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MULTISCALE PIECEWISE DETERMINISTIC MARKOV PROCESS IN INFINITE DIMENSION: CENTRAL LIMIT THEOREM AND LANGEVIN APPROXIMATION *

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Abstract. In [A. Genadot and M. Thieullen, Averaging for a fully coupled piecewise-deterministic markov process in infinite dimensions. *Adv. Appl. Probab.* **44** (2012) 749–773], the authors addressed the question of averaging for a slow-fast Piecewise Deterministic Markov Process (PDMP) in infinite dimensions. In the present paper, we carry on and complete this work by the mathematical analysis of the fluctuations of the slow-fast system around the averaged limit. A central limit theorem is derived and the associated Langevin approximation is considered. The motivation for this work is the study of stochastic conductance based neuron models which describe the propagation of an action potential along a nerve fiber.

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1. Introduction

In [17], the authors addressed the question of averaging for a class of multiscale spatially extended stochastic conductance based neuron models, also known as spatially extended stochastic generalized Hodgkin–Huxley models. These models describe the evolution of an action potential or nerve impulse along the axon of a neuron at the scale of ionic channels. More generally, in electro-physiology, these equations describe the evolution of an action potential in excitable membranes. Mathematically, these spatially extended stochastic conductance based models belong to the class of Hilbert-valued Piecewise Deterministic Markov Processes (PDMP) with multiple time scales. In [17], we obtained averaging results for this class of models. The averaged models are still Hilbert-valued PDMPs but of lower dimensions in the sense that the dynamic of the jump components of the slow-fast PDMP is simplified. In the present paper, we study the fluctuations of the original slow-fast systems around their averaged limit. A central limit theorem is derived and the associated Langevin approximation is considered. A numerical example based on a spatially extended stochastic Morris–Lecar model is provided at the end of the paper.

Keywords and phrases. Piecewise deterministic Markov process, averaging principle, neuron model.

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The mathematical analysis of PDMPs, and more generally of hybrid systems, constitutes a very active area of research since a few years. A hybrid system can be defined as a dynamical system describing the interactions between a continuous macroscopic dynamic and a discrete microscopic one. For PDMPs, between the jumps, the motion of the macroscopic component is given by a deterministic flow. This deterministic flow is in its turn given by solutions of a partial differential equation (PDE) in the infinite dimensional case and an ordinary differential equations (ODE) in the finite dimensional one. If the macroscopic component were held fixed, then the microscopic component would follow the dynamic of a continuous time Markov chain. It is essentially these two properties, a deterministic flow punctuated by markovian jumps, which give to PDMPs their most important characteristic as hybrid systems: they enjoy the Markov property.

Markovian hybrid systems such as PDMPs are the object of a great attention because they offer an accurate description of a large class of phenomena arising in various domains such as physics or biology. Moreover, their Markovian structure allows to use the very well developed theory of Markov processes to their study. In mathematical neurosciences, a domain the authors are more particularly interested in, PDMP models arise naturally in the description of the propagation of a nerve impulse. Point models of generation (but not propagation) of action potentials have been studied in [26,33] in the framework of finite dimensional PDMPs. Similarly, a wide class of conductance based neuron models describing the generation and propagation of an action potential may be described as infinite dimensional PDMPs, *cf.* [8]. In these models, the macroscopic components is continuous and describes the evolutions of the action potential on the neuronal membrane. It follows a PDE with parameters which are randomly updated. These switches corresponds to jumps of the microscopic components: the ionic channels, present all over the neuronal membrane, open and close stochastically. The stochastic neural field equations have also been considered as limits of underlying hybrid mechanisms, *cf.* [30]. These mathematical descriptions of neuron models are consistent with classical deterministic models such as the Hodgkin–Huxley model and the compartment type models, see [1, 30, 31].

PDMPs have been introduced by Davis in [12,13] in the finite dimensional setting and generalized in [8] to the infinite dimensional case and more particularly to Hilbert-valued PDMPs. Recently, the asymptotic behavior of finite dimensional PDMPs has been investigated in [3,4,32] through the research of existence and uniqueness of invariant measures for PDMPs. Let us mention that also for finite dimensional PDMPs, control problems have been studied in [10,18], numerical methods in [6,29], time reversal in [23] and to end up this list with no claim of completeness, estimation of the jump rates for PDMPs in [2,14].

Limit theorems for infinite dimensional PDMPs have been derived in [1,31]: a Law of Large Numbers and a Central Limit Theorem for sequences of Hilbert-valued PDMPs are obtained. In this way, the authors show the consistency of spatially extended stochastic conductance based neuron models with deterministic models. Indeed, in [1,31], the authors show that when the number of ionic channels increases, stochastic conductance based neuron models converge to their corresponding deterministic version. In the present paper, as in [17], we work with a fixed number of ionic channels but with neuron models exhibiting intrinsically two different timescales: the microscopic jump component is accelerated and thus considered as fast with respect to the macroscopic continuous one. With a number of ionic channels held fixed, we obtain limit theorems when the speed of acceleration of the fast dynamic goes to infinity. The studied models correspond to the one studied in [1, 17] and considered as application in [8, 31]. Averaging is of first importance for these models because it allows to simplify the dynamic of the system. Moreover, the averaged limit preserves the qualitative behavior of the original dynamical system such as the excitable properties of the model, see [27,36]. In the finite dimensional case, these questions have been addressed in [16] and [26]. In particular, we extend the results of [26] to spatially extended conductance based neuron models. A question remains, namely, how to treat simultaneously the phenomena of acceleration of the speed of the fast component ($\varepsilon \to 0$) and increase of the number of ionic channels ($N \to \infty$)? For example, do we obtain the same models in permuting the order of these two different limits?

To prove that the renormalized difference between the non-averaged and averaged models converges in law, we use a tightness argument based on the Prohorov procedure developed in the Hilbert space setting in [24]. For this purpose, we use technical estimates of the same kind as in [17], but more involved. To characterize the limit, we extend the arguments of [33] Chapter 6 to the infinite dimensional setting. Then, the Langevin approximation is shown to be a good approximation of the averaged process by arguments coming from SPDE theory. To the best of the authors's knowledge, these methods have never been applied to study the fluctuations of a two timescales infinite dimensional PDMP around its averaged limit.

We notice at this point that the study of slow-fast systems of Stochastic Partial Differential Equations (SPDEs), a framework different from ours but very instructive, is an area of very active research. Averaging results have been derived in [7,9,35] and fluctuations around the limit and large deviations have been studied in [34]. The results of these papers are of the same flavor than the results obtained in the present paper: a Hilbert-valued slow motion is averaged against the invariant measure of a fast dynamic.

The paper is organized as follows. In Section 2 and 3 we recall as briefly as possible the model and the main results of [17] and in particular the different properties of the averaged process. Section 4 introduces the main results of the present paper: the Central Limit Theorem and the attached Langevin approximation are stated. The description of the general class of PDMP which can be included in our framework is described in Section 2.3. In Section 5, we begin by proving the Central Limit Theorem in the so-called all-fast case of Section 4.1. In the all fast case, we divide the proof in two parts: tightness in Section 5.1 and identification of the limit in Section 5.2. Properties of the diffusion operator related to the fluctuations are investigated in Section 5.3. In Section 5.4, the Langevin approximation associated to the averaged model and its fluctuations is considered. In Section 6, as an example, we consider a spatially extended stochastic Morris–Lecar model and provide numerical experiments.

2. The models

2.1. Stochastic Hodgkin–Huxley models

In this section, we introduce the stochastic generalized Hodgkin–Huxley model also known in the literature as stochastic conductance based neuron models. This model was first considered in [1], and later in [8,17,31]. Although we are interested in multiscale stochastic conductance based neuron models, we start by describing the model without different time scales, for the sake of clarity. We begin by stating all our mathematical definitions and assumptions before providing the biological interpretation of our model.

Let T be a fixed finite time horizon, $I = [0,1]$ and E a finite set. We fix an integer $N \geq 1$ and consider the subset $\mathcal{N} = \{z_i, i = 1,2,\dots,N\}$ of $\mathring{I} = (0,1)$. We write $\mathcal{R}$ for the finite set $E^{\mathcal{N}}$ and $H = H_0^1(I)$ for the space of functions in $L^2(I)$ with first distributional derivative also belonging to $L^2(I)$. We recall some basic facts on the Hilbert spaces H and $L^2(I)$, the Laplacian operator Δ and the Dirac delta function on H in Appendix A. Let us simply recall that H and $L^2(I)$ are both Hilbert spaces with respective scalar products denoted by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{L^2(I)}$.

For $(u,r) \in H \times \mathcal{R}$ we define the generalized function $G_r(u)$ (or reaction term) in H^* by

$$G_r(u) = \frac{1}{N}\sum_{i=1}^{N} c_{r(i)}(v_{r(i)} - u(z_i))\delta_{z_i}, \tag{2.1}$$

where $H^* = H^{-1}(I)$ is the dual space of H. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between H and H^*. For $\xi \in E$, c_ξ and v_ξ are two real constants, the first being positive. We omit N in the notation of $G_r(u)$ because, in contrary to [1,31], N is held fixed all along the paper. Notice that, in contrary to the model developed in Section 2.2, $G_r(u)$ does not belong to H, thus, the model of the present section does not enter in the general framework of Section 2.3. However, we prefer to present in a first part the model given by (2.1) with the Dirac delta functions because it corresponds exactly to the model studied in [1,17].

