, R. Nadarajan and R. Arumuganathan, Analysis of a bulk queue with N-policy, multiple vacations and setup times. *Comput. Oper. Res.* **25** (1998) 957–967.

[15] H.W. Lee, S.S Lee, J.O Park and K.C. Chae, Analysis of the $M^x/G/1$ queue with N-policy and multiple vacations. *J. Appl. Prob.* **31** (1994) 476–496.

[16] J. Li and N. Tian, The discrete-time GI/Geo/1 queue with working vacations and vacation interruption. *Appl. Math. Comput.* **185** (2007) 1–10.

[17] J. Medhi, *Recent Developments in Bulk Queueing Models*. Wiley Eastern Ltd. New Delhi (1984).

[18] Mian Zhang and Zhengting Hou, Performance analysis of M/G/1 queue with working vacations and vacation interruption. *J. Comput. Appl. Math.* **234** (2010) 2977–2985.

[19] Mian Zhang and Zhengting Hou, Performance analysis of MAP/G/1 queue with working vacations and vacation interruption. *Appl. Math. Modell.* **35** (2011) 1551–1560.

[20] N. Limnios and Gheorghe Oprisan, *Semi-Markov processes and reliability- Statistics for Industry and Technology* Birkhauser Boston, Springer (2001).

[21] H. Takagi, Queueing Analysis: A foundation of Performance Evaluation, Vacation and Priority Systems. North Holland, Amsterdam (1991), Vol. 1.

[22] N. Tian and S.G. Zhang, *Vacation Queueing Models: Theory and Applications*. Springer, New York (2006).

[23] Y. Baba, The M/PH/1 queue with working vacations and vacation interruption. *J. Syst. Sci. Eng.* **19** (2010) 496–503.