# TRANSIENT RANDOM WALK IN $\mathbb{Z}^{2}$ WITH STATIONARY ORIENTATIONS 

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#### Abstract

In this paper, we extend a result of Campanino and Pétritis [Markov Process. Relat. Fields 9 (2003) 391-412]. We study a random walk in $\mathbb{Z}^{2}$ with random orientations. We suppose that the orientation of the $k$ th floor is given by $\xi_{k}$, where $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ is a stationary sequence of random variables. Once the environment fixed, the random walk can go either up or down or can stay in the present floor (but moving with respect to its orientation). This model was introduced by Campanino and Pétritis in [Markov Process. Relat. Fields 9 (2003) 391-412] when the $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ is a sequence of independent identically distributed random variables. In [Theory Probab. Appl. 52 (2007) 815-826], Guillotin-Plantard and Le Ny extend this result to a situation where the orientations of the floors are independent but chosen with stationary probabilities (not equal to 0 and to 1 ). In the present paper, we generalize the result of [Markov Process. Relat. Fields 9 (2003) 391-412] to some cases when $\left(\xi_{k}\right)_{k}$ is stationary. Moreover we extend slightly a result of [Theory Probab. Appl. 52 (2007) 815-826].


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## 1. Introduction

Random walks in random environment in $\mathbb{Z}^{d}$ have been studied by many authors. For a general reference on this subject, we refer to chapter 6 of Hughes [10]. Random walks with random orientations have been less studied. However these two subjects are not far from each other. Indeed, random walks with random orientations can be viewed as a degenerate case of random walks in random environment in the sense that transition probabilities are allowed to be null. But this difference is significant. Moreover random walks in $\mathbb{Z}^{2}$ with random orientations can also be viewed as a question of oriented percolation (see Sect. 12.8 of Grimmett [8]).

The present paper contains an extension of the model introduced by Campanino and Pétritis in [5] in another direction than the one chosen by Guillotin-Plantard and Le Ny in [9]. But our result will also apply to random walks of the form studied in [9]. Now, let us present the different models introduced in [5], in [9] and in the present paper with their common ideas and their differences. Let us construct a random walk $\left(M_{n}=\left(\tilde{X}_{n}, \tilde{Y}_{n}\right)\right)_{n \geq 0}$ in $\mathbb{Z}^{2}$ with random orientations as follows. Let $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ be a stationary sequence of centered random variables with values in $\{-1 ; 1\}$. The orientations of the $k$ th horizontal floor of $\mathbb{Z}^{2}$ is given by $\xi_{k}$. Once the environment fixed, the random walk $\left(M_{n}=\left(\tilde{X}_{n}, \tilde{Y}_{n}\right)\right)_{n}$ will be such that $M_{0}=(0,0)$ and such that the distribution of $M_{n+1}-M_{n}$ conditioned to $\sigma\left(M_{k} ; k=0, \ldots, n\right)$ is uniform on $\left\{(0,1) ;(0,-1) ;\left(\xi_{\tilde{Y}_{n}}, 0\right)\right\}$.

[^0]In [5], Campanino and Pétritis prove the transience of the random walk $\left(M_{n}\right)_{n}$ when $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ is sequence of independent identically distributed random variables. Moreover, they point out the fact that the random walk $\left(M_{n}\right)_{n \geq 0}$ is recurrent in the "alternate" case where $\xi_{k}$ only depends on the parity of $k$. Hence the behaviour of this random walk depends on the randomness of the orientations $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$.

In [9], Guillotin-Plantard and Le Ny give a first generalization of the work of Campanino and Pétritis. They envisage the case when the orientations of the floors are taken independently with stationary probabilities. More precisely, they consider the following situation: let $\left(f_{k}\right)_{k \in \mathbb{Z}}$ be a stationary sequence of random variables with values in $[0 ; 1]$ and with expectation equal to $\frac{1}{2}$ defined on some probability space $(M, \mathcal{F}, \nu)$. Let us consider the probability space given by $\left(\Omega_{1}:=M \times[0 ; 1]^{\mathbb{Z}}, \mathcal{F}_{1}:=\mathcal{F} \otimes(\mathcal{B}([0 ; 1]))^{\otimes \mathbb{Z}}, \nu_{1}:=\nu \otimes(\lambda) \otimes \mathbb{Z}\right)$, where $\lambda$ is the Lebesgue measure on $[0 ; 1]$. We define $\left(\tilde{\xi}_{k, f_{k}}\right)_{k \in \mathbb{Z}}$ on this space as follows:

$$
\tilde{\xi}_{k, f_{k}}\left(\omega,\left(z_{m}\right)_{m \in \mathbb{Z}}\right):=2.1_{\left\{z_{k} \leq f_{k}(\omega)\right\}}-1 .
$$

This means that, once a realization of $\left(f_{k}\right)_{k}$ given, the horizontal floors are oriented independently; the $k$ th floor being oriented to the right with probability $f_{k}$. We will use this notation $\tilde{\xi}_{k, f_{k}}$ later in the paper. In [9], Guillotin-Plantard and Le Ny prove that, if $\left(\xi_{k}\right)_{k}=\left(\tilde{\xi}_{k, f_{k}}\right)_{k}$, then the corresponding random walk $\left(M_{n}\right)_{n}$ is transient under the following condition: $\int_{M} \frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}} \mathrm{d} \nu<+\infty$ (this implies that $0<f_{0}<1$ a.s.).

Let us notice that the $\left(\xi_{k}\right)_{k}$ studied in [9] is stationary. Conversely, if $\left(\xi_{k}\right)_{k}$ is stationary, then it can be described by the approach of [9] by taking $f_{k}:=\mathbf{1}_{\left\{\xi_{k}=1\right\}}=\frac{1}{2}\left(\xi_{k}+\mathbf{1}\right)$. But the method of [9] cannot be applied to a function $f_{0}$ that can be equal to 0 or 1 with a non-null probability.

In this paper, we are interested in the case when $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ is a stationary sequence of random variables satisfying some strong decorrelation properties. We state our main result in Section 2 and prove it in Section 3. Examples are given in Section 2 and detailed in the appendix. Our examples satisfy a strong mixing condition. We complete this paper with a short discussion in Section 4 about the model envisaged by Guillotin-Plantard and Le Ny. We prove that their result remains true if the condition $\int_{M} \frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}} \mathrm{d} \nu<+\infty$ is replaced by $\int_{M} \frac{1}{\left[f_{0}\left(1-f_{0}\right)\right]^{p}} \mathrm{~d} \nu<+\infty$, for some $p>0$.

## 2. Main result, examples, strong mixing property

Theorem 1. Let $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ be a stationary sequence of centered random variables with values in $\{-1 ; 1\}$ such that:
(1) We have: $\sum_{p \geq 0} \sqrt{1+p}\left|\mathbb{E}\left[\xi_{0} \xi_{p}\right]\right|<+\infty$ and $c_{0}^{\prime}:=\sup _{N \geq 1} N^{-2} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=0, \ldots, N-1}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right|<$ $+\infty$.
(2) There exist some $C>0$, some $\left(\varphi_{p, s}\right)_{p, s \in \mathbb{N}}$ and some integer $r \geq 1$ such that for all positive integers $p$ and $s$, we have $\varphi_{p+1, s} \leq \varphi_{p, s}$, such that we have $\lim _{s \rightarrow+\infty} s^{6} \varphi_{r s, s}=0$ and such that, for all integers $n_{1}, n_{2}, n_{3}, n_{4}$ with $0 \leq n_{1} \leq n_{2} \leq n_{3} \leq n_{4}$, for all real numbers $\alpha_{n_{1}}, \ldots, \alpha_{n_{2}}$ and $\beta_{n_{3}}, \ldots, \beta_{n_{4}}$, we have:

$$
\left|\operatorname{Cov}\left(\mathrm{e}^{i \sum_{k=n_{1}}^{n_{2}} \alpha_{k} \xi_{k}}, \mathrm{e}^{i \sum_{k=n_{3}}^{n_{4}} \beta_{k} \xi_{k}}\right)\right| \leq C\left(1+\sum_{k=n_{1}}^{n_{2}}\left|\alpha_{k}\right|+\sum_{k=n_{3}}^{n_{4}}\left|\beta_{k}\right|\right) \varphi_{n_{3}-n_{2}, n_{4}-n_{3}} .
$$

Then the random walk $\left(M_{n}\right)_{n}$ is transient.
This result is proved in Section 3. We will see in its proof that this question is linked with $\sum_{k=0}^{n-1} \xi_{S_{k}}$ where $\left(S_{m}\right)_{m \geq 0}$ is a simple symmetric random walk on $\mathbb{Z}$ independent of $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$. Let us give some examples of stationary sequences $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ to which this result applies.

Theorem 2. ( $\alpha$-mixing condition) Let $\left(g_{k}\right)_{k \in \mathbb{Z}}$ be a stationary sequence of bounded real-valued random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying the following $\alpha$-mixing condition:

$$
\sup _{n \geq 1} n^{6} \alpha_{n}<+\infty, \text { with } \alpha_{n}:=\sup _{p \geq 0 ; m \geq 0} \sup _{A \in \sigma\left(g_{-p}, \ldots, g_{0}\right)} \sup _{B \in \sigma\left(g_{n}, \ldots, g_{n+m}\right)}|\mathbb{P}(A \cap B)-\mathbb{P}(A) \mathbb{P}(B)|
$$

Then:
(a) If $g_{k}$ takes its values in $\{-1 ; 1\}$, if $\int_{M} g_{k} \mathrm{~d} \nu=0$ and if $\left(\xi_{k}:=g_{k}\right)_{k \in \mathbb{Z}}$, then $\left(M_{n}\right)_{n}$ is transient.
(b) If $g_{k}$ takes its values in $[0 ; 1]$, if $\int_{M} g_{k} \mathrm{~d} \nu=\frac{1}{2}$ and if $\left(\xi_{k}:=\tilde{\xi}_{k, g_{k}}\right)_{k \in \mathbb{Z}}$, then $\left(M_{n}\right)_{n}$ is transient.

We will prove that the hypotheses of Theorem 1 are satisfied in the general context of strongly mixing dynamical systems. We say that $(M, \mathcal{F}, \nu, T)$ is an invertible dynamical system if $(M, \mathcal{F}, \nu)$ is a probability space endowed with an invertible bi-measurable transformation $T: M \rightarrow M$.
Definition 3. We say that an invertible dynamical $\operatorname{system}(M, \mathcal{F}, \nu, T)$ is strongly mixing if there exists $c_{0}>0$, there exist two real sequences $\left(\varphi_{n}\right)_{n \geq 0}$ and $\left(\kappa_{m}\right)_{m \geq 0}$ and, for any function $g: M \rightarrow \mathbb{C}$, there exist $K_{g}^{(1)} \in[0 ;+\infty]$ and $K_{g}^{(2)} \in[0 ;+\infty]$ such that, for all bounded functions $g, h: M \rightarrow \mathbb{C}$ :
(1) for all integer $n \geq 0$, we have: $\left|\operatorname{Cov}_{\nu}\left(g, h \circ T^{n}\right)\right| \leq c_{0}\left(\|g\|_{\infty}\|h\|_{\infty}+\|h\|_{\infty} K_{g}^{(1)}+\|g\|_{\infty} K_{h}^{(2)}\right) \varphi_{n} ;$
(2) for all integer $m \geq 0$, we have: $K_{g \circ T^{-m}}^{(1)} \leq c_{0} K_{g}^{(1)}$ and: $K_{h \circ T^{m}}^{(2)} \leq c_{0} K_{h}^{(2)}\left(1+\kappa_{m}\right)$;
(3) we have: $K_{g \times h}^{(1)} \leq\|g\|_{\infty} K_{h}^{(1)}+\|h\|_{\infty} K_{g}^{(1)}$ and: $K_{g \times h}^{(2)} \leq\|g\|_{\infty} K_{h}^{(2)}+\|h\|_{\infty} K_{g}^{(2)}$;
(4) the sequence $\left(\varphi_{n}\right)_{n \geq 0}$ is decreasing, the sequence $\left(\kappa_{m}\right)_{m \geq 0}$ is increasing and there exists an integer $r \geq 1$ such that: $\sup _{n \geq 1} n^{6}\left(1+\kappa_{n}\right) \varphi_{r n}<+\infty$.
Proposition 4. Let $(M, \mathcal{F}, \nu, T)$ be a strongly mixing dynamical system. Let the sequence $\left(\xi_{k}\right)$ be of one the two following kinds:
(a) $\xi_{k}=f \circ T^{k}$ with $f: M \rightarrow\{-1 ; 1\}$ a $\nu$-centered function such that $K_{f}^{(1)}+K_{f}^{(2)}<+\infty$. We suppose that there exists some real number $c_{1}>0$ such that, for any real number $\alpha$, we have: $K_{\exp (i \alpha f)}^{(1)}+K_{\exp (i \alpha f)}^{(2)} \leq$ $c_{1}|\alpha|$.
(b) $\xi_{k}=\tilde{\xi}_{k, f \circ T^{k}}$ with $f: M \rightarrow[0 ; 1]$ such that $\int_{M} f \mathrm{~d} \nu=\frac{1}{2}$ and such that there exists some $c_{1}>0$ such that, for any $a, b \in \mathbb{C}$, we have $K_{a f+b}^{(1)}+K_{a f+b}^{(2)} \leq c_{1}|a|$.
Then $\left(\xi_{k}\right)_{k}$ satisfies the hypothesis of Theorem 1.
Proposition 4 is proved in Appendix A. Theorem 2 will appear as a direct consequence (see App. B). Our strong mixing property is satisfied by a large class of dynamical systems (endowed with some metric) with $K_{f}^{(1)}$ and $K_{f}^{(2)}$ dominated by the Hölder constant of $f$ of order $\eta$. Interesting examples are given by hyperbolic or quasi-hyperbolic dynamical systems. We quickly give some examples of such dynamical systems. In the case of the billiard transformation, because of the discontinuity of the transformation, our class of allowed functions will contain discontinuous functions.
Examples 2.1. (1) Let $(M, \mathcal{F}, \nu, T)$ where $T$ is an ergodic algebraic automorphism of the torus or a diagonal transformation on a compact quotient of $S l_{d_{0}}(\mathbb{R})$ by a discrete group. Let $\eta>0$. According to [16], the strong mixing property holds with $K_{g}^{(1)}$ some $\eta$-Hölder constant of $g$ along the unstable manifolds and with $K_{h}^{(2)}$ some $\eta$-Hölder constant of $h$ along the stable-central manifolds and with $\varphi_{n}=\alpha^{n}$ for some $\alpha \in(0,1)$ and $\kappa_{m}=m^{\beta}$ for some $\beta \geq 0$. Moreover $K_{g}^{(1)}$ and $K_{g}^{(2)}$ are dominated by the Hölder constant of order $\eta$ of $g$.
(2) Let $(M, \mathcal{F}, \nu, T)$ where $T$ is the Sinai billiard transformation (in $\mathbb{T}^{2}$ ) with $C^{3}$-convex scatterers and with finite horizon and where $\nu$ is the $T$ invariant measure absolutely continuous with respect to the Lebesgue measure [17]. Let $m_{0} \in \mathbb{Z}_{+}$and $\eta>0$. According to [6] (Th. 4.3), the strong mixing property holds with $\varphi_{n}=\alpha^{n}$ for some $\alpha \in(0,1)$ and $\kappa_{m}=m^{\beta}$ for some $\beta \geq 0, K_{g}^{(1)}$ being some Hölder constant of
$g$ along the $T^{-m_{0}}\left(\gamma^{u}\right)$ 's (where the $\gamma^{u}$ 's are the unstable curves) and $K_{h}^{(2)}$ being some Hölder constant of $h$ along the $T^{m_{0}}\left(\gamma^{s}\right.$ 's (where the $\gamma^{s}$ 's are the stable curves). The quantities $K_{h}^{(1)}$ and $K_{h}^{(2)}$ will be dominating by $C_{h}^{\left(\eta, m_{0}\right)}=\sup _{C \in \mathcal{C}_{m}} \sup _{x, y \in C, x \neq y} \frac{|h(x)-h(y)|}{\max \left(d\left(T^{k}(x), T^{k}(y)\right) ; k=-m, \ldots, m\right)^{\eta}}$, where $\mathcal{C}_{m}$ is a set of open subsets of $M$ on which $T^{m}$ and $T^{-m}$ are $C^{1}$.