For two states $\xi,\zeta \in E$, we define by $\alpha_{\xi,\zeta}$ the jump intensity or transition rate function from the state ξ to the state ζ. The function $\alpha_{\xi,\zeta}$ is a real valued function of a real variable supposed to be, as its derivative, Lipschitz-continuous. We assume moreover that $0 \leq \alpha_{\xi,\zeta} \leq \alpha^+$ for any $\xi,\zeta \in E$ and either $\alpha_{\xi,\zeta}$ is constant equal to zero or is positive bounded below by a positive constant α_-. That is, the non-zero rate functions are bounded below and above by positive constants.

Then, for $u \in H$ and $(r, \tilde{r})$ two different states of $\mathcal{R}$, we define by $q_{r\tilde{r}}$ the jump intensity or transition rate function from the state r to the state $\tilde{r}$. This is a real valued function defined on H by

$$q_{r\tilde{r}}(u) = \begin{cases} \qquad 0 & \text{if } r \text{ and } \tilde{r} \text{ differ from more than one component,} \\ \dfrac{\alpha_{r(i)\tilde{r}(i)}(u(z_i))}{\alpha_{r(i)}(u(z(i)))} & \text{if } r(i) \neq \tilde{r}(i) \text{ and all the other components are equal.} \end{cases} \tag{2.2}$$

The quantity $\alpha_{r(i)}(u(z_i)) = \sum_{\xi \in E \setminus \{r(i)\}} \alpha_{r(i)\xi}(u(z_i))$ represents the total rate of leaving the state $r(i) \in E$.

The stochastic conductance based model for excitable cells we consider consists in the following evolution problem on I

$$\begin{cases} \partial_t u_t = \Delta u_t + G_{r_t}(u_t), \\ \mathbb{P}(r_{t+h} = \tilde{r} | r_t = r) = q_{r\tilde{r}}(u_t)h + \mathrm{o}(h) \end{cases} \tag{2.3}$$

for $t \in [0, T]$ and zero Dirichlet boundary conditions. That is $u_t(0) = u_t(1) = 0$ for all $t \in [0, T]$. We are interested in the stochastic process $(u_t, r_t)_{t \in [0,T]}$.

The spatially extended stochastic Hodgkin–Huxley model (2.3) describes the propagation of an action potential along an axon at the scale of ionic channels. The axon, or nerve fiber, is the component of a neuron which allows the propagation of an incoming signal from the soma to another neuron on long distances. The length of the axon is large relative to its radius, thus, for mathematical convenience, we consider the axon as a segment I. All along the axon are the ion channels which allow and amplify the propagation of the incoming impulse. We assume that there are N ion channels along the axon located in the subset $\mathcal{N} = \{z_i, i = 1, 2, \ldots, N\}$ of $\mathring{I} = (0, 1)$. In [1, 17] for instance, $\mathcal{N} = \{\frac{i}{N}, i = 1, \ldots, N-1\}$ which means that the ion channels are regularly spaced. Each ion channel can be in several states $\xi \in E$, for instance, in the Hodgkin–Huxley model, a state can be: "receptive to sodium ions and open". When a ion channel is open, it allows some ionic species to enter or leave the cell, generating in this way a current. For a greater insight into the underlying biological phenomena governing the model, the authors refer to [21], Chapter 2.

The ion channels switch between states according to a continuous time Markov chain whose jump intensities depend on the local potential of the axon membrane. For a given channel, the rate function describes the rate at which it switches from one state to another.

A possible configuration of all the N ion channels is denoted by $r = (r(i), i \in \mathcal{N})$, a point in the space of all configurations $\mathcal{R} = E^{\mathcal{N}}$: $r(i)$ is the state of the channel located at z_i, for $i \in \mathcal{N}$. The channels, or stochastic processes $r(i)$, are supposed to evolve independently over infinitesimal time-scales. Denoting by $u_t(z_i)$ the local potential at point z_i at time t, we have

$$\mathbb{P}(r_{t+h}(i) = \zeta | r_t(i) = \xi) = \alpha_{\xi,\zeta}\left(u_t\left(z_i\right)\right) h + \mathrm{o}(h). \tag{2.4}$$

For any $\xi \in E$, c_ξ represents the maximal conductance and v_ξ the steady state potentials, or driven potentials, of a channel in state ξ.

The transmembrane potential $u_t(x)$, that is the difference of electrical potential between the outside and the inside of the axon, evolves according to the following hybrid reaction-diffusion PDE

$$\partial_t u_t = \Delta u_t + \frac{1}{N} \sum_{i=1}^{N} c_{r_t(i)}(v_{r_t(i)} - u_t(z_i))\delta_{z_i}. \tag{2.5}$$

The zero Dirichlet boundary conditions for this PDE corresponds to the case of a clamped axon [21].

2.2. Stochastic Hodgkin–Huxley model with mollifiers

For technical reasons, in the present paper, we will work with a slightly different model where the Dirac distributions δ_{z_i} in (2.5) are replaced by approximations ϕ_{z_i} in the sense of distributions, in the same way as in so called compartment models (see Sect. 4 for more details). In such a model the reaction term is given by

$$G_r(u) = \frac{1}{N}\sum_{i=1}^{N} c_{r(i)}(v_{r(i)} - \bar{u}_i)\phi_{z_i} \tag{2.6}$$

for $(r, u) \in \mathcal{R} \times L^2(I)$ where for any $h \in L^2(I)$, $\bar{h}_i = (h, \phi_{z_i})_{L^2(I)}$. For $i \in \mathcal{N}$, the function ϕ_{z_i} which belongs to $L^2(I)$ approximates the Dirac distribution δ_{z_i}. For $i \in \{1, \dots, N\}$ the functions ϕ_{z_i} are defined on I by

$$\phi_{z_i}(x) = \frac{1}{\kappa} M\left(\frac{x - z_i}{\kappa}\right)$$

with κ small enough such that ϕ_{z_i} is compactly supported in I. The mollifier M is defined on $\mathbb{R}$ by

$$M(x) = \mathrm{e}^{-\frac{1}{1-x^2}} 1_{[-1,1]}(x).$$

Replacing δ_{z_i} by ϕ_{z_i} corresponds to consider that when the channel located at z_i is open and allows a current to pass, not only the voltage at the point z_i is affected, but also the voltage on a small area around z_i, see [8], Section 3.1. The family of functions ϕ_{z_i} is indexed by a parameter κ related to the considered membrane area: the smaller κ is, the smaller is the area. When u is held fixed, the dynamic of the ion channel at location z_i is given by

$$\mathbb{P}(r_{t+h}(i) = \zeta | r_t(i) = \xi) = \alpha_{\xi,\zeta}\left(\bar{u}_{t,i}\right) h + \mathrm{o}(h) \tag{2.7}$$

for $\xi, \zeta \in E$ and $t, h \geq 0$. The present paper is thus concerned with the following evolution problem, for $t \in [0, T]$

$$\begin{cases} \partial_t u_t = \Delta u_t + G_{r_t}(u_t), \\ \mathbb{P}(r_{t+h}(i) = \zeta | r_t(i) = \xi) = \alpha_{\xi\zeta}(\bar{u}_{t,i})h + \mathrm{o}(h). \end{cases} \tag{2.8}$$

2.3. A general framework

The previous stochastic Hodgkin–Huxley model with mollifiers actually belong to a more general framework that we now describe.

Let A be a self-adjoint linear operator on a separable Hilbert space H such that there exists a Hilbert basis $\{e_k, k \geq 1\}$ of H made up with eigenvectors of A

$$Ae_k = -l_k e_k \tag{2.9}$$

for $k \geq 1$ and such that

$$\sup_{k\geq 1}\sup_{y\in I} |e_k(y)| < \infty. \tag{2.10}$$

The eigenvalues $\{l_k, k \geq 1\}$ are assumed to form an increasing sequence of positive numbers enjoying the following property

$$\sum \frac{1}{l_k} < \infty. \tag{2.11}$$

Let $\mathcal{R}$ be a finite set. For any $r \in \mathcal{R}$, the reaction term $G_r : H \mapsto H$ is globally Lipschitz on H uniformly on $r \in \mathcal{R}$. That is to say, there exists a constant $L_G > 0$ such that for any $(r, u, \tilde{u}) \in \mathcal{R} \times H \times H$ we have

$$\|G_r(u) - G_r(\tilde{u})\|_H \leq L_G \|u - \tilde{u}\|_H. \tag{2.12}$$

For fixed $u \in H$ let $Q(u) := (q_{r\tilde{r}}(u))_{(r,\tilde{r})\in\mathcal{R}\times\mathcal{R}}$ be the generator of a continuous time Markov chain $(r_t, t \geq 0)$ on $\mathcal{R}$. We assume that for $r \neq \tilde{r}$, the intensity rate functions $q_{r\tilde{r}} : H \mapsto \mathbb{R}_+$ are uniformly bounded and Lipschitz continuous. There exist two constants B_q, L_q such that for any $(r, \tilde{r}, u, \tilde{u}) \in \mathcal{R}\times\mathcal{R}\times H\times H$ we have

$$\sup_{(r,\tilde{r})\in\mathcal{R}\times\mathcal{R}} \sup_{u\in H} q_{r\tilde{r}}(u) \leq B_q, \quad |q_{r\tilde{r}}(u) - q_{r\tilde{r}}(\tilde{u})| \leq L_q \|u - \tilde{u}\|_H. \tag{2.13}$$

Moreover, we assume that there exists a positive constant q_- such that

$$\inf_{u\in H} \lambda(u) \geq q_-, \tag{2.14}$$

where $\lambda(u)$ is the first non-zero eigenvalue of $Q(u)$. We also assume that there exists a unique pseudo-invariant measure $\mu(u)$ associated to the generator $Q(u)$ which is bounded and Lipschitz continuous with respect to u.