The first example is a direct consequence of [16]. The second example is a consequence of [6]. In Appendix C, we give a precise definition of $K_{f}^{(1)}$ and of $K_{f}^{(2)}$ for these examples (and a definition of $\mathcal{C}_{m}$ for the Sinai billiard). For these systems, we can say a little more:
Theorem 5. Let $\eta \in(0,1)$ and let $(M, \mathcal{F}, \nu, T)$ be a strongly mixing dynamical system (endowed with some metric) such that there exists $\alpha \in(0,1)$ and $\beta \geq 0$ such that $\varphi_{n}=\alpha^{n}$ and $\kappa_{m}=m^{\beta}$ and such that $K_{h}^{(1)}$ and $K_{h}^{(2)}$ are both dominated by the $\eta$-Hölder constant of $h$. Then:
(A) If $\left(\xi_{k}:=\tilde{\xi}_{k, g_{0} \circ T^{k}}\right)_{k \in \mathbb{Z}}$ with $g_{0}: M \rightarrow[0 ; 1]$ a Hölder continuous function (of order $\eta$ ) such that $\int_{M} g_{0} \mathrm{~d} \nu=\frac{1}{2}$, then $\left(M_{n}\right)_{n}$ is transient.
(B) If $\left(\xi_{k}=2 \mathbf{1}_{A} \circ T^{k}-1\right)_{k \in \mathbb{Z}}$ with $\nu(A)=1 / 2$ and with $A$ such that there exist $c_{A}>0$ and $\zeta>0$ such that, for every $\varepsilon \in] 0 ; 1\left[\right.$, we have: $\nu(\{x \in M: d(x, A)<\varepsilon\}) \leq c_{A} \varepsilon^{\zeta}$, then $\left(M_{n}\right)_{n}$ is transient.
Conclusion (A) of Theorem 5 follows directly from Proposition 4. Conclusion (B) of Theorem 5 is proved in Appendix D.

## 3. Proof of Theorem 1

Let us define $T_{0}:=0$ and, for all $n \geq 1: T_{n+1}:=\inf \left\{k>T_{n}: \tilde{Y}_{k} \neq \tilde{Y}_{k-1}\right\}$.
Now, following [5], we construct a realization of $\left(M_{T_{n}}\right)_{n}$. Let us consider a symmetric random walk $\left(S_{n}\right)_{n}$ on $\mathbb{Z}$ independent of $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$. For any integer $m \geq 1$ and any integer $k$, we define:

$$
N_{m}(k):=\operatorname{Card}\left\{j=0, \ldots, m: S_{j}=k\right\} .
$$

Let us also consider a sequence of independent random variables $\left(\zeta_{i}^{(y)}\right)_{i \geq 1, y \in \mathbb{Z}}$ independent of $\left(\left(\xi_{y}\right)_{y \in \mathbb{Z}},\left(S_{p}\right)_{p \geq 1}\right)$ and such that $\mathbb{P}\left(\zeta_{i}^{(y)}=k\right)=\frac{2}{3^{k+1}}$ for every integer $k \geq 0$.
Lemma 6. The process $\left(X_{n}, S_{n}\right)_{n \geq 1}$ with $X_{n}:=\sum_{y \in \mathbb{Z}} \xi_{y} \sum_{i=1}^{N_{n-1}(y)} \zeta_{i}^{(y)}$ has the same distribution as $\left(M_{T_{n}}\right)_{n \geq 1}$.
In this lemma, $\zeta_{i}^{(y)}$ corresponds to the duration of the stay at the $y$ th horizontal floor during the $i$ th visit to this floor. According to the Borel-Cantelli lemma, it suffices to prove that: $\sum_{n \geq 1} \mathbb{P}\left(\left\{X_{n} \leq 0 \leq X_{n+1}\right.\right.$ and $\left.\left.S_{n}=0\right\}\right)<+\infty$. We follow the scheme of the proof of [5]. The difference will be in our way of estimating $I_{n}^{(1)}$ and in the introduction of the sets $U_{n}$. We will consider $\delta_{1}, \delta_{2}, \delta_{3}$, and $\gamma$ such that: $0<\delta_{1}<2 \delta_{2}$, $\delta_{1}+\left(\frac{27}{2}+16\right) \delta_{2}<\frac{1}{8}, \delta_{3}>0, \frac{1}{4}-3 \delta_{2}<\delta_{3}<\frac{1}{4}-\frac{5}{2} \delta_{2}-\delta_{1}, \frac{\delta_{3}}{2}-2 \delta_{2}<\beta<\frac{\delta_{3}}{2}-\delta_{2}, \max \left(\delta_{1}, \delta_{2}\right)<\gamma<$ $\frac{1}{2}-22 \max \left(\delta_{1}, \delta_{2}\right)$. The idea is that $\delta_{1}, \delta_{2}, \frac{1}{4}-\delta_{3}$ and $\frac{1}{8}-\beta$ are positive numbers very close to zero. As in [5,9], let us define: $A_{n}:=\left\{\omega \in \Omega: \max _{\ell \in \mathbb{Z}} N_{n-1}(\ell) \leq n^{\frac{1}{2}+\delta_{2}}\right.$ and $\left.\max _{k=0, \ldots, n}\left|S_{k}\right|<n^{\frac{1}{2}+\delta_{1}}\right\}$. Moreover, we define: $U_{n}:=\left\{\omega \in A_{n}: \forall x, y \in \mathbb{Z},\left|N_{n-1}(x)-N_{n-1}(y)\right| \leq \sqrt{|x-y| n^{\frac{1}{2}+\gamma}}\right\}$. The sketch of the proof is the following:
(1) As in Proposition 4.1 of [5], we have: $\sum_{n \geq 1} \mathbb{P}\left(\left\{X_{n} \leq 0 \leq X_{n+1}\right.\right.$ and $\left.\left.S_{n}=0\right\} \backslash A_{n}\right)<+\infty$. Actually we have: $\sum_{n \geq 1} \mathbb{P}\left(\left\{S_{n}=0\right\} \backslash A_{n}\right)<+\infty$.
(2) We will see in Lemma 7 of the present paper that we have: $\sum_{n \geq 1} \mathbb{P}\left(A_{n} \backslash U_{n}\right)<+\infty$. Therefore, we have: $\sum_{n \geq 0} \mathbb{P}\left(\left\{X_{n} \leq 0 \leq X_{n+1}\right.\right.$ and $\left.\left.S_{n}=0\right\} \backslash U_{n}\right)<+\infty$.
(3) Let us define $B_{n}:=\left\{\omega \in U_{n}:\left|\sum_{y \in \mathbb{Z}} \xi_{y} N_{n-1}(y)\right|>n^{\frac{1}{2}+\delta_{3}}\right\}$. As in Proposition 4.3 of [5], we have: $\sum_{n \geq 0} \mathbb{P}\left(B_{n} \cap\left\{X_{n} \leq 0 \leq X_{n+1}\right.\right.$ and $\left.\left.S_{n}=0\right\}\right)<+\infty$ since $\mathbb{P}\left(\left\{X_{n} \leq 0 \leq X_{n+1}\right\} \mid\left(S_{p}\right)_{p},\left(\xi_{y}\right)_{y}\right)=$ $\sum_{q \geq 0} \mathbb{P}\left(\left\{X_{n}=-q\right\} \mid\left(S_{p}\right)_{p},\left(\xi_{y}\right)_{y}\right) \frac{1}{3^{q}}$. It remains to prove that:
$\sum_{n \geq 0} \mathbb{P}\left(U_{n} \cap\left\{X_{n} \leq 0 \leq X_{n+1}\right.\right.$ and $\left.\left.S_{n}=0\right\} \backslash B_{n}\right)<+\infty$.
(a) As in Lemma 4.5 of [5], there exists a real number $C>0$ such that:

$$
\begin{gathered}
\sup _{\omega \in U_{n} \mid B_{n}} \mathbb{P}\left(\left\{X_{n} \leq 0 \leq X_{n+1}\right\} \mid\left(S_{p}\right)_{p \geq 1},\left(\xi_{k}\right)_{k \in \mathbb{Z}}\right) \leq C \sqrt{\frac{\ln (n)}{n}} \\
\text { since } \mathbb{P}\left(\left\{X_{n} \leq 0 \leq X_{n+1}\right\} \mid\left(S_{p}\right)_{p},\left(\xi_{y}\right)_{y}\right)=\sum_{q \geq 0} \mathbb{P}\left(\left\{X_{n}=-q\right\} \mid\left(S_{p}\right)_{p},\left(\xi_{y}\right)_{y}\right) \frac{1}{3^{q}} .
\end{gathered}
$$

(b) We will prove that there exists some $\tilde{\delta}>0$ and some $C^{\prime}>0$ such that:

$$
\forall \omega \in U_{n}, \quad \mathbb{P}\left(U_{n} \backslash B_{n} \mid\left(S_{p}\right)_{p}\right)(\omega) \leq C^{\prime} n^{-\tilde{\delta}}
$$

(i) This probability is bounded by $c^{\prime} n^{\frac{1}{2}+\delta_{3}} I_{n}(\omega)$ with $I_{n}(\omega)=I_{n}^{(1)}(\omega)+I_{n}^{(2)}(\omega)$ and

$$
I_{n}^{(1)}(\omega):=\int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\mathrm{e}^{i t \sum_{y \in \mathbb{Z}} \xi_{y} N_{n-1}(y)(\omega)} \mid\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t
$$

and

$$
I_{n}^{(2)}(\omega):=\int_{\left\{|t|>n^{\left.-\frac{1}{2}-\delta_{3}+\delta_{2}\right\}}\right.} \mathbb{E}\left[\mathrm{e}^{i t \sum_{y \in \mathbb{Z}} N_{n-1}(y)(\omega)} \mid\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t
$$

(ii) we will prove that $n^{\frac{1}{2}+\delta_{3}} \sup _{U_{n}} I_{n}^{(1)}=O\left(n^{-\delta}\right)$ for some $\delta>0$ (see our Lem. 8);
(iii) on the other hand, following [5], we have:

$$
n^{\frac{1}{2}+\delta_{3}} I_{n}^{(2)} \leq \int_{\left\{|s|>n^{\delta_{2}}\right\}} \mathrm{e}^{-\frac{s^{2}}{2}} \mathrm{~d} s \leq 2 n^{-\delta_{2}} \mathrm{e}^{-\frac{n^{2 \delta_{2}}}{2}}
$$

(c) We have $\mathbb{P}\left(S_{n}=0\right) \leq C^{\prime \prime} n^{-\frac{1}{2}}$.
(d) Hence we have: $\mathbb{P}\left(U_{n} \cap\left\{X_{n} \leq 0 \leq X_{n+1}\right.\right.$ and $\left.\left.S_{n}=0\right\} \backslash B_{n}\right) \leq C^{\prime \prime \prime} n^{-1-\tilde{\delta}} \sqrt{\ln (n)}$.

We have to prove that points 2 and $3(\mathrm{~b})(\mathrm{ii})$ are true with our choices of parameters. Indeed, all the other points are true for any positive $\delta_{1}, \delta_{2}, \delta_{3}$ and for any sequence of random variables $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ independent of $\left(S_{p}\right)_{p}$. We notice that, for any integer $n \geq 1$, we have: $\sum_{j=0}^{n-1} \xi_{S_{j}}=\sum_{k \in \mathbb{Z}} \xi_{k} N_{n-1}(k)$. In our proof, we need some real numbers $\delta_{1}, \delta_{2}, \delta_{3}, \delta_{4}, \beta, \gamma$ and $\varepsilon>0$. We will suppose that:
$\delta_{1}>0, \delta_{2}>0, \delta_{1}+\left(\frac{27}{2}+16\right) \delta_{2}<\frac{1}{8}, \delta_{3}>0, \delta_{1}<\delta_{4}<\frac{1}{4}-\delta_{3}-\frac{5}{2} \delta_{2}, \frac{1}{4}-3 \delta_{2}<\delta_{3}<\frac{1}{4}-\frac{5}{2} \delta_{2}, \frac{5}{3} \delta_{2}<\frac{1}{2} \delta_{3}$, $\frac{\delta_{3}}{2}-2 \delta_{2}<\beta<\frac{\delta_{3}}{2}-\delta_{2}, \frac{5}{2} \delta_{3}>\frac{1}{2}+6 \delta_{2}+\delta_{1}, \max \left(\delta_{1}, \delta_{2}\right)<\gamma<\frac{1}{2}-22 \max \left(\delta_{1}, \delta_{2}\right)$ and:

$$
n^{\delta_{1}+11 \delta_{2}} \sum_{m \geq \frac{(r+1) n^{\beta}}{2}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|=O\left(n^{-\varepsilon}\right) .
$$

(we have: $\left.\sum_{m \geq N}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right| \leq N^{-\frac{1}{2}} \sum_{m \geq N} \sqrt{m}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|\right)$. All these inequalities are true with the following choices of parameters:

$$
\delta_{1}=\frac{1}{3000}, \delta_{2}=\frac{1}{500}, \delta_{3}=\frac{1}{4}-\frac{11}{4} \delta_{2}=489 / 2000, \delta_{4}=1 / 2500, \beta=\frac{\delta_{3}}{2}-\frac{3}{2} \delta_{2}=477 / 4000, \gamma=\frac{1}{4}
$$

Lemma 7. We have: $\sum_{n \geq 1} \mathbb{P}\left(A_{n} \backslash U_{n}\right)<+\infty$.

Proof. Let us consider any $x, y \in \mathbb{Z}$ with $x \neq y$ and $|x-y| \leq 3 n^{\frac{1}{2}+\delta_{1}}$. For any integer $j \geq 1$, we define the time $\tau_{j}(x)$ of the $j$ th visit of $\left(S_{p}\right)_{p}$ to $x$ and the number $\mathcal{N}_{j}(x, y)$ of visits of $\left(S_{p}\right)_{p}$ to $y$ between the times $\tau_{j}(x)$ and $\tau_{j+1}(x)$. According to $[14,18]$ (see [14] Lem. 2), for any integer $p \geq 1$, there exists $K_{p}>0$ such that, for any $x^{\prime} \neq y$,' we have: $\mathbb{E}\left[\left(\mathcal{N}_{j}\left(x^{\prime}, y^{\prime}\right)\right)^{p}\right] \leq K_{p}\left|x^{\prime}-y^{\prime}\right|^{p-1}$. According to $[14]$, on the set $\left\{\tau_{1}(x) \leq \tau_{1}(y)\right\}$, we have:

$$
\left(N_{n-1}(x)-N_{n-1}(y)\right)=\sum_{j=1}^{N_{n-1}(x)}\left(1-\mathcal{N}_{j}(x, y)\right)+\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)} \mathbf{1}_{\left\{S_{k}=y\right\}} .
$$

Let $p$ be any positive integer. We have:

$$
\left(N_{n-1}(x)-N_{n-1}(y)\right)^{2 p} \mathbf{1}_{\left\{\tau_{1}(x) \leq \tau_{1}(y)\right\}} \leq 2^{2 p}\left[\left(\sum_{j=1}^{N_{n-1}(x)}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2 p}+\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)} \mathbf{1}_{\left\{S_{k}=y\right\}}\right)^{2 p}\right]
$$

But, on $A_{n}$, since we have $N_{n-1}(x) \leq n^{\frac{1}{2}+\delta_{2}}$, we get:

$$
\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)} \mathbf{1}_{\left\{S_{k}=y\right\}}\right)^{2 p} \leq\left(\mathcal{N}_{N_{n-1}(x)}(x, y)\right)^{2 p} \leq \sum_{j=1}^{\left\lfloor n^{\frac{1}{2}+\delta_{2}}\right\rfloor}\left(\mathcal{N}_{j}(x, y)\right)^{2 p}
$$

Hence we have:

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{k=n}^{\tau_{N_{n-1}(x)+1}(x)} \mathbf{1}_{\left\{S_{k}=y\right\}}\right)^{2 p} \mathbf{1}_{A_{n}}\right] & \leq n^{\frac{1}{2}+\delta_{2}} K_{2 p}|x-y|^{2 p-1} \\
& \leq K_{2 p} 3^{p-1}|x-y|^{p}\left(n^{\frac{1}{2}+\max \left(\delta_{1}, \delta_{2}\right)}\right)^{p}
\end{aligned}
$$

Moreover, on $A_{n}$, we have:

$$
\left(\sum_{j=1}^{N_{n-1}(x)}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2 p} \leq \max _{k=1, \ldots,\left\lfloor n^{\frac{1}{2}+\delta_{2}}\right\rfloor}\left(\sum_{j=1}^{k}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2 p}
$$

Since $\left(\sum_{j=1}^{k}\left(1-\mathcal{N}_{j}(x, y)\right)\right)_{k \geq 1}$ is a martingale (see [14] Lem. 2), according to a maximal inequality, we have:

$$
\left\|\max _{k=1, \ldots,\left\lfloor n^{\frac{1}{2}+\delta_{2}}\right\rfloor}\left(\sum_{j=1}^{k}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2}\right\|_{L^{p}} \leq \frac{p}{p-1} \max _{k=1, \ldots,\left\lfloor n^{\frac{1}{2}+\delta_{2}}\right\rfloor}\left\|\left(\sum_{j=1}^{k}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2}\right\|_{L^{p}}
$$

Hence we have:

$$
\mathbb{E}\left[\left(\sum_{j=1}^{N_{n-1}(x)}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2 p} \mathbf{1}_{A_{n}}\right] \leq\left(\frac{p}{p-1}\right)^{p} \max _{k=1, \ldots,\left\lfloor n^{\frac{1}{2}+\delta_{2}}\right\rfloor} \mathbb{E}\left[\left(\sum_{j=1}^{k}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2 p}\right]
$$

Let us write $\mathcal{M}_{\nu_{1}, \ldots, \nu_{l}}^{2 p}=\frac{(2 p)!}{\prod_{i=1}^{l} \nu_{i}!}$. For any $k=1, \ldots,\left\lfloor n^{\frac{1}{2}+\delta_{2}}\right\rfloor$, since the $\mathcal{N}_{j_{m}}$ 's are independent and since $\mathbb{E}\left[1-\mathcal{N}_{j}(x, y)\right]=0$, we have:

$$
\begin{aligned}
& \mathbb{E}\left[\left(\sum_{j=1}^{k}\left(1-\mathcal{N}_{j}(x, y)\right)\right)^{2 p}=\sum_{l=1}^{2 p} \sum_{\nu_{1}+\ldots+\nu_{l}=2 p ; \min _{i} \nu_{i} \geq 1} \mathcal{M}_{\nu_{1}, \ldots, \nu_{l}}^{2 p} \sum_{j_{1}<\ldots<j_{l}} \prod_{m=1}^{l} \mathbb{E}\left[\left(1-\mathcal{N}_{j_{m}}(x, y)\right)^{\nu_{m}}\right]\right. \\
& \quad \leq \sum_{l=1}^{2 p} \sum_{\nu_{1}+\ldots+\nu_{l}=2 p ; \min _{i} \nu_{i} \geq 2} \mathcal{M}_{\nu_{1}, \ldots, \nu_{l}}^{2 p} \sum_{1 \leq j_{1}<\ldots<j_{l} \leq k} \prod_{m=1}^{l}\left(2^{\nu_{m}} \mathbb{E}\left[1+\left(\mathcal{N}_{j_{m}}(x, y)\right)^{\nu_{m}}\right]\right) \\
& \leq \sum_{l=1}^{2 p} \sum_{\nu_{1}+\ldots+\nu_{l}=2 p ; \min _{i} \nu_{i} \geq 2}^{l} \mathcal{M}_{\nu_{1}, \ldots, \nu_{l}}^{2 p} \sum_{1 \leq j_{1}<\ldots<j_{l} \leq k}^{l} \sum_{m=1}^{\nu_{m}}\left(1+K_{\nu_{m}}|x-y|^{\nu_{m}-1}\right) \\
& \leq \tilde{C}_{p} \sum_{l=1}^{2 p}|x-y|^{2 p-l}\left(n^{\frac{1}{2}+\delta_{2}}\right)^{l} \leq 2 p 3^{p} \tilde{C}_{p}|x-y|^{p}\left(n^{\frac{1}{2}+\max \left(\delta_{1}, \delta_{2}\right)}\right)^{p}
\end{aligned}
$$

Hence we get: $\mathbb{E}\left[\left(N_{n-1}(x)-N_{n-1}(y)\right)^{2 p} \mathbf{1}_{A_{n}}\right] \leq \tilde{C}_{p}^{\prime}|x-y|^{p}\left(n^{\frac{1}{2}+\max \left(\delta_{1}, \delta_{2}\right)}\right)^{p}$. Therefore, according to the Markov inequality, for any integer $p \geq 1$, we have:

$$
\begin{aligned}
\mathbb{P}\left(A_{n} \backslash U_{n}\right) & \leq \sum_{x, y=-\left\lceil n^{\frac{1}{2}+\delta_{1}}\right\rceil}^{\left\lceil n^{\frac{1}{2}+\delta_{1}}\right\rceil} \mathbb{P}\left(A_{n} \cap\left\{\left|N_{n-1}(x)-N_{n-1}(y)\right|>\sqrt{|x-y| n^{\frac{1}{2}+\gamma}}\right\}\right) \\
& \leq \sum_{x, y=-\left\lceil n^{\frac{1}{2}+\delta_{1}}\right\rceil}^{\left\lceil n^{\frac{1}{2}+\delta_{1}}\right\rceil} \frac{\mathbb{E}\left[\left(N_{n-1}(x)-N_{n-1}(y)\right)^{2 p} \mathbf{1}_{A_{n}}\right]}{|x-y|^{p}\left(n^{\frac{1}{2}+\gamma}\right)^{p}} \leq c_{p}\left(5 n^{\frac{1}{2}+\delta_{1}}\right)^{2}\left(n^{\max \left(\delta_{1}, \delta_{2}\right)-\gamma}\right)^{p} .
\end{aligned}
$$

By taking $p$ large enough, we get: $\sum_{n \geq 1} \mathbb{P}\left(A_{n} \backslash U_{n}\right)<+\infty$.

### 3.1. Estimates on $\boldsymbol{U}_{\boldsymbol{n}}$

In this section, we suppose that we are in $U_{n}$. We will estimate:

$$
I_{n}^{(1)}(\omega):=\int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}}\left(\mathbb{E}\left[\mathrm{e}^{i t \sum_{y \in \mathbb{Z}} \xi_{y} N_{n-1}(y)} \mid\left(S_{p}\right)_{p}\right](\omega)\right) \mathrm{e}^{-\frac{t^{2} n_{n}^{1+2 \delta_{3}}}{2}} \mathrm{~d} t .
$$

Lemma 8. There exists a real number $\delta>0$ such that: $\sup _{n \geq 1} n^{\delta} \sup _{\omega \in U_{n}} n^{\frac{1}{2}+\delta_{3}} I_{n}^{(1)}(\omega)<+\infty$.
To prove this lemma, we will use the following formula:

$$
n^{\frac{1}{2}+\delta_{3}} I_{n}^{(1)}(\omega)=n^{\delta_{2}} \int_{\{|u| \leq 1\}}\left(\mathbb { E } \left[\mathrm{e}^{\left.\left.\left.i u n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \sum_{y \in \mathbb{Z}} \xi_{y} N_{n-1}(y) \right\rvert\,\left(S_{p}\right)_{p}\right](\omega)\right) \mathrm{e}^{-\frac{u^{2} n^{2} \delta_{2}}{2}} \mathrm{~d} u . . . . . . .}\right.\right.
$$

The main idea is to prove that, in this formula, we can replace the term:

$$
B_{n}(u)(\omega):=\mathbb{E}\left[\left.\mathrm{e}^{i u n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \sum_{y \in \mathbb{Z}} \xi_{y} N_{n-1}(y)} \right\rvert\,\left(S_{p}\right)_{p}\right](\omega)
$$

by the term: $A_{n}(u)(\omega):=\mathrm{e}^{-\frac{u^{2}}{2 n^{1+2 \delta} \delta_{3}-2 \delta_{2}} \sum_{y, z} \mathbb{E}\left[\xi_{y} \xi_{z}\right]\left(N_{n-1}(y)(\omega)\right)^{2}}$. More precisely let us prove that we have:

Lemma 9. There exists a real number $\delta_{0}>0$ such that we have:

$$
\begin{equation*}
\sup _{n \geq 1} n^{\delta_{0}} \sup _{\omega \in U_{n}} n^{\delta_{2}} \int_{|u| \leq 1}\left|B_{n}(u)(\omega)-A_{n}(u)(\omega)\right| \mathrm{e}^{-\frac{u^{2} n^{2} \delta_{2}}{2}} \mathrm{~d} u<+\infty \tag{3.1}
\end{equation*}
$$

After proving 9 , we will prove that Lemma 8 is a consequence of it. We will use the following notation: $\sigma_{\xi}^{2}:=\sum_{m \in \mathbb{Z}} \mathbb{E}\left[\xi_{0} \xi_{m}\right]$.