The present paper is concerned with the following evolution problem, for $t \in [0, T]$

$$\begin{cases} \partial_t u_t = A u_t + G_{r_t}(u_t), \\ \mathbb{P}(r_{t+h} = \tilde{r} | r_t = r) = q_{r\tilde{r}}(u_t)h + o(h). \end{cases} \tag{2.15}$$

Let us mention that in this framework, the model with mollifiers corresponds to $H = L^2(I)$ and $\mathcal{R} = E^N$. With $A = \Delta$, the Hilbert space basis $\{f_k, k \geq 1\}$ of $L^2(I)$ defined in Appendix A and $l_k = (k\pi)^2$ for $k \geq 1$, assumptions (2.9)–(2.14) are satisfied.

2.4. Basic properties of stochastic Hodgkin–Huxley models

The following result states that there exists a stochastic process satisfying system (2.8). Let u_0 be in $\mathcal{D}(\Delta)$ such that $\min_{\xi\in E} v_\xi \leq u_0 \leq \max_{\xi\in E} v_\xi$, the initial potential of the axon. Let $q_0 \in \mathcal{R}$ be the initial configuration of the ion channels.

Proposition 2.1 ([8])**.** *Fix $N \geq 1$. There exists a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ on which a pair $(u_t, r_t)_{0\leq t\leq T}$ of càdlàg adapted stochastic processes satisfies that each sample path of u is in $\mathcal{C}([0, T], L^2(I))$, r_t is in $\mathcal{R}$ for all $t \in [0, T]$ and $(u_t, r_t)_{0\leq t\leq T}$ is solution of (2.8). Moreover $(u_t, r_t)_{0\leq t\leq T}$ is a so called Piecewise Deterministic Markov Process.*

The existence of a stochastic process solution of (2.3) has been first proved in [1]. The proof in [1] is in two parts. First, the Schaeffer fixed point theorem implies that when the jump process r jumps at rate 1, there exists a solution to (2.3). Then the original dynamic of r is recovered using the Girsanov theorem for càdlàg processes with finite state space. Another approach has been developed in [8]. There, the process (u, r) is constructed explicitly as a piecewise deterministic Markov process generalizing in this way the theory developed by Davis [12,13] from the finite to the infinite dimensional setting. In [8], the authors prove that their process is markovian and moreover characterize its generator. Still, another approach based on the marked point process theory is also possible, see for instance [22], Chapter 7 and the extension to our framework in [28].

We proceed now by recalling the form of the generator of the process $(u_t, r_t)_{0\leq t\leq T}$ solution of (2.8). For $(u_0, r) \in L^2(I)\times\mathcal{R}$, we denote by $(\psi_r(t, u_0), t \in [0, T])$ the unique solution starting from u_0 of the PDE

$$\partial_t u_t = \Delta u_t + \frac{1}{N}\sum_{i=1}^{N} c_{r(i)}(v_{r(i)} - \bar{u}_{t,i})\phi_{z_i} \tag{2.16}$$

with zero Dirichlet boundary conditions.

Proposition 2.2. *Let f be a locally bounded measurable function on $L^2(I)\times\mathcal{R}$ such that the map $t \mapsto f(\psi_r(t, u_0), r)$ is continuous for all $(u_0, r) \in L^2(I)\times\mathcal{R}$. Then f is in the domain $\mathcal{D}(\mathcal{A})$ of the extended generator of the process (u, r). The extended generator is given for almost all t by*

$$\mathcal{A}f(u_t, r_t) = \frac{\mathrm{d}f}{\mathrm{d}t}(u_., r_t)(t) + \mathcal{B}(u_t)f(u_t, \cdot)(r_t), \tag{2.17}$$

where

$$\mathcal{B}(u_t)f(u_t,\cdot)(r_t) = \sum_{i=1}^{N}\sum_{\zeta\in E}[f(u_t, r_t(r_t(i)\to\zeta)) - f(u_t,r_t)]\alpha_{r_t(i),\zeta}(\bar{u}_{t,i}).$$

The element $r_t(r_t(i)\to\zeta)$ of $\mathcal{R}$ is equal to $r_t(j)$ if $j\neq i$ and to ζ if $j=i$. The notation $\frac{\mathrm{d}}{\mathrm{d}t}f(u_\cdot,r_t)(t)$ means that the function $s\mapsto f(u'_s,r)$ is differentiated at $s=t$, where u' is the solution of the PDE (2.16) with the channel state r_t held fixed equal to r. When f is continuously Fréchet differentiable with respect to its first argument and such that the Riesz representation $f_u\in H$ of the Fréchet derivative satisfies $f_u(u,r)\in H$ for $u\in H$ and is a locally bounded composition operator in $L^2((0,T),H)$ then

$$\frac{\mathrm{d}f}{\mathrm{d}t}(u_\cdot,r_t)(t) = \langle f_u(u_t,r_t), \varDelta u_t + G_{r_t}(u_t)\rangle\,.$$

See Appendix A for the definition and main properties of Fréchet differentiable functions.

3. Multiscale models, singular perturbation and averaging

In this section, we introduce a slow-fast dynamic in the stochastic Hodgkin–Huxley model described in Section 2.2: some states of the ion channels communicate faster between each other than others. This is biologically relevant as remarked for example in [21], Chapter 18. Mathematically, this leads to the introduction of an additional small parameter $\varepsilon>0$ in our previously described model: the states which communicate at a faster rate communicate at the previous rate $\alpha_{\xi,\zeta}$ divided by ε. For an introduction on slow-fast systems, we refer to [27], for a general theory of slow-fast continuous time Markov chain, see [36] and for the case of slow-fast systems with diffusion, see [5].

In the context of Section 2.1, we make a partition of the state space E according to the different orders in ε of the rate functions

$$E = E_1\sqcup\cdots\sqcup E_l,$$

where $l\in\{1,2,\cdots\}$ is the number of classes. Inside a class E_j, the states communicate faster at jump rates of order $\frac{1}{\varepsilon}$. States in different classes communicate at the usual rate of order 1. For $\varepsilon>0$ fixed, we denote by $(u^\varepsilon,r^\varepsilon)$ the modification of the PDMP introduced in the previous section with now two time scales. Its generator is, for $f\in\mathcal{D}(\mathcal{A}^\varepsilon)$

$$\mathcal{A}^\varepsilon f\left(u_t^\varepsilon,r_t^\varepsilon\right) = \frac{\mathrm{d}f}{\mathrm{d}t}\left(u_\cdot^\varepsilon,r_t^\varepsilon\right)(t) + \mathcal{B}^\varepsilon\left(u_t^\varepsilon\right)f\left(u_t^\varepsilon,\cdot\right)\left(r_t^\varepsilon\right). \tag{3.1}$$

The term $\mathcal{B}^\varepsilon$ is the component of the generator related to the continuous time Markov chain r^ε. According to (2.17) and our slow-fast description, we have the two time scales decomposition of this generator

$$\mathcal{B}^\varepsilon = \frac{1}{\varepsilon}\mathcal{B} + \hat{\mathcal{B}}, \tag{3.2}$$

where the “fast” generator $\mathcal{B}$ is given by

$$\mathcal{B}\left(u_t^\varepsilon\right)f\left(u_t^\varepsilon,r_t^\varepsilon\right) = \sum_{i=1}^{N}\sum_{j=1}^{l}1_{E_j}\left(r_t^\varepsilon(i)\right)\sum_{\zeta\in E_j}\left[f\left(u_t^\varepsilon,r_t^\varepsilon\left(r_t^\varepsilon(i)\to\zeta\right)\right) - f\left(u_t^\varepsilon,r_t^\varepsilon\right)\right]\alpha_{r_t^\varepsilon(i),\zeta}\left(\bar{u}_{t,i}^\varepsilon\right)$$

and the “slow” generator $\hat{\mathcal{B}}$ is given by

$$\hat{\mathcal{B}}\left(u_t^\varepsilon\right)f\left(u_t^\varepsilon,r_t^\varepsilon\right) = \sum_{i=1}^{N}\sum_{j=1}^{l}1_{E_j}\left(r_t^\varepsilon(i)\right)\sum_{\zeta\notin E_j}\left[f\left(u_t^\varepsilon,r_t^\varepsilon\left(r_t^\varepsilon(i)\to\zeta\right)\right) - f\left(u_t^\varepsilon,r_t^\varepsilon\right)\right]\alpha_{r_t^\varepsilon(i),\zeta}\left(\bar{u}_{t,i}^\varepsilon\right).$$

For $y \in \mathbb{R}$ fixed and $g : \mathbb{R} \times E \to \mathbb{R}$, we denote by $\mathcal{B}_j(y)$, $j \in \{1, \cdots, l\}$ the following generator

$$\mathcal{B}_j(y)g(\xi) = 1_{E_j}(\xi) \sum_{\zeta \in E_j} [g(y,\zeta) - g(y,\xi)]\alpha_{\xi,\zeta}(y).$$

For any $y \in \mathbb{R}$ fixed, and any $j \in \{1, \cdots, l\}$, we assume that the fast generator $\mathcal{B}_j(y)$ is weakly irreducible on E_j, *i.e.* has a unique quasi-stationary distribution denoted by $\mu_j(y)$. This quasi-stationary distribution is supposed to be Lipschitz-continuous in y, as well as its derivative.