### 3.1.1. Proof of Lemma 9

Our proof uses a method introduced by Jan (cf. [12,13]). This method also gives a result of convergence in distribution for $\left(n^{-3 / 4} \sum_{k=0}^{n-1} \xi_{S_{n}}\right)_{n \geq 1}$ (see [15]). Let $n$ be an integer such that $n^{\beta} \geq 2$. Let us fix $\omega \in U_{n}$ and $u \in[-1 ; 1]$. Let us recall that $0<\beta<\frac{\delta_{3}}{2}-\delta_{2}$ et let us define: $L_{n}:=\left\lfloor\frac{2\left\lfloor n^{\frac{1}{2}+\delta_{1}}\right\rfloor+1}{\left\lfloor n^{\beta}\right\rfloor}\right]$ (we have: $L_{n} \leq 4 n^{\frac{1}{2}+\delta_{1}-\beta}$ ) and, for all integer $k=0, \ldots, L_{n}: \alpha_{(k)}:=-\left\lfloor n^{\frac{1}{2}+\delta_{1}}\right\rfloor+k\left\lfloor n^{\beta}\right\rfloor$ and $\alpha_{\left(L_{n}+1\right)}:=\left\lfloor n^{\frac{1}{2}+\delta_{1}}\right\rfloor+1$; $b_{k}:=\mathrm{e}^{i u n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \sum_{y=\alpha_{(k)}}^{\alpha(k+1)^{-1}} \xi_{y} N_{n-1}(y)}$ and $a_{k}:=\mathrm{e}^{-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}} \sum_{y=\alpha_{(k)}}^{\alpha(k+1)-1}} \sigma_{\xi}^{2}\left(N_{n-1}(y)\right)^{2}}$. We have to estimate: $n^{\delta_{2}}\left|\mathbb{E}\left[\prod_{k=0}^{L_{n}} b_{k} \mid\left(S_{p}\right)_{p}\right](\omega)-\prod_{k=0}^{L_{n}} a_{k}(\omega)\right|$. Hence it is enough to estimate:

$$
n^{\delta_{2}} \sum_{k=0}^{L_{n}}\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_{m}\right)\left(b_{k}-a_{k}\right)\left(\prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}}\right) \mid\left(S_{p}\right)_{p}\right](\omega)\right| .
$$

- We explain how we can restrict our study to the sum over the $k$ such that $(r+1)^{4} \leq k \leq L_{n}-1$. Let $k \in\left\{0, \ldots, L_{n}\right\}$. We have:

$$
\begin{aligned}
\mathbb{E}\left[\left(\sum_{\ell=\alpha+1}^{\alpha+\theta} \xi_{\ell} N_{n-1}(\ell)\right)^{2} \mid\left(S_{p}\right)_{p}\right](\omega) & \leq \sum_{\ell=\alpha+1}^{\alpha+\theta} \sum_{m=\alpha+1}^{\alpha+\theta}\left|\mathbb{E}\left[\xi_{\ell} \xi_{m}\right]\right| N_{n-1}(\ell)(\omega) N_{n-1}(m)(\omega) \\
& \leq \theta \sum_{m \in \mathbb{Z}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right| n^{1+2 \delta_{2}}
\end{aligned}
$$

Hence we have:

$$
\begin{aligned}
& \mathbb{E}\left[\left|b_{k}-1\right| \mid\left(S_{p}\right)_{p}\right](\omega) \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\left(\mathbb{E}\left[\left|\sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \xi_{y} N_{n-1}(y)\right| \mid\left(S_{p}\right)_{p}\right](\omega)\right) \\
& \quad \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} n^{\frac{\beta}{2}} \sqrt{\sum_{m \in \mathbb{Z}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right| n^{\frac{1}{2}+\delta_{2}} \leq n^{-\frac{3}{4} \delta_{3}+\frac{3}{2} \delta_{2}} \sqrt{\sum_{m \in \mathbb{Z}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|},} .
\end{aligned}
$$

since we have $\beta<\frac{\delta_{3}}{2}-\delta_{2}$. Moreover we have:

$$
\left|a_{k}(\omega)-1\right| \leq \frac{\sigma_{\xi}^{2} \sum_{y=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(N_{n-1}(y)(\omega)\right)^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} \leq \frac{\sigma_{\xi}^{2} n^{1+2 \delta_{2}}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} n^{\beta} \sigma_{\xi}^{2} n^{1+2 \delta_{2}} \leq \frac{n^{-\frac{3}{2} \delta_{3}+3 \delta_{2}} \sigma_{\xi}^{2}}{2}
$$

From which, we get:

$$
\begin{equation*}
n^{\delta_{2}} \sum_{k=0}^{(r+1)^{4}-1} \mathbb{E}\left[\left|b_{k}-a_{k}\right| \mid\left(S_{p}\right)_{p}\right](\omega)+\mathbb{E}\left[\left|b_{L_{n}}-a_{L_{n}}\right| \mid\left(S_{p}\right)_{p}\right](\omega) \leq c_{0}\left(n^{-\frac{3}{4} \delta_{3}+\frac{5}{2} \delta_{2}}+n^{-\frac{3}{2} \delta_{3}+4 \delta_{2}}\right), \tag{3.2}
\end{equation*}
$$

with $c_{0}:=\left((r+1)^{4}+1\right) \sqrt{\sum_{m \in \mathbb{Z}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|}+\frac{1}{2} \sigma_{\xi}^{2}$. Let us recall that $\frac{5}{3} \delta_{2}<\frac{1}{2} \delta_{3}$.

Hence, it remains to estimate:

$$
\begin{equation*}
n^{\delta_{2}} \sum_{k=(r+1)^{4}}^{L_{n}-1}\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-1} b_{m}\right)\left(b_{k}-a_{k}\right) \prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}} \mid\left(S_{p}\right)_{p}\right]\right| . \tag{3.3}
\end{equation*}
$$

- Let us introduce some holes in the indices $m$ in order to use our decorrelation hypothesis. Let us control the following quantity:

$$
\begin{aligned}
\tilde{B}_{n}:=n^{\delta_{2}} \sum_{k=(r+1)^{4}}^{L_{n}-1} \mid \mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^{4}} b_{m}\right) \prod_{j=1}^{3}\left(\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^{j}} b_{m}\right)-1\right)\right. \\
\left.\times \prod_{m^{\prime \prime}=k-r}^{k-1} b_{m^{\prime \prime}}\left(b_{k}-a_{k}\right) \prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}} \mid\left(S_{p}\right)_{p}\right] \mid
\end{aligned}
$$

We have:

$$
\tilde{B}_{n}(\omega) \leq n^{\delta_{2}} \sum_{k=(r+1)^{4}}^{L_{n}-1} \prod_{j=1}^{3}\left\|\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^{j}} b_{m}\right)-1\right\|_{L^{\infty}\left(U_{n}\right)}\left\|b_{k}-a_{k}\right\|_{L^{\infty}\left(U_{n}\right)} .
$$

On $U_{n}$, we have: $\left|b_{k}-1\right| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} n^{\beta} n^{\frac{1}{2}+\delta_{2}} \leq n^{-\delta_{3}+2 \delta_{2}+\beta}$.
Analogously, we get: $\left|\left(\prod_{m=k-(r+1)^{j+1}+1}^{k-(r+1)^{j}} b_{m}\right)-1\right| \leq r(r+1)^{j} n^{-\delta_{3}+2 \delta_{2}+\beta}$. On the other hand, we have: $\left|a_{k}-1\right| \leq \frac{1}{2} n^{-2 \delta_{3}+4 \delta_{2}+\beta} \sigma_{\xi}^{2}$. Therefore, since we have $\beta<\frac{\delta_{3}}{2}-\delta_{2}$, we get:

$$
\tilde{B}_{n} \leq 4 n^{\delta_{2}} n^{\frac{1}{2}+\delta_{1}-\beta} r^{3}(r+1)^{6}\left(1+\frac{1}{2} \sigma_{\xi}^{2}\right)\left(n^{-\delta_{3}+2 \delta_{2}+\beta}\right)^{4}=O\left(n^{\frac{1}{2}-\frac{5}{2} \delta_{3}+6 \delta_{2}+\delta_{1}}\right)
$$

The control of the quantity $\tilde{B}_{n}$ comes from the fact that $\frac{5}{2} \delta_{3}>\frac{1}{2}+6 \delta_{2}+\delta_{1}$.
It remains to estimate: $n^{\delta_{2}} \sum_{k=(r+1)^{4}+1}^{L_{n}-1} \sum_{1 \leq j_{0}<j_{1} \leq j_{2} \leq 4} C_{n, k, j_{0}, j_{1}, j_{2}}$, where $C_{n, k, j_{0}, j_{1}, j_{2}}$ is the following quantity:

$$
\left|\mathbb{E}\left[\left(\prod_{m=0}^{k-(r+1)^{4}} b_{m}\right)\left(\prod_{m=k-(r+1)^{j_{2}+1}}^{k-(r+1)^{j_{1}}} b_{m}\right)\left(\prod_{m=k-(r+1)^{j_{0}+1}}^{k-1} b_{m}\right)\left(b_{k}-a_{k}\right) \prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}} \mid\left(S_{p}\right)_{p}\right]\right|,
$$

with the convention: $\prod_{m=\alpha}^{\beta} b_{m}=1$ if $\beta<\alpha$. Let $j_{0}, j_{1}, j_{2}$ be fixed. We have: $C_{n, k, j_{0}, j_{1}, j_{2}} \leq$ $D_{n, k, j_{0}, j_{1}, j_{2}}+E_{n, k, j_{0}, j_{1}, j_{2}}$, with:

$$
D_{n, k, j_{0}, j_{1}, j_{2}}:=\left|\operatorname{Cov}_{\mid\left(S_{p}\right)_{p}}\left(\Delta_{n, k, j_{1}, j_{2}}, \Gamma_{n, k, j_{0}}\right) \prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}}\right|
$$

$$
\text { and } \quad E_{n, k, j_{0}, j_{1}, j_{2}}:=\left|\mathbb{E}\left[\Delta_{n, k, j_{1}, j_{2}} \mid\left(S_{p}\right)_{p}\right] \mathbb{E}\left[\Gamma_{n, k, j_{0}} \mid\left(S_{p}\right)_{p}\right] \prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}}\right|
$$

with $\Delta_{n, k, j_{1}, j_{2}}:=\prod_{m=0}^{k-(r+1)^{4}} b_{m} \prod_{m^{\prime}=k-(r+1)^{j_{2}}+1}^{k-(r+1)^{j_{1}}} b_{m^{\prime}}$ and $\Gamma_{n, k, j_{0}}:=\left(\prod_{m=k-(r+1)^{j_{0}+1}}^{k-1} b_{m}\right)\left(b_{k}-a_{k}\right)$.

- Control of the covariance terms (thanks to our decorrelation hypothesis). Let $j_{0}, j_{1}, j_{2}$ be fixed. Let $k=(r+1)^{4}, \ldots, L_{n}-1$. We have:

$$
\begin{aligned}
D_{n, k, j_{0}, j_{1}, j_{2}} \leq & \left|\operatorname{Cov}_{\mid\left(S_{p}\right)_{p}}\left(\Delta_{n, k, j_{1}, j_{2}}, \prod_{m=k-(r+1)^{j_{0}+1}}^{k} b_{m}\right) \prod_{m^{\prime}=k+1}^{L_{n}} a_{m^{\prime}}\right| \\
& +\left|\operatorname{Cov}_{\mid\left(S_{p}\right)_{p}}\left(\Delta_{n, k, j_{1}, j_{2}}, \prod_{m=k-(r+1)^{j_{0}+1}}^{k-1} b_{m}\right) \prod_{m^{\prime}=k}^{L_{n}} a_{m^{\prime}}\right| .
\end{aligned}
$$

But we have: $\prod_{m=\theta_{1}+1}^{\theta_{1}+\theta_{2}} b_{m}=\mathrm{e}^{i u n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \sum_{\ell=\alpha_{( }}^{\left.\alpha_{1}\right)}{ }^{\alpha_{\left(\theta_{1}+\theta_{2}+1\right)}-1} \xi_{\ell} N_{n-1}(\ell)}$. Therefore, according to point 2 of the hypothesis of our theorem, we have:

$$
D_{n, k, j_{0}, j_{1}, j_{2}} \leq 2 C\left(1+n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \sum_{\ell \in \mathbb{Z}} N_{n-1}(\ell)\right) \varphi_{p, s}
$$

with $p:=\left\lfloor n^{\beta}\right\rfloor\left((r+1)^{j_{1}}-(r+1)^{j_{0}}\right)$ and $s:=\left\lfloor n^{\beta}\right\rfloor(r+1)^{j_{0}}-1$. Let us notice that we have: $p \geq r s$. Since $\sum_{\ell \in \mathbb{Z}} N_{n-1}(\ell)=n$, we have:

$$
\begin{aligned}
n^{\delta_{2}} \sum_{k=(r+1)^{4}}^{L_{n}-1} D_{n, k, j_{0}, j_{1}, j_{2}} & \leq 4 C\left(n^{1-\delta_{3}+\delta_{1}-\beta+2 \delta_{2}}\right) n^{-6 \beta} \sup _{s \geq n^{\beta}} s^{6} \varphi_{r s, s} \\
& \leq 4 C\left(n^{1-\frac{9}{8}+\delta_{1}+\left(\frac{27}{2}+16\right) \delta_{2}}\right) \sup _{s \geq n^{\beta}} s^{6} \varphi_{r s, s}
\end{aligned}
$$

since $\beta>\frac{\delta_{3}}{2}-2 \delta_{2}$ and $\delta_{3}>\frac{1}{4}-3 \delta_{2}$. We end this point by noticing that $\delta_{1}+\left(\frac{27}{2}+16\right) \delta_{2}<\frac{1}{8}$.