Following [36], the states in E_j can be considered as equivalent. For any $i = 1, \ldots, N$ we define a new stochastic process $(\bar{r}^\epsilon_t)_{t\geq 0}$ by $\bar{r}^\varepsilon_t(i) = j$ when $r^\varepsilon_t(i) \in E_j$ and abbreviate E_j by j. We then obtain an aggregate process $\bar{r}^\varepsilon(i)$ with values in $\{1, \cdots, l\}$. This process is also often called the coarse-grained process. It is not a Markov process for $\varepsilon > 0$ but a Markovian structure is recovered at the limit when ε goes to 0. More precisely, we have the following proposition.

Proposition 3.1 ([36], Chap. 7)**.** *For any $y \in \mathbb{R}$, $i = 1, \ldots, N$, the process $\bar{r}^\varepsilon(i)$ converges weakly when ε goes to 0 to a Markov process $\bar{r}(i)$ generated by*

$$\bar{\mathcal{B}}(y)g(\bar{r}(i)) = \sum_{j=1}^{l} 1_j(\bar{r}(i)) \sum_{k=1,k\neq j}^{l} (g(k) - g(j)) \sum_{\xi\in E_j} \sum_{\zeta\in E_k} \alpha_{\zeta,\xi}(y)\mu_j(y)(\zeta)$$

with $g : \{1, \cdots, l\} \to \mathbb{R}$ measurable and bounded.

We proved in [17] (in the context of the model with Dirac mass) that the limit of $(u^\varepsilon, \bar{r}^\varepsilon)$ when ε goes to zero requires to average the reaction term $G_r(u)$ against the quasi-invariant distributions. That is we consider that each cluster of states E_j has reached its stationary behavior. This leads to the averaged reaction term of the following form: for any $\bar{r} \in \bar{\mathcal{R}} = \{1, \cdots, l\}^N$

$$F_{\bar{r}}(u) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{l} 1_j(\bar{r}(i)) \sum_{\zeta\in E_j} c_\zeta \mu_j(\bar{u}_i))(\zeta)(v_\zeta - \bar{u}_i)\phi_{z_i}. \tag{3.3}$$

Therefore, we call the following hybrid PDE

$$\partial_t u_t = \Delta u_t + F_{\bar{r}_t}(u_t), \tag{3.4}$$

the *averaged equation*. We take zero Dirichlet boundary conditions and initial conditions u_0 and $\bar{q}_0$ where $\bar{q}_0$ is the aggregation of the initial channel configuration q_0: $\bar{q}_0 = \sum_{j=1}^{l} j 1_{E_j}(q_0)$. In equation (3.4), each coordinate of $(\bar{r}_t)_{t\in[0,T]}$ evolves independently over infinitesimal time intervals and according to the averaged jump rates between the subsets E_j of E. For j and k in $\{1, \cdots, l\}$, the average jump rate from class E_j to class E_k is given by

$$\bar{\alpha}_{jk}(y) = \sum_{\zeta\in E_j} \sum_{\xi\in E_k} \alpha_{\zeta,\xi}(y)\mu_j(y)(\zeta). \tag{3.5}$$

We can now state the averaging result proved for the model with Dirac mass in [17] but easily adaptable to the model with mollifiers.

Theorem 3.2. *When ε goes to 0 the stochastic process $(u^\varepsilon, \bar{r}^\varepsilon)$ solution of (3.1) converges in distribution in the space $\mathcal{C}([0,T], L^2(I)) \times \mathbb{D}([0,T], \mathcal{R})$ to $(u, \bar{r})$, solution of (3.4)–(3.5).*

Let us recall a result of first importance to prove Theorem 3.2 and in the present paper as well. We refer the interested reader to [17] for the proof. This result establishes the uniform boundedness in ε of the process u^ε.

Proposition 3.3. *For any $T > 0$, there is a deterministic positive constant C independent of $\varepsilon \in]0,1]$ such that*

$$\sup_{t\in[0,T]} \|u_t^\varepsilon\|_{L^2(I)} \le C,$$

almost-surely.

For the sake of completeness, we recall a second result which states that the averaged model is well posed and is still a PDMP.

Proposition 3.4. *For any $T > 0$ there exists a probability space such that equations (3.4)–(3.5) define a PDMP $(u_t, \bar{r}_t)_{t\in[0,T]}$ in infinite dimension in the sense of [8]. Moreover, there is a constant C such that*

$$\sup_{t\in[0,T]} \|u_t\|_{L^2(I)} \le C$$

and $u \in \mathcal{C}([0,T], L^2(I))$ almost-surely.

4. Main results

We present in this section the main results of the present paper. The averaging result of Theorem 3.2 above may be seen as a Law of Large Numbers. The natural next step is then to study the fluctuations of the slow-fast system around its averaged limit, in other words, to look for a Central Limit Theorem.

4.1. Fluctuations for the stochastic Hodgkin–Huxley models

For the sake of clarity in our presentation, we first present our result in the so called all-fast case that we proceed to define.

When all states in E communicate at fast rates, there is a single class as described in Section 3, which is equal to the whole set E. For each $\varepsilon > 0$, the generator of the process $(u^\varepsilon, r^\varepsilon)$ is given by

$$\mathcal{A}^\varepsilon f\left(u_t^\varepsilon, r_t^\varepsilon\right) = \frac{\mathrm{d}f}{\mathrm{d}t}\left(u_\cdot^\varepsilon, r_t^\varepsilon\right)(t) + \frac{1}{\varepsilon}\mathcal{B}\left(u_t^\varepsilon\right) f\left(u_t^\varepsilon, \cdot\right)\left(r_t^\varepsilon\right), \tag{4.1}$$

where the slow part of the generator reduces to zero, $\hat{\mathcal{B}} \equiv 0$ in Section 3, and

$$\mathcal{B}\left(u_t^\varepsilon\right) f\left(u_t^\varepsilon, r_t^\varepsilon\right) = \sum_{i=1}^N \sum_{\xi\in E} \left[f\left(u_t^\varepsilon, r_t^\varepsilon\left(r_t^\varepsilon(i) \to \xi\right)\right) - f\left(u_t^\varepsilon, r_t^\varepsilon\right)\right] \alpha_{r_t^\varepsilon(i),\xi}\left(\bar{u}_{t,i}^\varepsilon\right).$$

When $u \in L^2(I)$ is held fixed, the Markov process $r(i)$ has a unique stationary distribution $\mu(\bar{u}_i)$ for any $i = 1, \ldots, N$. Then the process $(r(i), i = 1, \ldots, N)$ has the following stationary distribution

$$\mu(u) = \bigotimes_{i=1}^N \mu(\bar{u}_i).$$

The averaged reaction term reduces to

$$F(u) = \int_{\mathcal{R}} G_r(u)\mu(u)(dr) = \frac{1}{N}\sum_{\xi\in E}\sum_{i=1}^N c_\xi \mu(\bar{u}_i)(\xi)(v_\xi - \bar{u}_i)\phi_{z_i}. \tag{4.2}$$

The averaged limit u is solution of the PDE

$$\partial_t u_t = \Delta u_t + F(u_t)$$

with initial condition u_0 and zero Dirichlet boundary conditions. Note that in this case, the limit PDE is no longer hybrid in contrast with (3.4). For $\varepsilon > 0$, we denote by z^ε the renormalized difference between u^ε and u:

$$z^\varepsilon = \frac{u^\varepsilon - u}{\sqrt{\varepsilon}}\cdot \tag{4.3}$$

Recall that we denote by $\{f_k, k \geq 1\}$ a Hilbert basis of $L^2(I)$. The main result of the present paper is the following.

Theorem 4.1. *When ε goes to 0 the process z^ε converges in distribution in $\mathcal{C}([0,T], L^2(I))$ towards a process z. For $u \in L^2(I)$, let $C(u) : L^2(I) \to L^2(I)$ be a diffusion operator characterized by*

$$(C(u)f_j, f_i) = \int_{\mathcal{R}} (G_r(u) - F(u), f_i)_{L^2(I)} (\Phi(r,u), f_j)_{L^2(I)} \mu(u)(\mathrm{d}r).$$

Φ is the unique solution of the equation

$$\begin{cases} \mathcal{B}(u)\Phi(r,u) = -(G_r(u) - F(u)), \\ \int_{\mathcal{R}} \Phi(r,u)\mu(u)(\mathrm{d}r) = 0. \end{cases} \tag{4.4}$$

Let us also define an operator $\bar{\mathcal{G}}^1$ by, for $t \in [0,T]$ and a measurable, bounded and twice Fréchet differentiable function $\psi : L^2(I) \to \mathbb{R}$,

$$\bar{\mathcal{G}}^1(t)\psi(z) = \frac{\mathrm{d}\psi}{\mathrm{d}z}(z)\left[\Delta z + \frac{\mathrm{d}F}{\mathrm{d}u}(u_t)[z]\right] + \mathrm{Tr}\left[\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}(z)C(u_t)\right].$$

The process z is uniquely determined as the solution of the following martingale problem. For any measurable, bounded and twice Fréchet differentiable function $\psi : L^2(I) \to \mathbb{R}$, the process

$$\bar{N}_\psi(t) := \psi(z_t) - \int_0^t \bar{\mathcal{G}}^1(s)\psi(z_s)\mathrm{d}s \tag{4.5}$$

for $t \in [0,T]$, is a martingale.

The evolution equation associated to the martingale problem (4.5) is the following SPDE (see [11])

$$\mathrm{d}z_t = \left(\Delta z_t + \frac{\mathrm{d}F}{\mathrm{d}u}(u_t)[z_t]\right)\mathrm{d}t + \Gamma(u_t)\mathrm{d}W_t \tag{4.6}$$

with initial condition 0 and zero Dirichlet boundary conditions. The operator $\Gamma(u)$ is the square root of $C(u)$: $C(u) = \Gamma(u)\Gamma(u)^*$, which is well defined by Proposition 5.9. W denotes the standard cylindrical Wiener process on the Hilbert space $L^2(I)$. Formally, the cylindrical Wiener process W is defined as follows: let $((\beta_k(t))_{t\geq 0}, k \geq 1)$ be a family of independent Brownian motions, then

$$W_t = \sum_{k\geq 1} \beta_k(t) f_k,$$

where the definition of the Hilbert basis $\{f_k, k \geq 1\}$ of $L^2(I)$ is recalled in Appendix A. See [11], Sections 2.2.3 and 3.6 for more information about the construction of W. A complete description of the diffusion operator C is provided is Section 5.3. For any $u \in \mathcal{C}([0,T], L^2(I))$ and $t > 0$, the operator

$$Q_t : \psi \mapsto \int_0^t \mathrm{e}^{\Delta(t-s)} C(u_s) \mathrm{e}^{\Delta(t-s)} \psi \mathrm{d}s$$

is of trace class in $L^2(I)$. Thus we can apply classical results from the theory of SPDE in Hilbert spaces to deduce the existence and uniqueness of a mild solution to equation (4.6), see the classical reference [11] on this topic.

Theorem 4.1 extends to the multi-scale case. In this case there are at least two classes E_j as described in Section 3.

Theorem 4.2. *When ε goes to 0, the process z^ε converges in distribution in $\mathcal{C}([0,T],L^2(I))$ towards a process z uniquely defined as the solution of the following martingale problem: for any measurable, bounded and twice Fréchet differentiable function $\psi : L^2(I) \to \mathbb{R}$, the process*

$$\bar{N}_\psi(t) := \psi(z_t) - \int_0^t \bar{\mathcal{G}}^1(u_s,\bar{r}_s)\psi(z_s)\mathrm{d}s \tag{4.7}$$

is a martingale for $t \in [0,T]$. The operator $\bar{\mathcal{G}}^1$ is given by

$$\bar{\mathcal{G}}^1(u,\bar{r})\psi(z) = \frac{\mathrm{d}\psi}{\mathrm{d}z}(z)\left[\Delta z + \frac{\mathrm{d}F_{\bar{r}}}{\mathrm{d}u}(u_t)[z]\right] + \mathrm{Tr}\left[\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}(z)C_{\bar{r}}(u)\right]. \tag{4.8}$$

Note that the evolution of the limit process z is coupled with the evolution of $(\bar{r}_t, t\in[0,T])$ and $(u_t, t\in[0,T])$ in contrary to Theorem 4.1 where no jumps remain.

The diffusion operator $C_{\bar{r}}(u) : L^2(I) \to L^2(I)$ is characterized by the quantities $(C_{\bar{r}}(u)f_j, f_i)_{L^2(I)}$ which are given by

$$\int_{\mathcal{R}} (G_r(u) - F_{\bar{r}}(u), f_i)_{L^2(I)}(\Phi(u,r), f_j)_{L^2(I)} \otimes_{i=1}^N \mu_{\bar{r}(i)}(u)(\mathrm{d}r),$$

Moreover $\Phi : L^2(I)\times\mathcal{R} \to L^2(I)$ is the unique solution of

$$\begin{cases} \mathcal{B}(u)\Phi(u,r) = -(G_r(u) - F_{\bar{r}}(u)), \forall (u,r) \in L^2(I)\times\mathcal{R} \\ \int_{\mathcal{R}} \Phi(u,r)\otimes_{i=1}^N \mu_{j_i}(u)(\mathrm{d}r) = 0, \qquad \forall (j_1,\cdots,j_N)\in\{1,\cdots,l\}^N, \end{cases} \tag{4.9}$$

where $\mathcal{B}$ is the "fast" generator introduced in (3.2).

The evolution equation associated to the martingale problem (4.7) is no longer an SPDE but a hybrid SPDE satisfying

$$\mathrm{d}z_t = \left(\Delta z_t + \frac{\mathrm{d}F_{\bar{r}_t}}{\mathrm{d}u}(u_t)[z_t]\right)\mathrm{d}t + \Gamma_{\bar{r}_t}(u_t)\mathrm{d}W_t \tag{4.10}$$

with initial condition 0 and zero Dirichlet boundary conditions. For $(u,\bar{r})$ held fixed, $\Gamma_{\bar{r}}(u)$ is the square root of $C_{\bar{r}}(u)$: $C_{\bar{r}}(u) = \Gamma_{\bar{r}}(u)\Gamma_{\bar{r}}(u)^*$. Hence, two noise sources are present in the multiscale case: the ionic channel noise represented by the random jumps of the process $\bar{r}$ and the Gaussian noise due to the fluctuations induced by the white noise W. In between each jump of the component $\bar{r}$, the process z follows a classical SPDE parametrized by the current value of the process $\bar{r}$. The hybrid SPDE (4.10) is well defined if for each $\bar{r} = j \in \{1,\cdots,l\}$ held fixed, the SPDE

$$\mathrm{d}z_t = \left(\Delta z_t + \frac{\mathrm{d}F_j}{\mathrm{d}u}(u_t)[z_t]\right)\mathrm{d}t + \Gamma_j(u_t)\mathrm{d}W_t$$

is well defined. For any $(j,u)\in\{1,\cdots,l\}\times\mathcal{C}([0,T],L^2(I))$ and $t>0$, the operator

$$Q_t^j : \psi \mapsto \int_0^t \mathrm{e}^{\Delta(t-s)}C_j(u_s)\mathrm{e}^{\Delta(t-s)}\psi\mathrm{d}s$$

is of trace class in $L^2(I)$. This allows us to apply classical results from the theory of SPDE in Hilbert spaces to deduce existence and uniqueness of a mild solution to equation (4.10). See also [37] for an introduction to switching diffusions.

Theorem 4.1 (*all-fast* case) is proved in full details in Section 5. The proof of Theorem 4.2 (*multi-scale* case) follows the same structure with an additional complication in the notations and the following necessary adaptations. Regarding the proof of tightness, the argument in Section 5.1 below relies on the Poisson equation. We refer the reader to [17] Section 3.2, which explains how the Poisson equation may be extended to the multiscale setting. Regarding the identification of the limit, we adapt the method of [33], Chapter 5, Section 4.3, where the multiscale case is considered in the finite dimensional setting. The key point is to be able to write down the generator of the process $(z^\varepsilon, u^\varepsilon, \bar{r}^\varepsilon)$. For another instructive example dealing with slow-fast continuous Markov chain, see [36], Chapter 7.

4.2. The Langevin approximation

A natural step after having obtained a Central Limit Theorem corresponding to an averaged model is to look for the associated Langevin approximation. Formally, the Langevin approximation corresponds to the averaged model *plus* fluctuations. In our case, this results in the study of the process $\tilde{u}^\varepsilon$ defined in the all-fast case for $\varepsilon > 0$ by

$$\mathrm{d}\tilde{u}^\varepsilon = [\Delta\tilde{u}^\varepsilon + F_{\bar{r}}(\tilde{u}^\varepsilon)]\mathrm{d}t + \sqrt{\varepsilon}\Gamma_{\bar{r}}(\tilde{u}^\varepsilon)\mathrm{d}W_t \tag{4.11}$$

with initial condition u_0 and zero Dirichlet boundary conditions. We would like to compare the averaged equation with the above Langevin approximation.