- Control of the term with the product of the expectations. Let $j_{0}, j_{1}, j_{2}$ be fixed. Let $k=(r+1)^{4}, \ldots, L_{n}-$ 1. We can notice that $E_{n, k, j_{0}, j_{1}, j_{2}}$ is bounded by the following quantity:

$$
F_{n, k, j_{0}}:=\left|\mathbb{E}\left[\prod_{m=k-(r+1)^{j_{0}+1}}^{k} b_{m}-\left(\prod_{m=k-(r+1)^{j_{0}+1}}^{k-1} b_{m}\right) a_{k} \mid\left(S_{p}\right)_{p}\right]\right| .
$$

We approximate the terms with exponential using Taylor expansions.

- First we explain that, in $F_{n, k, j_{0}}$, we can replace
by the formula given by the second order Taylor expansion of the exponential function:

$$
\begin{equation*}
1+i u n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \sum_{\ell=\alpha}^{\alpha_{(k+1)}-1} \xi_{\ell} N_{n-1}(\ell)-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left(\sum_{\left.\ell=\alpha_{(k-(r+1)}\right)^{\left.j_{0}+1\right)}}^{\alpha_{(k+1)}-1} \xi_{\ell} N_{n-1}(\ell)\right)^{2} . \tag{3.4}
\end{equation*}
$$

Indeed, the induced error is less than: $\frac{1}{6} n^{-\frac{3}{2}-3 \delta_{3}+3 \delta_{2}} \mathbb{E}\left[\left|\sum_{\ell=\alpha_{(k-(r+1)}^{\left.j_{0}+1\right)}}^{\alpha_{(k+1)}-1} \xi_{\ell} N_{n-1}(\ell)\right|^{3} \mid\left(S_{p}\right)_{p}\right]$. Moreover, we have:
$\mathbb{E}\left[\left|\sum_{\ell=\alpha_{\left(k-(r+1)^{j} 0+1\right)}}^{\alpha_{(k+1)}-1} \xi_{\ell} N_{n-1}(\ell)\right|^{4} \mid\left(S_{p}\right)_{p}\right] \leq \sum_{\left.y_{1}, y_{2}, y_{3}, y_{4}=\alpha_{(k-(r+1)} j_{0}+1\right)}^{\alpha_{(k+1)}-1}\left|\mathbb{E}\left[\xi_{y_{1}} \xi_{y_{2}} \xi_{y_{3}} \xi_{y_{4}}\right]\right|\left(n^{\frac{1}{2}+\delta_{2}}\right)^{4}$

$$
\leq \quad c_{0}^{\prime} n^{2+4 \delta_{2}}(r+1)^{6} n^{2 \beta}
$$

according to the hypothesis of our theorem. Hence, taking the sum over $k=(r+1)^{4}, \ldots, L_{n}-1$ and multiplying by $n^{\delta_{2}}$, this substitution induces a total error bounded by:

$$
\frac{\left(c_{0}^{\prime}\right)^{3 / 4}}{6} n^{\delta_{2}+\frac{1}{2}+\delta_{1}-\beta} n^{-\frac{3}{2}-3 \delta_{3}+3 \delta_{2}} n^{\frac{3}{2}+3 \delta_{2}}(r+1)^{\frac{9}{2}} n^{\frac{3}{2} \beta}
$$

and so by: $\frac{\left(c_{0}^{\prime}\right)^{3 / 4}}{6} n^{7 \delta_{2}+\frac{1}{2}+\delta_{1}-3 \delta_{3}+\frac{1}{2} \beta}(r+1)^{\frac{9}{2}}$. Since $\beta<\frac{\delta_{3}}{2}-\delta_{2}, \delta_{3}>\frac{1}{4}-3 \delta_{2}$ and $\delta_{1}+\left(\frac{27}{2}+16\right) \delta_{2}<\frac{1}{8}$, we have: $7 \delta_{2}+\frac{1}{2}+\delta_{1}-3 \delta_{3}+\frac{1}{2} \beta \leq-\frac{1}{16}$.

- Let us introduce $Y_{k}:=\sum_{\ell=\alpha_{(k-(r+1)} j_{0+1)}}^{\alpha_{(k)}-1} \xi_{\ell} N_{n-1}(\ell)$ and $Z_{k}:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sigma_{\xi}^{2} N_{n-1}(\ell)^{2}$. We explain that, in $F_{n, k, j_{0}}$, we can replace $\left(\prod_{m=k-(r+1)^{j_{0}}+1}^{k-1} b_{m}\right) a_{k}=\mathrm{e}^{\frac{i u}{\frac{1}{2}+\delta_{3}-\delta_{2}} Y_{k}-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}}$ by the formula given by the Taylor expansion of the exponential function at the second order:

$$
\begin{equation*}
1+\frac{i u}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}} Y_{k}-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}+\frac{1}{2}\left(\frac{i u}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}} Y_{k}-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}\right)^{2} \tag{3.5}
\end{equation*}
$$

Indeed the modulus of the error between these two quantities is less than:

$$
\frac{1}{6} \mathbb{E}\left[\left.\left|\frac{i u}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}} Y_{k}-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}\right|^{3} \right\rvert\,\left(S_{p}\right)_{p}\right] \leq \frac{4}{3} \mathbb{E}\left[\left.\left|\frac{1}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}} Y_{k}\right|^{3}+\left|\frac{1}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}\right|^{3} \right\rvert\,\left(S_{p}\right)_{p}\right]
$$

We control the first term as in the preceding point. Moreover, we have:

$$
\left|n^{-1-2 \delta_{3}+2 \delta_{2}} Z_{k}\right|^{3} \leq n^{-3-6 \delta_{3}+6 \delta_{2}}\left(\sigma_{\xi}^{2}\right)^{3} n^{3 \beta} n^{3+6 \delta_{2}} \leq n^{-6 \delta_{3}+12 \delta_{2}+3 \beta}\left(\sigma_{\xi}^{2}\right)^{3} .
$$

Hence, taking the sum over $k=(r+1)^{4}, \ldots, L_{n}-1$ and multiplying by $n^{\delta_{2}}$, we get a quantity bounded by: $2 n^{\frac{1}{2}+\delta_{1}-6 \delta_{3}+13 \delta_{2}+2 \beta}\left(\sigma_{\xi}^{2}\right)^{3}$ and we have: $\frac{1}{2}+\delta_{1}-6 \delta_{3}+13 \delta_{2}+2 \beta<0$.

- Now, we show that in formula (3.5), we can omit the term with $\left(Z_{k}\right)^{2}$. Indeed, we have:

$$
\begin{aligned}
n^{\delta_{2}} \sum_{(r+1)^{4}}^{L_{n}-1}\left(n^{-1-2 \delta_{3}+2 \delta_{2}} Z_{k}\right)^{2} & \leq 2 n^{\delta_{2}+\frac{1}{2}+\delta_{1}-\beta-2-4 \delta_{3}+4 \delta_{2}} n^{2 \beta}\left(\sigma_{\xi}^{2}\right)^{2} n^{2+4 \delta_{2}} \\
& \leq 2 n^{-\frac{1}{5}-\frac{2}{5} \delta_{1}-\frac{2}{5} \delta_{2}}\left(\sigma_{\xi}^{2}\right)^{2}
\end{aligned}
$$

since $\beta<\frac{\delta_{3}}{2}-\delta_{2}$ and $\frac{5}{2} \delta_{3}>\frac{1}{2}+6 \delta_{2}+\delta_{1}$.

- Hence, it remains to estimate the following quantity called $G_{n, k, j_{0}}$ :

$$
\begin{aligned}
& \left\lvert\, \mathbb{E}\left[\frac{i u}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}}\left(Y_{k}+W_{k}\right)-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left(Y_{k}+W_{k}\right)^{2}-\frac{i u}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}} Y_{k}+\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}\right.\right. \\
&\left.\left.+\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left(Y_{k}\right)^{2}+\frac{i u}{n^{\frac{1}{2}+\delta_{3}-\delta_{2}}} Y_{k} \frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k} \right\rvert\,\left(S_{p}\right)_{p}\right] \mid,
\end{aligned}
$$

with $W_{k}:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \xi_{\ell} N_{n-1}(\ell)$. We get:

$$
\begin{aligned}
G_{n, k, j_{0}} & =\left|\mathbb{E}\left[\left.-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left(Y_{k}+W_{k}\right)^{2}+\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} Z_{k}+\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left(Y_{k}\right)^{2} \right\rvert\,\left(S_{p}\right)_{p}\right]\right| \\
& =\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left|\mathbb{E}\left[\left(W_{k}\right)^{2}+2 W_{k} Y_{k}-Z_{k} \mid\left(S_{p}\right)_{p}\right]\right|
\end{aligned}
$$

Let us notice that we have:

$$
Z_{k}:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}\left[\left(\xi_{\ell}\right)^{2}\right] N_{n-1}(\ell)^{2}+2 \sum_{m \leq \ell-1} \mathbb{E}\left[\xi_{\ell} \xi_{m}\right] N_{n-1}(\ell)^{2}\right)
$$

- Let us show that, in the last expression of $G_{n, k, j_{0}}$, we can replace $Z_{k}$ by:

$$
\tilde{Z}_{k}:=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}\left[\left(\xi_{\ell}\right)^{2}\right] N_{n-1}(\ell)^{2}+2 \sum_{m \leq \ell-1} \mathbb{E}\left[\xi_{\ell} \xi_{m}\right] N_{n-1}(\ell) N_{n-1}(m)\right) .
$$

Indeed, by definition of $U_{n}$, we have:

$$
\begin{aligned}
\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} \mathbb{E}\left[\left|Z_{k}-\tilde{Z}_{k}\right| \mid\left(S_{p}\right)_{p}\right] & \leq \frac{1}{n^{1+2 \delta_{3}-2 \delta_{2}}} \sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}^{-1}} \sum_{m \leq \ell-1}\left|\mathbb{E}\left[\xi_{\ell} \xi_{m}\right]\right| N_{n-1}(\ell)\left|N_{n-1}(m)-N_{n-1}(\ell)\right| \\
& \leq n^{-\frac{1}{4}-2 \delta_{3}+3 \delta_{2}+\beta+\frac{\gamma}{2}} \sum_{m \geq 1} \sqrt{m}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|
\end{aligned}
$$

Hence, taking the sum over $k=(r+1)^{4}, \ldots, L_{n}-1$ and multiplying by $n^{\delta_{2}}$, we get a quantity bounded by: $4 n^{\frac{1}{4}+\delta_{1}-2 \delta_{3}+4 \delta_{2}+\frac{\gamma}{2}} \sum_{m \geq 1} \sqrt{m}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|$. But, since $\delta_{3}>\frac{1}{4}-3 \delta_{2}$ and $\gamma<\frac{1}{2}-$ $22 \max \left(\delta_{1}, \delta_{2}\right)$, we have: $\frac{1}{4}+\delta_{1}-2 \delta_{3}+4 \bar{\delta}_{2}+\frac{\gamma}{2}<0$.

- Hence we have to estimate:

$$
\tilde{G}_{n, k, j_{0}}=\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}\left|\mathbb{E}\left[\left(W_{k}\right)^{2}+2 W_{k} Y_{k}-\tilde{Z}_{k} \mid\left(S_{p}\right)_{p}\right]\right| .
$$

We have:
$\mathbb{E}\left[\left(W_{k}\right)^{2} \mid\left(S_{p}\right)_{p}\right]=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}^{-1}}\left(\mathbb{E}\left[\left(\xi_{\ell}\right)^{2}\right]\left(N_{n-1}(\ell)\right)^{2}+2 \sum_{m=\alpha_{(k)}}^{\ell-1} \mathbb{E}\left[\xi_{\ell} \xi_{m}\right] N_{n-1}(\ell) N_{n-1}(m)\right)$.

Hence we have:
$\mathbb{E}\left[\left(W_{k}\right)^{2}+2 W_{k} Y_{k} \mid\left(S_{p}\right)_{p}\right]=\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1}\left(\mathbb{E}\left[\left(\xi_{\ell}\right)^{2}\right]\left(N_{n-1}(\ell)\right)^{2}+2 \sum_{\left.m=\alpha_{(k-(r+1)} j_{0}+1\right)}^{\ell-1} \mathbb{E}\left[\xi_{\ell} \xi_{m}\right] N_{n-1}(\ell) N_{n-1}(m)\right)$.
We get:

$$
\begin{aligned}
& \tilde{G}_{n, k, j_{0}}=\frac{u^{2}}{n^{1+2 \delta_{3}-2 \delta_{2}}}\left|\sum_{\ell=\alpha_{(k)}}^{\alpha_{(k+1)}-1} \sum_{m \leq \alpha} \mathbb{E}\left[\xi_{\ell} \xi_{m}\right] N_{n-1}(\ell) N_{n-1}(m)\right| \\
\leq & \frac{u^{2}}{n^{1+2 \delta_{3}-2 \delta_{2}}} n^{\beta} \sum_{m \geq \frac{(r+1))^{\left.j_{0} \beta+1\right)}}{2}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right| n^{1+2 \delta_{2}} \leq n^{-2 \delta_{3}+4 \delta_{2}+\beta} \sum_{m \geq \frac{(r+1) n^{\beta}}{2}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right| .
\end{aligned}
$$

Hence, taking the sum over $k=(r+1)^{4}, \ldots, L_{n}-1$ of these quantities and multiplying by $n^{\delta_{2}}$, we get a quantity bounded by:

$$
4 n^{\frac{1}{2}+\delta_{1}-2 \delta_{3}+5 \delta_{2}} \sum_{m \geq \frac{(r+1) n^{\beta}}{2}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right| \leq 4 n^{\delta_{1}+11 \delta_{2}} \sum_{m \geq \frac{(r+1) n^{\beta}}{2}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|,
$$

since $\delta_{3}>\frac{1}{4}-3 \delta_{2}$. To conclude it suffices to notice that: $n^{\delta_{1}+11 \delta_{2}} \sum_{m \geq \frac{(r+1) n^{\beta}}{2}}\left|\mathbb{E}\left[\xi_{0} \xi_{m}\right]\right|=O\left(n^{-\varepsilon}\right)$.

### 3.1.2. Proof of Lemma 8

Let us consider $n \geq 2$. According to Lemma 9, it suffices to prove that there exists a real number $\delta^{\prime}>0$ such that we have:

$$
\sup _{n \geq 1} n^{\delta^{\prime}} \sup _{\omega \in U_{n}} n^{\delta_{2}} \int_{|u| \leq 1} \exp \left(-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} \sum_{y, z} \mathbb{E}\left[\xi_{y} \xi_{z}\right]\left(N_{n-1}(y)(\omega)\right)^{2}\right) \mathrm{e}^{-\frac{u^{2} n^{2} \delta_{2}}{2}} \mathrm{~d} u<+\infty
$$

Let us take $\omega \in U_{n}$. We have:

$$
\exp \left(-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} \sum_{y, z} \mathbb{E}\left[\xi_{y} \xi_{z}\right]\left(N_{n-1}(y)(\omega)\right)^{2}\right)=\exp \left(-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} \sigma_{\xi}^{2} \sum_{y}\left(N_{n-1}(y)(\omega)\right)^{2}\right) .
$$

Let us define: $p_{n}:=\operatorname{Card}\left\{y \in \mathbb{Z}: N_{n-1}(y) \geq \frac{n^{\frac{1}{2}-\delta_{4}}}{3}\right\}$. We have:

$$
\begin{aligned}
n & =\sum_{y=-\left\lfloor n^{\frac{1}{2}+\delta_{1}}\right\rfloor}^{\left\lfloor n^{\frac{1}{2}+\delta_{1}}\right\rfloor} N_{n-1}(y) \leq p_{n} n^{\frac{1}{2}+\delta_{2}}+\frac{n^{\frac{1}{2}-\delta_{4}}}{3}\left(3 n^{\frac{1}{2}+\delta_{1}}-p_{n}\right) \\
& \leq p_{n} n^{\frac{1}{2}+\delta_{2}}\left(1-\frac{n^{-\left(\delta_{2}+\delta_{4}\right)}}{3}\right)+n^{1+\delta_{1}-\delta_{4}} .
\end{aligned}
$$

Since $\delta_{1}<\delta_{4}$, we have: $p_{n} \geq n^{-\frac{1}{2}-\delta_{2}}\left(n-n^{1-\left(\delta_{4}-\delta_{1}\right)}\right) \geq n^{\frac{1}{2}-\delta_{2}}\left(1-n^{-\left(\delta_{4}-\delta_{1}\right)}\right) \geq c_{0} n^{\frac{1}{2}-\delta_{2}}$, with $c_{0}:=1-$ $2^{-\left(\delta_{4}-\delta_{1}\right)}$. Hence we have: $\sum_{y \in \mathbb{Z}}\left(N_{n-1}(y)(\omega)\right)^{2} \geq p_{n}\left(\frac{n^{\frac{1}{2}-\delta_{4}}}{3}\right)^{2} \geq \frac{c_{0} n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}}}{9}$ and

$$
\begin{aligned}
\mathrm{e}^{-\frac{u^{2} \sum_{y} \sigma_{\xi}^{2}\left(N_{n-1}(y)(\omega)\right)^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}}} & \leq \mathrm{e}^{-\frac{u^{2} \sigma_{\xi}^{2} c_{0} n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}}}{18 n^{1+2 \delta_{3}-2 \delta_{2}}}} \\
& \leq \mathrm{e}^{-\frac{u^{2}}{18} \sigma_{\xi}^{2} c_{0} n^{\frac{1}{2}+\delta_{2}-2 \delta_{3}-2 \delta_{4}}}
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
n^{\delta_{2}} \int_{|u| \leq 1} \mathrm{e}^{-\frac{u^{2}}{2 n^{1+2 \delta_{3}-2 \delta_{2}}} \sum_{y, z} \mathbb{E}\left[\xi_{y} \xi_{z}\right]\left(N_{n-1}(y)(\omega)\right)^{2}} \mathrm{e}^{-\frac{u^{2} n^{2} \delta_{2}}{2}} \mathrm{~d} u & \leq n^{\delta_{2}} \int_{|u| \leq 1} \mathrm{e}^{-\frac{u^{2}}{18} \sigma_{\xi}^{2} c_{0} n^{\frac{1}{2}+\delta_{2}-2 \delta_{3}-2 \delta_{4}}} \mathrm{~d} u \\
& \leq n^{-\frac{1}{4}+\delta_{4}+\frac{1}{2} \delta_{2}+\delta_{3}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{v^{2}}{18} \sigma_{\xi}^{2} c_{0}} \mathrm{~d} v
\end{aligned}
$$

This ends the proof since $\delta_{4}+\delta_{3}+\frac{1}{2} \delta_{2}<\frac{1}{4}$.

## 4. About the model of Guillotin-Plantard and Le Ny

In this section, we prove that the hypothesis $\int_{M} \frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}} \mathrm{d} \nu<+\infty$ of Guillotin-Plantard and Le Ny in [9] can be replaced by the existence of $p \geq 1$ such that $\int_{M} \frac{1}{\left(f_{0}\left(1-f_{0}\right)\right)^{p}} \mathrm{~d} \nu<+\infty$, for some $p>0$. In this situation, there is no need to introduce the set $U_{n}$; we take $U_{n}=A_{n}$. If we take $\delta_{1}>0, \delta_{2}>0$ and $\delta_{3}>0$, all the points (of the sketch of the proof of Sect. 3) except the point 3 (b)(ii) come in the same way without the need of the hypothesis $\int_{M} \frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}} \mathrm{d} \nu<+\infty$. It remains to estimate:

$$
\sup _{\omega \in A_{n}} n^{\frac{1}{2}+\delta_{3}} I_{n}^{(1)}(\omega):=n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\mathrm{e}^{i t \sum_{y \in \mathbb{Z}} \xi_{y} N_{n-1}(y)} \mid\left(S_{p}\right)_{p}\right](\omega) \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t
$$

Let us take $\omega \in A_{n}$. We suppose $\delta_{3}>2 \delta_{2}$ and $\delta_{1}<\delta_{4}<\frac{1}{4}-\delta_{3}-\frac{\delta_{2}}{2}$. The idea of Guillotin-Plantard and Le Ny is to write:

$$
\begin{aligned}
n^{\frac{1}{2}+\delta_{3}}\left|I_{n}^{(1)}\right| & \leq n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\prod_{y \in \mathbb{Z}}\left|\cos \left(t N_{n-1}(y)\right)+i\left(2 f_{y}-1\right) \sin \left(t N_{n-1}(y)\right)\right| \mid\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t \\
& \leq n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\prod_{y \in \mathbb{Z}} \sqrt{1-4 f_{y}\left(1-f_{y}\right) \sin ^{2}\left(t N_{n-1}(y)\right)} \mid\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t \\
& \leq n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\left.\prod_{y \in \mathbb{Z}} \sqrt{1-f_{y}\left(1-f_{y}\right) \frac{16}{\pi^{2}}\left(t N_{n-1}(y)\right)^{2}} \right\rvert\,\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t \\
& \leq n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\prod_{y \in \mathbb{Z}} \mathrm{e}^{-\frac{8}{\pi^{2}} f_{y}\left(1-f_{y}\right) t^{2} N_{n-1}(y)^{2}}\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t
\end{aligned}
$$

since $\left|t N_{n-1}(y)\right| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} n^{\frac{1}{2}+\delta_{2}}=n^{2 \delta_{2}-\delta_{3}}$. Hence, if $n$ is large enough, then $\left|t N_{n-1}(y)\right|$ will be uniformly less than $\frac{\pi}{2}$ and $\left|\sin \left(t N_{n-1}(y)\right)\right| \geq \frac{2}{\pi}\left|t N_{n-1}(y)\right|$. We also use the fact that, for positive $u$, we have: $1-u \leq \mathrm{e}^{-u}$.

According to the Hölder inequality with $\sum_{y} \frac{N_{n-1}(y)^{2}}{\sum_{k} N_{n-1}(k)^{2}}=1$, we have:

$$
n^{\frac{1}{2}+\delta_{3}}\left|I_{n}^{(1)}\right| \leq n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\left.\mathrm{e}^{-\frac{8}{\pi^{2}} f_{0}\left(1-f_{0}\right) t^{2} \sum_{k} N_{n-1}(k)^{2}} \right\rvert\,\left(S_{p}\right)_{p}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t
$$

Now, we use the fact that, since $\delta_{4}>\delta_{1}$, there exists a constant $c$ such that we have:

$$
\forall \omega^{\prime} \in A_{n}, \quad \sum_{y \in \mathbb{Z}}\left(N_{n-1}(y)\right)^{2}\left(\omega^{\prime}\right) \geq c n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}} .
$$

This has been proved in the previous Section 3.1.2. Hence, under the hypothesis $\int_{M} \frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}} \mathrm{d} \nu<+\infty$ of Guillotin-Plantard and Le Ny, we have:

$$
\begin{aligned}
n^{\frac{1}{2}+\delta_{3}}\left|I_{n}^{(1)}(\omega)\right| & \leq n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{-\frac{1}{2}-\delta_{3}+\delta_{2}}\right\}} \mathbb{E}\left[\mathrm{e}^{-\frac{8 c}{\pi^{2}} f_{0}\left(1-f_{0}\right) t^{2} n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}}}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t \\
& \leq n^{-\frac{1}{4}+\delta_{3}+\frac{\delta_{2}}{2}+\delta_{4}} \int_{\mathbb{R}} \mathbb{E}\left[\frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}}\right] \mathrm{e}^{-\frac{8}{\pi^{2}} v^{2}} \mathrm{~d} v
\end{aligned}
$$

with the change of variable $v=t \sqrt{f_{0}\left(1-f_{0}\right) n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}}}$. This gives the result of Guillotin-Plantard and Le Ny since $-\frac{1}{4}+\delta_{3}+\frac{\delta_{2}}{2}+\delta_{4}<0$. We adapt this argument to our hypothesis. Now let us replace the hypothesis $\int_{M} \frac{1}{\sqrt{f_{0}\left(1-f_{0}\right)}} \mathrm{d} \nu<+\infty$ by $\int_{M} \frac{1}{\left[f_{0}\left(1-f_{0}\right)\right]^{p}} \mathrm{~d} \nu<+\infty$ for some $p>0$. Let us take $\delta_{3}>2 \delta_{2}$ and $\delta_{1}<\delta_{4}<$ $\frac{1}{4}-\delta_{3}-\frac{\delta_{2}}{2}-\frac{\delta_{2}}{p}$. We have:

$$
\begin{aligned}
& n^{\frac{1}{2}+\delta_{3}} \int_{\left\{|t| \leq n^{\left.-\frac{3}{4}+\frac{\delta_{2}}{2}+\delta_{4}+\frac{\delta_{2}}{p}\right\}}\right.} \mathbb{E}\left[\mathrm{e}^{\left.-\frac{8}{\pi^{2}} f_{0}\left(1-f_{0}\right) t^{2} n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}}\right] \mathrm{e}^{-\frac{t^{2} n^{1}+2 \delta_{3}}{2}} \mathrm{~d} t} \leq 2 n^{\frac{1}{2}+\delta_{3}} n^{-\frac{3}{4}+\frac{\delta_{2}}{2}+\delta_{4}+\frac{\delta_{2}}{p}}\right. \\
& \leq 2 n^{-\frac{1}{4}+\delta_{3}+\frac{\delta_{2}}{2}+\delta_{4}+\frac{\delta_{2}}{p}}
\end{aligned}
$$

On the other hand, let $c_{p}=\sup _{u>0} u^{p} \mathrm{e}^{-u}$, we have:

$$
\begin{aligned}
& n^{\frac{1}{2}+\delta_{3}} \int_{\left\{n^{-\frac{3}{4}+\frac{\delta_{2}}{2}+\delta_{4}+\frac{\delta_{2}}{p}}<|t|<n^{\left.-\frac{1}{2}-\delta_{3}+\delta_{2}\right\}}\right.} \mathbb{E}\left[\mathrm{e}^{\left.-\frac{8}{\pi^{2}} f_{0}\left(1-f_{0}\right) t^{2} n^{\frac{3}{2}-\delta_{2}-2 \delta_{4}}\right] \mathrm{e}^{-\frac{t^{2} n^{1+2 \delta_{3}}}{2}} \mathrm{~d} t \leq}\right. \\
& 2 n^{\frac{1}{2}+\delta_{3}} n^{-\frac{1}{2}-\delta_{3}+\delta_{2}} \int_{M} \mathrm{e}^{-\frac{8}{\pi^{2}} f_{0}\left(1-f_{0}\right) n^{\frac{2 \delta_{2}}{p}}} \mathrm{~d} \nu \leq n^{-\delta_{2}} c_{p}\left(\frac{\pi^{2}}{8}\right)^{p} \int_{M}\left[f_{0}\left(1-f_{0}\right)\right]^{-p} \mathrm{~d} \nu
\end{aligned}
$$

## Appendix A: Proof of Proposition 4

In cases $(a)$ and $(b),\left(\xi_{k}\right)_{k}$ is a stationary sequence of bounded centered random variables.