Proposition 4.3. *The following estimate holds, where the trace is taken in the $L^2(I)$-sense*

$$\mathrm{Tr}\int_0^t \mathrm{e}^{\Delta(t-s)}C_{\bar{r}}(u_s)\mathrm{e}^{\Delta(t-s)}\mathrm{d}s \leq \sum_{k\geq 1}\int_0^t (\alpha\|u_s\|^2_{L^2(I)} + \beta\|u_s\|_{L^2(I)} + \gamma)\mathrm{e}^{-2(k\pi)^2(t-s)}\mathrm{d}s$$

for any $t \in [0,T]$, any functions $u \in \mathcal{C}([0,T], L^2(I))$ and averaged state $\bar{r} \in \{1, \cdots, l\}$ with α, β, γ three constants.

In particular, the operators Q_t^j are of trace class in $L^2(I)$ and the Langevin approximation of u is then well defined.

Proposition 4.4. *Let $\varepsilon > 0$. The hybrid SPDE:*

$$\mathrm{d}\tilde{u}^\varepsilon = [\Delta\tilde{u}^\varepsilon + F_{\bar{r}}(\tilde{u}^\varepsilon)]\mathrm{d}t + \sqrt{\varepsilon}\Gamma_{\bar{r}}(\tilde{u}^\varepsilon)\mathrm{d}W_t \tag{4.12}$$

with initial condition u_0 and zero Dirichlet boundary condition, has a unique solution with sample paths in $\mathcal{C}([0,T], L^2(I))$. Moreover

$$\sup_{t\in[0,T]} \mathbb{E}(\|\tilde{u}_t^\varepsilon\|^2_{L^2(I)}) < +\infty. \tag{4.13}$$

We can now compare the Langevin approximation to the averaged model u.

Theorem 4.5. *Let $T > 0$ held fixed. There exists a deterministic constant c depending only on T but otherwise not on ε such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u_t - \tilde{u}_t^\varepsilon\|^2_{L^2(I)}\right) \leq c\varepsilon. \tag{4.14}$$

Therefore, the Langevin approximation is indeed an approximation of u.

Remark 4.6. The arguments developed for the averaging in [17] as well as those leading to the Central Limit Theorem (Thm. 4.1) and to the Langevin approximation (Thm. 4.5) described in the previous sections are also valid in the general setting present in Section 2.3. As mentioned in the introduction, we provide detailed proofs for the stochastic generalized Hodgkin–Huxley model with mollifiers only (2.2).

5. Proofs

In Theorem 4.1, we want to prove the convergence in distribution of the process z^ε when ε goes to zero. As usual in this context such a proof can be divided in two parts: the proof of tightness of the family $\{z^\varepsilon, \varepsilon \in]0,1]\}$ which implies that there exists a convergent subsequence and the identification of the limit which allows us to characterize the limit of any converging subsequence and prove its uniqueness. We write in full details the proof in the all fast case corresponding to Section 4.1, that is when all the states in E communicate at fast rates of order $\frac{1}{\varepsilon}$. In this case there is a unique class of fast communications which is the whole state space E (that is $l = 1$ w.r.t. the notation of Sect. 3). As already noticed, the multiscale case (when $l > 1$) considered in Theorem 4.2 may be deduced from the all fast case and amounts mainly in additional complication in the notations.

5.1. Tightness

To show that the family $\{z^\varepsilon, \varepsilon \in]0,1]\}$ is tight in $\mathbb{D}([0,T], L^2(I))$, we use Aldous criterion (*cf.* [24]) which can be splitted in two parts as follows.

Criterion 5.1 (general criterion for tightness [24])**.** *Let us assume that the family $\{z^\varepsilon, \varepsilon \in]0,1]\}$ satisfies Aldous's condition: for any $\delta, M > 0$, there exist $\eta, \varepsilon_0 > 0$ such that for all stopping times τ with $\tau + \eta < T$,*

$$\sup_{\varepsilon \in]0,\varepsilon_0]} \sup_{\theta \in]0,\eta[} \mathbb{P}(\|z^\varepsilon_{\tau+\theta} - z^\varepsilon_\tau\|_{L^2(I)} \geq M) \leq \delta \tag{5.1}$$

and moreover, for each $t \in [0,T]$, the family $\{z^\varepsilon_t, \varepsilon \in]0,1]\}$ is tight in $L^2(I)$. Then $\{z^\varepsilon, \varepsilon \in]0,1]\}$ is tight in $\mathbb{D}([0,T], L^2(I))$.

Criterion 5.2 (tightness in a Hilbert space [24])**.** *Recall that $L^2(I)$ is a separable Hilbert space endowed with its basis $\{f_k, k \geq 1\}$ and for $k \geq 1$ define*

$$L_k = \mathrm{span}\{f_i, 1 \leq i \leq k\}.$$

Then, for t held fixed, $(z^\varepsilon_t, \varepsilon \in]0,1])$ is tight in $L^2(I)$ if, and only if, for any $\delta, \eta > 0$ there exist $\rho, \varepsilon_0 > 0$ and $L_{\delta,\eta} \subset \{L_k, k \geq 1\}$ such that

$$\sup_{\varepsilon \in]0,\varepsilon_0]} \mathbb{P}(\|z^\varepsilon_t\|_{L^2(I)} > \rho) \leq \delta, \tag{5.2}$$

$$\sup_{\varepsilon \in]0,\varepsilon_0]} \mathbb{P}(d(z^\varepsilon_t, L_{\delta,\eta}) > \eta) \leq \delta, \tag{5.3}$$

where $d(z^\varepsilon_t, L_{\delta,\eta}) = \inf_{v \in L_{\delta,\eta}} \|z^\varepsilon_t - v\|_{L^2(I)}$ is the distance of z^ε to the subspace $L_{\delta,\eta}$.

We begin by showing that for a fixed $t \in [0,T]$, the family $\{z^\varepsilon_t, \varepsilon \in]0,1]\}$ is uniformly bounded in $L^2(\Omega, L^2(I))$. We recall the definition of the Hilbert basis $\{f_k, k \geq 1\}$ of $L^2(I)$

$$f_k(x) = \sqrt{2}\sin(k\pi x), \quad x \in I.$$

Proposition 5.3. *There exists a constant C depending only on T but otherwise neither on $t \in [0,T]$ nor on $\varepsilon \in]0,1]$ such that*

$$\mathbb{E}\left(\|z^\varepsilon_t\|^2_{L^2(I)}\right) \leq C$$

In particular, for any fixed $t \in [0,T]$, condition (5.2) is satisfied by the family $\{z^\varepsilon_t, \varepsilon \in]0,1]\}$.

Proof. Let $t \in [0,T]$ and $\varepsilon \in]0,1]$ be fixed. Using the evolution equations on u^ε and u and plugging F given by (4.2) in the calculation, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t^\varepsilon - u_t\|^2_{L^2(I)} &= 2\langle \partial_t(u_t^\varepsilon - u_t), u_t^\varepsilon - u_t\rangle \\ &= 2\langle \Delta(u_t^\varepsilon - u_t), u_t^\varepsilon - u_t\rangle + 2\left\langle G_{r_t^\varepsilon}(u_t^\varepsilon) - F(u_t), u_t^\varepsilon - u_t\right\rangle \\ &= -2\|D(u_t^\varepsilon - u_t)\|^2_{L^2(I)} + 2\left(G_{r_t^\varepsilon}(u_t^\varepsilon) - F(u_t^\varepsilon), u_t^\varepsilon - u_t\right)_{L^2(I)} \\ &\quad + 2\left(F(u_t^\varepsilon) - F(u_t), u_t^\varepsilon - u_t\right)_{L^2(I)},\end{aligned}$$

almost surely. We treat each of the above terms separately. Regarding the third term, we notice that the application $u \mapsto (F(u), u)_{L^2(I)}$ is locally Lipschitz on $L^2(I)$ and that the quantities u_t^ε and u_t are uniformly bounded w.r.t. $t \in [0,T]$ and $\varepsilon \in]0,1]$ thanks to Propositions 3.2 and 3.3. Thus there exists a constant C, depending only on T but otherwise not on $t \in [0,T]$ and $\varepsilon \in]0,1]$, such that

$$2\left(F(u_t^\varepsilon) - F(u_t), u_t^\varepsilon - u_t\right)_{L^2(I)} \le C\|u_t^\varepsilon - u_t\|^2_{L^2(I)}.$$

Integrating over $[0,t]$ and taking expectation yields the following inequality

$$\begin{aligned}\mathbb{E}\left(\|u_t^\varepsilon - u_t\|^2_{L^2(I)}\right) &\le \mathbb{E}\left(\|u_0^\varepsilon - u_0\|^2_{L^2(I)}\right) + 2C\int_0^t \mathbb{E}\left(\|u_s^\varepsilon - u_s\|^2_{L^2(I)}\right)\mathrm{d}s \\ &\quad + \mathbb{E}\left(\int_0^t 2\left(G_{r_s^\varepsilon}(u_s^\varepsilon) - F(u_s^\varepsilon), u_s^\varepsilon - u_s\right)_{L^2(I)}\mathrm{d}s\right).\end{aligned}$$