## A. 1 Proof of (a)

We have: $\sum_{p \geq 0} \sqrt{1+p}\left|\mathbb{E}\left[\xi_{0} \xi_{p}\right]\right|=\sum_{p \geq 0} \sqrt{1+p}\left|\mathbb{E}_{\nu}\left[f . f \circ T^{p}\right]\right|$ which is less than:

$$
c_{0}\|f\|_{\infty}\left(\|f\|_{\infty}+K_{f}^{(1)}+K_{f}^{(2)}\right) \sum_{p \geq 0} \sqrt{1+p} \varphi_{p}
$$

and hence is finite. Let us consider an integer $N \geq 1$. We have:

$$
\frac{1}{N^{2}} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=0, \ldots, N-1}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right| \leq \frac{24}{N^{2}} \sum_{0 \leq k_{1} \leq k_{2} \leq k_{3} \leq k_{4} \leq N-1}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right|
$$

Let us consider the set $E_{N}^{(1)}$ of $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ such that $0 \leq k_{1} \leq k_{2} \leq k_{3} \leq k_{4} \leq N-1$ and $k_{4}-k_{3} \geq N^{\frac{1}{3}}$. We have:

$$
\begin{aligned}
\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(1)}}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right| & =\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(1)}}\left|\operatorname{Cov}_{\nu}\left(f \circ T^{k_{1}-k_{3}} f \circ T^{k_{2}-k_{3}} f, f \circ T^{k_{4}-k_{3}}\right)\right| \\
& \leq c_{0} N^{4}\left(\|f\|_{\infty}^{4}+\|f\|_{\infty}^{3}\left(K_{f}^{(2)}+3 c_{0} K_{f}^{(1)}\right)\right) \varphi_{\left\lceil N^{\frac{1}{3}}\right\rceil} \\
& \leq c_{0} N^{2}\left(\|f\|_{\infty}^{4}+\|f\|_{\infty}^{3}\left(K_{f}^{(2)}+3 c_{0} K_{f}^{(1)}\right)\right) \sup _{n \geq 1} n^{6} \varphi_{n} .
\end{aligned}
$$

Let us consider the set $E_{N}^{(2)}$ of $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ such that $0 \leq k_{1} \leq k_{2} \leq k_{3} \leq k_{4} \leq N-1$ and $k_{4}-k_{3}<N^{\frac{1}{3}}$ and $k_{3}-k_{2} \geq r N^{\frac{1}{3}}$. We have:

$$
\begin{aligned}
\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(2)}}\left|\operatorname{Cov}\left(\xi_{k_{1}} \xi_{k_{2}}, \xi_{k_{3}} \xi_{k_{4}}\right)\right| & =\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(2)}}\left|\operatorname{Cov}_{\nu}\left(f \circ T^{k_{1}-k_{2}} f,\left(f . f \circ T^{k_{4}-k_{3}}\right) \circ T^{k_{3}-k_{2}}\right)\right| \\
& \leq 2^{6} c_{0} N^{2}\left(\|f\|_{\infty}^{4}+2 c_{0}\|f\|_{\infty}^{3}\left(K_{f}^{(2)}+K_{f}^{(1)}\right)\right) \sup _{n \geq 1} n^{6}\left(1+\kappa_{n}\right) \varphi_{r n}
\end{aligned}
$$

Moreover, we have:

$$
\begin{aligned}
\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(2)}}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}}\right] \mathbb{E}\left[\xi_{k_{3}} \xi_{k_{4}}\right]\right| & \leq\left(\sum_{0 \leq k_{1} \leq k_{2} \leq N-1}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}}\right]\right|\right)^{2} \leq\left(N \sum_{k \geq 0}\left|\mathbb{E}_{\nu}\left[f \cdot f \circ T^{k}\right]\right|\right)^{2} \\
& \leq N^{2}\left(c_{0}\left(\|f\|_{\infty}^{2}+\|f\|_{\infty}\left(K_{f}^{(1)}+K_{f}^{(2)}\right)\right) \sum_{k \geq 0} \varphi_{k}\right)^{2}
\end{aligned}
$$

Let us consider the set $E_{N}^{(3)}$ of $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ such that $0 \leq k_{1} \leq k_{2} \leq k_{3} \leq k_{4} \leq N-1$ and $k_{4}-k_{3}<N^{\frac{1}{3}}$ and $k_{3}-k_{2}<r N^{\frac{1}{3}}$ and $k_{2}-k_{1} \geq r(1+r) N^{\frac{1}{3}}$. By the same method, we get:

$$
\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(3)}}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right| \leq N^{2} \frac{c_{0} 2^{6}}{(1+r)^{6}}\left(\|f\|_{\infty}^{4}+3 c_{0}\|f\|_{\infty}^{3}\left(K_{f}^{(2)}+K_{f}^{(1)}\right)\right) \sup _{n \geq 1} n^{6}\left(1+\kappa_{n}\right) \varphi_{r n}
$$

Since the number of $\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ such that $0 \leq k_{1} \leq k_{2} \leq k_{3} \leq k_{4} \leq N-1$ and that do not belong to $E_{N}^{(1)} \cup E_{N}^{(2)} \cup E_{N}^{(3)}$ is bounded by $N^{2} 2(r+1)^{3}$, we get:

$$
\sup _{N \geq 1} \frac{1}{N^{2}} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=0, \ldots, N-1}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right|<+\infty
$$

Now, let us prove the point 2 of the hypothesis of Theorem 1. Let $n_{1}, n_{2}, n_{3}$ and $n_{4}$ be four integers such that $0 \leq n_{1} \leq n_{2} \leq n_{3} \leq n_{4}$. Let us consider any real numbers $\alpha_{n_{1}}, \ldots, \alpha_{n_{2}}$ and $\beta_{n_{3}}, \ldots, \beta_{n_{4}}$. We have:

$$
\begin{aligned}
& \left|\operatorname{Cov}\left(\mathrm{e}^{i \sum_{k=n_{1}}^{\bar{n}_{2}} \alpha_{k} \xi_{k}}, \mathrm{e}^{i \sum_{k=n_{3}}^{n_{4}} \beta_{k} \xi_{k}}\right)\right|=\left|\operatorname{Cov}_{\nu}\left(\mathrm{e}^{i \sum_{k=n_{1}}^{n_{2}} \alpha_{k} f \circ T^{-\left(n_{2}-k\right)}},\left(\mathrm{e}^{i \sum_{k=n_{3}}^{n_{4}} \beta_{k} f \circ T^{k-n_{3}}}\right) \circ T^{n_{3}-n_{2}}\right)\right| \\
& \leq c_{0}\left(1+K_{\exp \left(i \sum_{k=n_{1}}^{n_{2}} \alpha_{k} f \circ T^{-\left(n_{2}-k\right)}\right)}^{(1)}+K_{\exp \left(i \sum_{k=n_{3}}^{n_{4}} \beta_{k} f \circ T^{k-n_{3}}\right)}^{(2)}\right) \varphi_{n_{3}-n_{2}} \\
& \leq c_{0}\left(1+\sum_{k=n_{1}}^{n_{2}} c_{0} c_{1}\left|\alpha_{k}\right|+\sum_{k=n_{3}}^{n_{4}} c_{0} c_{1}\left|\beta_{k}\right|\left(1+\kappa_{n_{4}-n_{3}}\right)\right) \varphi_{n_{3}-n_{2}} .
\end{aligned}
$$

This gives the point 2 of the hypothesis of Theorem 1 with $\varphi_{p, s}:=\left(1+\kappa_{s}\right) \varphi_{p}$.

## A. 2 Proof of (b)

Let us define the function $g=2 f-1$. This function is $\nu$-centered. More generally, for any integer $m \geq 1$, let us define: $g_{2 m}=1$ and $g_{2 m+1}=g$. We observe that, conditionally to $\omega \in M$, the expectation of $\left(\xi_{k}(\omega, \cdot)\right)^{m}$ is equal to $g_{m} \circ T^{k}(\omega)$. Using the Fubini theorem and starting by integrating over $[0 ; 1]^{\mathbb{Z}}$, we observe that, for any integers $p \geq 1$, we have: $\mathbb{E}\left[\xi_{0} \xi_{p}\right]=\mathbb{E}_{\nu}\left[g . g \circ T^{p}\right]$ and that, for any integers $k_{1}, k_{2}, k_{3}, k_{4}$, we have: $\mathbb{E}\left[\xi_{k_{1}}{ }^{n_{1}} \xi_{k_{2}}{ }^{n_{2}} \xi_{k_{3}}{ }^{n_{3}} \xi_{k_{4}}{ }^{n_{4}}\right]=\mathbb{E}_{\nu}\left[\prod_{j=1}^{4} g_{n_{j}} \circ T^{k_{j}}\right]$. Hence, we can prove the point 1 of Theorem 1 as we did for (a).

Now, let us prove the point 2 of the hypothesis of Theorem 1 . We observe that, conditionally to $\omega \in M$, the $\xi_{k}(\omega, \cdot)$ are independent and that the expectation of $\exp \left(i u \xi_{k}(\omega, \cdot)\right)$ is $h_{u} \circ T^{k}(\omega)$ with $\left(h_{u}:=\mathrm{e}^{-i u}+2 i \sin (u) f \circ T^{k}\right.$. The modulus of this function is bounded by 1 and we have: $\max \left(K_{h_{u}}^{(1)}, K_{h_{u}}^{(2)}\right) \leq 2 c_{1}|u|$. Let $n_{1}, n_{2}, n_{3}$ and $n_{4}$ be four integers such that $0 \leq n_{1} \leq n_{2}<n_{3} \leq n_{4}$. Let us consider any real numbers $\alpha_{n_{1}}, \ldots, \alpha_{n_{2}}$ and $\beta_{n_{3}}, \ldots, \beta_{n_{4}}$. We have:

$$
\begin{aligned}
\left|\operatorname{Cov}\left(\mathrm{e}^{i \sum_{k=n_{1}}^{n_{2}} \alpha_{k} \xi_{k}}, \mathrm{e}^{i \sum_{k=n_{3}}^{n_{4}} \beta_{k} \xi_{k}}\right)\right|=\left|\operatorname{Cov}_{\nu}\left(\prod_{k=n_{1}}^{n_{2}} h_{\alpha_{k}} \circ T^{k}, \prod_{k=n_{3}}^{n_{4}} h_{\beta_{k}} \circ T^{k}\right)\right| \\
\leq c_{0}\left(1+2 c_{0} c_{1}\left(\sum_{k=n_{1}}^{n_{2}}\left|\alpha_{k}\right|+\sum_{k=n_{3}}^{n_{4}}\left|\beta_{k}\right|\right)\right)\left(1+\kappa_{n_{4}-n_{3}}\right) \varphi_{n_{3}-n_{2}} .
\end{aligned}
$$

## Appendix B: Proof of Theorem 2: $\alpha$-mixing condition

Let us define $(M, \mathcal{F})=\left(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R})^{\otimes \mathbb{Z}}\right)$. Let $T: M \rightarrow M$ be such that $T\left(\left(\omega_{k}\right)_{k \in \mathbb{Z}}\right)=\left(\omega_{k+1}\right)_{k \in \mathbb{Z}}$. Let $\nu$ be the image probability measure on $(M, \mathcal{F})$ of $\Pi: \Omega \rightarrow \mathbb{R}^{\mathbb{Z}}$ with $\Pi(\omega)=\left(\xi_{k}(\omega)\right)_{k \in \mathbb{Z}}$. The process $\left(\xi_{k}\right)_{k \in \mathbb{Z}}$ (with respect to $\mathbb{P}$ ) has the same distribution as $\left(f \circ T^{k}\right)_{k \in \mathbb{Z}}$ (with respect to $\nu$ ) with $f: M \rightarrow \mathbb{R}$ given by $f\left(\left(\omega_{k}\right)_{k \in \mathbb{Z}}\right)=\omega_{0}$. According to [11], Lemma 1.2, $(M, \mathcal{F}, \nu, T)$ is strongly mixing (in the sense of our Def. 3) with the following choice of $K_{.}^{(1)}$ and of $K_{.}^{(2)}$. If $g$ is $\sigma\left(f \circ T^{k}, k \leq 0\right)$-measurable, we have $K_{g}^{(1)}:=0$; otherwise we have $K_{g}^{(1)}:=\infty$. If $h$ is $\sigma\left(f \circ T^{k}, k \geq 0\right)$-measurable, we have $K_{h}^{(2)}:=0$; otherwise we have $K_{h}^{(2)}:=\infty$. We conclude with Proposition 4.

## Appendix C: Proof of Example 2.1

## C. 1 Case 1

Let $\eta>0$. Let us denote by $\Gamma^{(s, e)}$ the set of stable-central manifolds and by $\Gamma^{u}$ the set of unstable manifolds. In [16], each $\gamma^{u} \in \Gamma^{u}$ is endowed with some metric $d^{u}$ and each $\gamma^{(s, e)} \in \Gamma^{(s, e)}$ is endowed with some metric $d^{(s, e)}$ such that there exist $\left.\tilde{c}_{0}>0, \delta_{0} \in\right] 0 ; 1\left[\right.$ and $\beta>0$ such that, for any integer $n \geq 0$, for any $\gamma^{u} \in \Gamma^{u}$ and any $\gamma^{(s, e)} \in \Gamma^{(s, e)}$, we have:

- For any $y, z \in \gamma^{u}, d^{u}(y, z) \geq d(y, z)$ and for any $y^{\prime}, z^{\prime} \in \gamma^{(s, e)}, d^{(s, e)}\left(y^{\prime}, z^{\prime}\right) \geq d\left(y^{\prime}, z^{\prime}\right)$.
- For any $y, z \in \gamma^{u}$, there exists $\gamma_{(n)}^{u} \in \Gamma^{u}$ such that $T^{-n}(y)$ and $T^{-n}(z)$ belong to $\gamma_{(n)}^{u}$ and we have: $d^{u}\left(T^{-n}(y), T^{-n}(z)\right) \leq \tilde{c}_{0}\left(\delta_{0}\right)^{n} d^{u}(y, z)$.
- For any $y, z \in \gamma^{(s, e)}$, there exists $\gamma_{(n)}^{(s, e)} \in \Gamma^{(s, e)}$ such that $T^{n}(y)$ and $T^{n}(z)$ belong to $\gamma_{(n)}^{(s, e)}$ and we have: $d^{(s, e)}\left(T^{n}(y), T^{n}(z)\right) \leq \tilde{c}_{0}\left(1+n^{\beta}\right) d^{(s, e)}(y, z)$.
We take:

$$
K_{f}^{(1)}:=\sup _{\gamma^{u} \in \Gamma^{u}} \sup _{y, z \in \gamma^{u}: y \neq z} \frac{|f(y)-f(z)|}{\left(d^{u}(y, z)\right)^{\eta}} \text { and } K_{f}^{(2)}:=\sup _{\gamma^{(s, e)} \in \Gamma^{(s, e)}} \sup _{y, z \in \gamma^{(s, e)}: y \neq z} \frac{|f(y)-f(z)|}{\left(d^{(s, e)}(y, z)\right)^{\eta}} .
$$

For these examples, the result follows from [16] (cf. Lem. 1.3.1 in [16]).

## C. 2 Case 2: Sinai billiard

Since the early work of Sinai [17], this billiard system has been studied by many authors ([1-4,7,19] and others). Let us recall that a point of $M$ is a couple $(q, v)$ corresponds to a reflected unit speed vector $v$ at the position $q$ on some obstacle $O_{i}$ and is parametrised by $(i, r \varphi)$ where $i$ is the index of the obstacle $O_{i}$, $r$ the curvilinear of $x$ on it and $\varphi$ the measure of the angle (taken in $[-\pi / 2 ; \pi / 2]$ ) made by $v$ with the unit normal vector $\vec{n}(q)$ to $O_{i}$ at $q$ directed to the outside of the obstacle. We endow $M$ with a metric $d$ such that: $d\left((i, r, \varphi),\left(i, r^{\prime}, \varphi^{\prime}\right)\right)=\left|r-r^{\prime}\right|+\left|\varphi-\varphi^{\prime}\right|$. Let us denote by $R_{0}$ the set of points in $M$ corresponding to a reflected vectors tangent to the obstacles, i.e. such that $\varphi= \pm \pi / 2$. The transformation $T^{n}$ defines a $C^{1}$-diffeomorphism from $M \backslash \bigcup_{k=0}^{n} T^{-k}\left(R_{0}\right)$ onto $M \backslash \bigcup_{k=0}^{n} T^{k}\left(R_{0}\right)$. Let us consider the set $\mathcal{C}_{m}$ of connected components of $M \backslash \bigcup_{k=-m}^{m} T^{k}\left(R_{0}\right)$. For all $k=-m, \ldots, m, T^{k}$ is $C^{1}$ on each $C$ belonging to $\mathcal{C}_{m}$. We will use the notations of Chernov in [6]. Let us consider the set $\Gamma^{s}$ of homogeneous stable curves and the set $\Gamma^{u}$ of homogeneous unstable curves and the two separation times $s_{+}(\cdot, \cdot)$ (in the future) and $s_{-}(\cdot, \cdot)$ (in the past) considered in [6]. We recall that there exist two constants $c_{1}>0$ and $\left.\delta_{1} \in\right] 0 ; 1[$ such that, for any nonnegative integer $n$, for any $y$ and $z$ in $M$, we have:

- If $y$ and $z$ belong to the same homogeneous unstable curve, then $s_{+}(x, y) \in \mathbb{Z}_{+}$, moreover $T^{-n}(y)$ and $T^{-n}(z)$ belong to a same homogeneous unstable curve and we have: $d\left(T^{-n}(y), T^{-n}(z)\right) \leq c_{1} \delta_{1}{ }^{n}$ and $s_{+}\left(T^{-n}(x), T^{-n}(y)\right) \geq n+s_{+}(x, y)$.
- If $y$ and $z$ belong to the same homogeneous stable curve, then $s_{-}(x, y) \in \mathbb{Z}_{+}$, moreover $T^{n}(y)$ and $T^{n}(z)$ belong to a same homogeneous stable curve and we have: $d\left(T^{n}(y), T^{n}(z)\right) \leq c_{1} \delta_{1}{ }^{n}$ and $s_{-}\left(T^{n}(x)\right.$, $\left.T^{n}(y)\right) \geq n+s_{-}(x, y)$.
With these notations, according to [6] (Th. 4.3 in [6] and the remark after Th. 4.3 in [6]), this system is strongly mixing with:

$$
K_{f}^{(1)}:=\sup _{\gamma^{u} \in \Gamma^{u}} \sup _{\substack{y, z \in \gamma^{u} ; y \neq z ;}} \frac{|f(y)-f(z)|}{\left(\delta_{1}\right)^{\eta_{s}(y, z)}} \text { and } K_{f}^{(2)}:=\sup _{\gamma^{s} \in \Gamma^{s}} \sup _{\substack{y, z \in \gamma^{s} ; y \neq z ; \\ s_{+}(y, z) \geq m+1}} \frac{|f(y)-f(z)|}{\left(\delta_{1}\right)^{\eta_{-}(y, z)}} .
$$

## Appendix D: Proof of conclusion (B) of Theorem 5

We will use $b$ and $\delta$ of Proposition 2.1. First let us notice that there exists $c_{A}^{\prime}>0$ such that, for every $\left.\varepsilon \in\right] 0 ; 1[$, there exists a Lipschitz continuous function $f_{\varepsilon}$ such that: $\left\|\mathbf{1}_{A}-f_{\varepsilon}\right\|_{L^{1}(\nu)} \leq c_{A} \varepsilon^{\zeta}, \quad\left\|f_{\varepsilon}\right\|_{\infty} \leq 1$ and $C_{f_{\varepsilon}}^{(1)} \leq \frac{c_{A}^{\prime}}{\varepsilon}$. It suffices to take $f_{\varepsilon}=\max \left(0,1-\frac{d(\cdot,, A)}{\varepsilon}\right)$.

- Let us prove that: $\sum_{p \geq 0} \sqrt{1+p}\left|\mathbb{E}\left[\xi_{0} \xi_{p}\right]\right|<+\infty$. This quantity can be rewritten:

$$
4 \sum_{p \geq 0} \sqrt{1+p}\left|\operatorname{Cov}_{\nu}\left(\mathbf{1}_{A}, \mathbf{1}_{A} \circ T^{p}\right)\right|
$$

and is less than: $4 \sum_{p \geq 0} \sqrt{1+p}\left|\operatorname{Cov}_{\nu}\left(f_{p^{-2 / \zeta}}, f_{p^{-2 / \zeta}} \circ T^{p}\right)+2 c_{A} p^{-2}\right|$. Moreover, we have:

$$
\left|\operatorname{Cov}_{\nu}\left(f_{p^{-2 / \zeta}}, f_{p^{-2 / \zeta}} \circ T^{p}\right)\right| \leq c_{0}\left(1+K_{f_{p^{-2 / \zeta}}^{(1)}}^{(1)}+K_{f_{p}-2 / \zeta}^{(2)}\right) \alpha^{p} \leq c_{0}\left(1+2 c_{A}^{\prime} p^{2 / \zeta}\right) \alpha^{p}
$$

- Let us prove that:

$$
\sup _{N \geq 1} N^{-2} \sum_{k_{1}, k_{2}, k_{3}, k_{4}=0, \ldots, N-1}\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right|<+\infty
$$

We use the notations $E_{N}^{(1)}, E_{N}^{(2)}$ and $E_{N}^{(3)}$ and the calculations done in Section A.1.
 $2\left(f_{N} \frac{-2}{\zeta}-\mathbb{E}_{\nu}\left[f_{N} \frac{-2}{\zeta}\right]\right)$. We have: $\left\|\xi_{k}-g_{N} \circ T^{k}\right\|_{L^{1}(\nu)} \leq 4 c_{A} N^{-2}$. This substitution makes a total error in $O\left(N^{2}\right)$. Moreover, according to the calculations of Section A.1, we have:

$$
\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(1)} \cup E_{N}^{(3)}}\left|\mathbb{E}\left[g_{N} \circ T^{k_{1}} g_{N} \circ T^{k_{2}} g_{N} \circ T^{k_{3}} g_{N} \circ T^{k_{4}}\right]\right| \leq C N^{4}\left(1+C_{g_{N}}^{(1)}\right)\left(1+N^{\beta / 3}\right) \alpha^{N^{\frac{1}{3}}}=O\left(N^{2}\right)
$$

- With the same technique, we get: $\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(2)}}\left|\operatorname{Cov}_{\nu}\left(\xi_{k_{1}} \xi_{k_{2}}, \xi_{k_{3}} \xi_{k_{4}}\right)\right|=O\left(N^{2}\right)$. Moreover, as in Section A.1, we have: $\sum_{\left(k_{1}, k_{2}, k_{3}, k_{4}\right) \in E_{N}^{(2)}}\left|\mathbb{E}_{\nu}\left[\xi_{k_{1}} \xi_{k_{2}}\right] \mathbb{E}_{\nu}\left[\xi_{k_{3}} \xi_{k_{4}}\right]\right| \leq N^{2}\left(\sum_{k \geq 0}\left|\mathbb{E}\left[\xi_{0} \xi_{k}\right]\right|\right)^{2}$ and we have already proved that: $\sum_{k>0}\left|\mathbb{E}\left[\xi_{0} \xi_{k}\right]\right|<+\infty$.
- The sum of $\left|\mathbb{E}\left[\xi_{k_{1}} \xi_{k_{2}} \xi_{k_{3}} \xi_{k_{4}}\right]\right|$ over the $k=\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$ such that $0 \leq k_{1} \leq k_{2} \leq k_{3} \leq k_{4}$ but that do not belong to $E_{N}^{(1)} \cup E_{N}^{(2)} \cup E_{N}^{(3)}$ is controlled as in Section A.1.
- Let us prove point 2 of hypothesis of Theorem 1. By replacing each $\xi_{k}$ by $\hat{\xi}_{k}^{\left(n_{3}-n_{2}\right)}:=h_{n_{3}-n_{2}} \circ T^{k}$, with $h_{N}:=2\left(f_{N} \frac{-7}{\zeta}-\mathbb{E}_{\nu}\left[f_{N} \frac{-7}{\zeta}\right]\right)$, we make a total error in $\left(1+\sum_{k=n_{1}}^{n_{2}}\left|\alpha_{k}\right|+\sum_{k=n_{3}}^{n_{4}}\left|\beta_{k}\right|\right)\left(n_{3}-n_{2}\right)^{-7}$. Moreover, according to the calculations done in Section A.1, we have:

$$
\begin{aligned}
\left|\operatorname{Cov}\left(\mathrm{e}^{i \sum_{k=n_{1}}^{n_{2}} \alpha_{k} \hat{\xi}_{k}^{n_{3}-n_{2}}}, \mathrm{e}^{i \sum_{k=n_{3}}^{n_{4}} \beta_{k} \hat{\xi}_{k}^{n_{3}-n_{2}}}\right)\right| & \leq c_{0}\left(1+\left(\sum_{k=n_{1}}^{n_{2}}\left|\alpha_{k}\right|+\sum_{k=n_{3}}^{n_{4}}\left|\beta_{k}\right|\right) c_{0} C_{h_{n_{3}-n_{2}}^{(1)}}\right)\left(1+\left(n_{4}-n_{3}\right)^{\beta}\right) \alpha^{n_{3}-n_{2}} \\
& \leq C\left(1+\sum_{k=n_{1}}^{n_{2}}\left|\alpha_{k}\right|+\sum_{k=n_{3}}^{n_{4}}\left|\beta_{k}\right|\right)\left(1+\left(n_{4}-n_{3}\right)^{\beta}\right)\left(n_{3}-n_{2}\right)^{\frac{7}{\zeta}} \alpha^{n_{3}-n_{2}}
\end{aligned}
$$

This gives the point 2 of the hypothesis of Theorem 1 with $\varphi_{p, s}=p^{-7}+\left(1+s^{\beta}\right) p^{7 / \zeta} \delta^{p}$.

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