Let us consider the latter of these terms. Using the same approach as the one developed for the identification of the limit in the proof of the averaging result in [17], we deduce the existence of a constant $C(T)$ depending only on T such that

$$\left|\mathbb{E}\left(\int_0^t 2\left(G_{r_s^\varepsilon}(u_s^\varepsilon) - F(u_s^\varepsilon), u_s^\varepsilon - u_s\right)_{L^2(I)}\mathrm{d}s\right)\right| \le C(T)\varepsilon.$$

For the sake of completeness, we review now briefly this approach and refer to [17] for more details. The key point is to show that there exists a measurable and bounded function $f : L^2(I)\times\mathcal{R}\times[0,T] \to \mathbb{R}$ such that $\int_{\mathcal{R}} f(u,r,t)\mu(u)(dr) = 0$ and for all $(u,r,t) \in L^2(I)\times\mathcal{R}\times[0,T]$

$$\mathcal{B}(u)f(u,\cdot,t)(r) = (G_r(u) - F(u), u - u_t)_{L^2(I)}. \tag{5.4}$$

Equation (5.4) is called the Poisson equation related to $\mathcal{B}$. Then using the regularity of the mappings $(u,r,t) \in L^2(I)\times\mathcal{R}\times[0,T] \mapsto (G_r(u) - F(u), u - u_t)_{L^2(I)}$ and the operator $\mathcal{B}(u)$ for $u \in L^2(I)$, we deduce that the application $(u,r,t) \in L^2(I)\times\mathcal{R}\times[0,T] \mapsto f(u,r,t)$ is bounded, Fréchet differentiable in u with bounded Fréchet derivative and differentiable in t with bounded derivative. Using the general theory of Markov processes, we deduce that there exists a martingale M^ε such that

$$\begin{aligned}f(u_t^\varepsilon, r_t^\varepsilon, t) &= f(u_0^\varepsilon, r_0^\varepsilon, 0) + \int_0^t \mathcal{A}^\varepsilon f(u_s^\varepsilon, r_s^\varepsilon, s)\,\mathrm{d}s + M_t^\varepsilon \\ &= f(u_0^\varepsilon, r_0^\varepsilon, 0) + \frac{1}{\varepsilon}\int_0^t \mathcal{B}(u_s^\varepsilon) f(u_s^\varepsilon, r_s^\varepsilon, s) + \frac{\mathrm{d}f}{\mathrm{d}s}(u_\cdot^\varepsilon, r_s^\varepsilon, s)(s) + \frac{\mathrm{d}f}{\mathrm{d}s}(u_s^\varepsilon, r_s^\varepsilon, \cdot)(s)\mathrm{d}s + M_t^\varepsilon \\ &= f(u_0^\varepsilon, r_0^\varepsilon, 0) + \frac{1}{\varepsilon}\int_0^t \left(G_{r_s^\varepsilon}(u_s^\varepsilon) - F(u_s^\varepsilon), u_s^\varepsilon - u_s\right)_{L^2(I)}\mathrm{d}s \\ &\quad + \int_0^t \frac{\mathrm{d}f}{\mathrm{d}s}(u_\cdot^\varepsilon, r_s^\varepsilon, s)(s) + \frac{\mathrm{d}f}{\mathrm{d}s}(u_s^\varepsilon, r_s^\varepsilon, \cdot)(s)\mathrm{d}s + M_t^\varepsilon.\end{aligned}$$

Therefore

$$\int_0^t \left(G_{r_s^\varepsilon}\left(u_s^\varepsilon\right) - F\left(u_s^\varepsilon\right), u_s^\varepsilon - u_s\right)_{L^2(I)} \mathrm{d}s = \varepsilon f\left(u_t^\varepsilon, r_t^\varepsilon, t\right) - \varepsilon f\left(u_0^\varepsilon, r_0^\varepsilon, 0\right)$$
$$- \varepsilon \int_0^t \frac{\mathrm{d}f}{\mathrm{d}s}\left(u_\cdot^\varepsilon, r_s^\varepsilon, s\right)(s) - \frac{\mathrm{d}f}{\mathrm{d}s}\left(u_s^\varepsilon, r_s^\varepsilon, \cdot\right)(s)\mathrm{d}s - \varepsilon M_t^\varepsilon.$$

Taking the expectation, using the fact that M^ε is a martingale and that f is regular, we obtain the desired estimate.

Assembling all the above estimates we obtain

$$\mathbb{E}\left(\|u_t^\varepsilon - u_t\|_{L^2(I)}^2\right) \leq \mathbb{E}\left(\|u_0^\varepsilon - u_0\|_{L^2(I)}^2\right) + C(T)\varepsilon + 2C\int_0^t \mathbb{E}\left(\|u_s^\varepsilon - u_s\|_{L^2(I)}^2\right)\mathrm{d}s.$$

Since $u_0^\varepsilon = u_0$ a standard application of Gronwall's lemma leads to the desired result. We end this proof by showing that for any fixed $t \in [0,T]$, the family $\{z_t^\varepsilon, \varepsilon \in]0,1]\}$ fulfills the requirement (5.2). Indeed, let $\delta > 0$ and denote by C the constant independent of ε and $t \in [0,T]$ such that

$$\mathbb{E}\left(\|z_t^\varepsilon\|_{L^2(I)}^2\right) \leq C.$$

By the Markov inequality we have, for $\rho > 0$,

$$\sup_{\varepsilon \in]0,1]} \mathbb{P}(\|z_t^\varepsilon\|_{L^2(I)} > \rho) \leq \sup_{\varepsilon \in]0,1]} \frac{\mathbb{E}(\|z_t^\varepsilon\|_{L^2(I)}^2)}{\rho^2} \leq \frac{C}{\rho^2}$$

and for ρ large enough, we obtain that $\sup_{\varepsilon \in]0,1]} \mathbb{P}(\|z_t^\varepsilon\|_{L^2(I)} > \rho) < \delta$. □

We now prove the tightness of the family $\{z_t^\varepsilon, \varepsilon \in]0,1]\}$ in $L^2(I)$ for any fixed $t \in [0,T]$. This is the object of the following propositions.

Proposition 5.4. *Let $t \in]0,T]$ and for $p \geq 1$ let us define the following truncation*

$$z_t^{\varepsilon,p} = \sum_{k=1}^p \left(z_t^\varepsilon, f_k\right) f_k.$$

Then

$$\lim_{p \to \infty} \mathbb{E}\left(\|z_t^\varepsilon - z_t^{\varepsilon,p}\|_{L^2(I)}^2\right) = 0,$$

uniformly in $\varepsilon \in]0,1]$.

Proof. For a fixed $k \geq 1$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(z_t^\varepsilon, e_k\right)^2 &= 2\left(z_t^\varepsilon, f_k\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(z_t^\varepsilon, f_k\right) \\
&= 2\left(z_t^\varepsilon, f_k\right)\left(-(k\pi)^2\left(z_t^\varepsilon, f_k\right) + \frac{1}{\sqrt{\varepsilon}}\left\langle G_{r_t^\varepsilon}\left(u_t^\varepsilon\right) - F(u_t), f_k\right\rangle\right) \\
&= -2(k\pi)^2\left(z_t^\varepsilon, f_k\right)^2 + \frac{2}{\sqrt{\varepsilon}}\left(z_t^\varepsilon, f_k\right)\left(F\left(u_t^\varepsilon\right) - F(u_t), f_k\right)_{L^2(I)} \\
&\quad + \frac{2}{\sqrt{\varepsilon}}\left(z_t^\varepsilon, f_k\right)\left(G_{r_t^\varepsilon}\left(u_t^\varepsilon\right) - F\left(u_t^\varepsilon\right), f_k\right)_{L^2(I)},
\end{aligned}$$

almost surely. A direct computation using the arguments developed in the proof of Proposition 5.3 leads to the existence of a constant $C(T)$ independent of $\varepsilon \in]0,1]$ such that

$$(z_t^\varepsilon, f_k)^2 \le C(T) - 2(k\pi)^2 \int_0^t (z_s^\varepsilon, f_k)^2 \, \mathrm{d}s,$$

almost surely. Using Gronwall's lemma we deduce that

$$(z_t^\varepsilon, f_k)^2 \le C(T)\mathrm{e}^{-2(k\pi)^2 t}.$$

The result follows since the series $\sum \mathrm{e}^{-2(k\pi)^2 t}$ is convergent for $t > 0$. □

We now check that the family $\{z^\varepsilon, \varepsilon \in]0,1]\}$ satisfies the first part of Criterion 5.1.

Proposition 5.5. *Let $\tau > 0$ be a stopping time and $\theta > 0$ such that $\tau + \theta \le T$. There exists a constant C depending only on T such that*

$$\mathbb{E}\left(\left\|z_{\tau+\theta}^\varepsilon - z_\tau^\varepsilon\right\|_{L^2(I)}^2\right) \le C\theta.$$

Proof. We notice that for $k \ge 1$, $t > 0$ and $\theta > 0$ such that $t + \theta \le T$ we have

$$\partial_\theta \left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)_{L^2(I)} = -(k\pi)^2 \left(z_{t+\theta}^\varepsilon, f_k\right)_{L^2(I)} + \frac{1}{\sqrt{\varepsilon}} \left(G_{r_{t+\theta}^\varepsilon}\left(u_{t+\theta}^\varepsilon\right) - F(u_{t+\theta}), f_k\right)_{L^2(I)}.$$

Thus, almost surely

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)_{L^2(I)}^2 &= -2(k\pi)^2 \left(z_{t+\theta}^\varepsilon, f_k\right)_{L^2(I)} \left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)_{L^2(I)} \\ &\quad + \frac{2}{\sqrt{\varepsilon}} \left\langle G_{r_{t+\theta}^\varepsilon}\left(u_{t+\theta}^\varepsilon\right) - F(u_{t+\theta}), f_k \right\rangle \left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)_{L^2(I)}.\end{aligned}$$

The first term satisfies

$$\begin{aligned}-2(k\pi)^2 \left(z_{t+\theta}^\varepsilon, f_k\right)_{L^2(I)} \left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)_{L^2(I)} &= -2(k\pi)^2 \left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)^2 + 2(k\pi)^2 \left(z_t^\varepsilon, f_k\right)_{L^2(I)}^2 \\ &\quad - 2(k\pi)^2 \left(z_{t+\theta}^\varepsilon, f_k\right)_{L^2(I)} \left(z_t^\varepsilon, f_k\right)_{L^2(I)} \\ &\le -2(k\pi)^2 \left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)^2 \\ &\quad + 3(k\pi)^2 \left\|z_t^\varepsilon\right\|_{L^2(I)}^2 + (k\pi)^2 \left\|z_{t+\theta}^\varepsilon\right\|_{L^2(I)}^2\end{aligned}$$

where $\|z_t^\varepsilon\|_{L^2(I)}^2$ and $\|z_{t+\theta}^\varepsilon\|_{L^2(I)}^2$ are bounded in expectation by a constant independent of t, θ and ε by Proposition 5.3. For the second term, the arguments developed in the proof of Proposition 5.3 lead to the existence of a constant C depending only on T such that

$$\mathbb{E}\left(\int_0^\theta \left(G_{r_{t+s}^\varepsilon}\left(u_{t+s}^\varepsilon\right) - F\left(u_{t+s}^\varepsilon\right), f_k\right)_{L^2(I)} \mathrm{d}s\right) \le C\theta\varepsilon.$$

Therefore, still denoting by C a constant depending only of T

$$\mathbb{E}\left(\left(z_{t+\theta}^\varepsilon - z_t^\varepsilon, f_k\right)_{L^2(I)}^2\right) \le -2(k\pi)^2 \int_0^\theta \mathbb{E}\left(\left(z_{t+s}^\varepsilon - z_t^\varepsilon, f_k\right)^2\right) \mathrm{d}s + C\left(1 + (k\pi)^2\right)\theta.$$

By application of the Gronwall's lemma and summation over k we obtain

$$\mathbb{E}\left(\left(\left\|z_{t+\theta}^\varepsilon - z_t^\varepsilon\right\|_{L^2(I)}^2\right) \le C\theta \sum_{k\ge1} \left(1 + (k\pi)^2\right) \mathrm{e}^{-2(k\pi)^2 t}$$

which yields the result for any $t > 0$ since the series $\sum_{k\ge1}(1 + (k\pi)^2)\mathrm{e}^{-2(k\pi)^2 t}$ is convergent for $t > 0$. The same arguments apply when replacing t by the stopping time τ. □

According to Criteria 5.1 and 5.2, Propositions 5.3, 5.4 and 5.5, the family $\{z^\varepsilon, \varepsilon \in]0,1]\}$ is tight in $\mathbb{D}([0,T], L^2(I))$. The continuity of each element of the family implies that $\{z^\varepsilon, \varepsilon \in]0,1]\}$ is tight in $\mathcal{C}([0,T], L^2(I))$.

5.2. Identification of the limit

In this section we want to prove that $(z^\varepsilon, \varepsilon \in]0,1])$ has a unique accumulation point that we identify as the unique solution of a martingale problem. For this purpose, we study the process $(z^\varepsilon, r^\varepsilon)$ for $\varepsilon \in]0,1]$.

Let us outline the strategy of the proof.

Step 1. Use the general theory on PDMP developed in [8] to write down the generator $\mathcal{G}^\varepsilon$ of the process $(z^\varepsilon, r^\varepsilon)$. The associated martingale problem gives rise to martingales M^ε_ϕ for appropriate functions ϕ.

Step 2. For a nice choice of ϕ, identify the terms of order one in ε of the martingale M^ε_ϕ. Since the difference between u and u^ε is renormalized by $\sqrt{\varepsilon}$, choose ϕ of the form $\psi + \sqrt{\varepsilon}\gamma$ (perturbed test function).

Step 3. Identify the generator $\bar{\mathcal{G}}$ of the limit process z. Prove that z is solution of the martingale problem associated to $\bar{\mathcal{G}}$.

Step 1. Notice first that the process z^ε satisfies the following equation

$$\begin{aligned}\partial_t z^\varepsilon_t &= \Delta z^\varepsilon_t + \frac{1}{\sqrt{\varepsilon}}(G_{r^\varepsilon_t}(u^\varepsilon_t) - F(u_t)) \\ &= \Delta z^\varepsilon_t + \frac{1}{\sqrt{\varepsilon}}(G_{r^\varepsilon_t}(u_t + \sqrt{\varepsilon} z^\varepsilon_t) - F(u_t)),\end{aligned} \tag{5.5}$$

by definition of z^ε. The initial condition for z^ε is 0 and the boundary conditions are still zero Dirichlet boundary conditions.

Let $\phi : L^2(I) \times \mathcal{R} \times \mathbb{R}_+$ be a real valued, measurable and bounded function of class $\mathcal{C}^2$ on $L^2(I)$ and $\mathcal{C}^1$ on $\mathbb{R}_+$. We write down the generator of the process $(z^\varepsilon, r^\varepsilon)$ against ϕ. Recall that in the all-fast case, the limit u of u^ε is deterministic so that $(z^\varepsilon, r^\varepsilon)$ is a classical PDMP with evolution equation given by (5.5) and dynamic of jumps given by (2.7). According to Theorem 4 of [8], for $(z, r, t) \in L^2(I) \times \mathcal{R} \times \mathbb{R}_+$, the generator $\mathcal{G}$ of $(z^\varepsilon, r^\varepsilon)$ is given by

$$\begin{aligned}\mathcal{G}(t)\phi(z,r,t) =& \frac{\mathrm{d}\phi}{\mathrm{d}z}(z,r,t)\left[\Delta z + \frac{1}{\sqrt{\varepsilon}}\left(G_r\left(u_t + \sqrt{\varepsilon}z\right) - F(u_t)\right)\right] \\ &+ \frac{1}{\varepsilon}\mathcal{B}(u_t + \sqrt{\varepsilon}z)\phi(z,r,t) + \partial_t\phi(z,r,t).\end{aligned} \tag{5.6}$$

Following the usual theory of Markov processes (see [15], Chap. 4), the process $(M^\varepsilon_\phi(t), t \in [0,T])$ defined for $t \geq 0$ by

$$M^\varepsilon_\phi(t) = \phi\left(z^\varepsilon_t, r^\varepsilon_t, t\right) - \int_0^t \mathcal{G}(s)\phi\left(z^\varepsilon_s, r^\varepsilon_s, s\right) \mathrm{d}s,$$

is a martingale for the natural filtration associated to the process $(z^\varepsilon, r^\varepsilon)$.

Step 2. We want to identify the terms of different orders in ε of M^ε_ϕ. For this purpose, we choose a function ϕ with the following decomposition

$$\phi(z,r,t) = \psi(z,r) + \sqrt{\varepsilon}\gamma(z,r,t),$$

where the functions ψ and γ have the same regularity as ϕ. We write the Taylor expansion in ε of the two following terms

$$\begin{aligned}G_r\left(u_t + \sqrt{\varepsilon}z\right) &= G_r(u_t) + \sqrt{\varepsilon}\frac{\mathrm{d}G_r}{\mathrm{d}u}(u_t)[z] + \sqrt{\varepsilon}\|z\|_{L^2(I)}\delta_1\left(\sqrt{\varepsilon}z\right) \\ \mathcal{B}\left(u_t + \sqrt{\varepsilon}z\right) &= \mathcal{B}(u_t) + \sqrt{\varepsilon}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}u}(u_t)[z] + \sqrt{\varepsilon}\|z\|_{L^2(I)}\delta_2\left(\sqrt{\varepsilon}z\right),\end{aligned}$$

where δ_1 and δ_2 are two $L^2(I)$-valued continuous functions such that $\delta_1(0_{L^2(I)}) = \delta_2(0_{L^2(I)}) = 0_{L^2(I)}$. Plugging this expansion in the expression of the generator (5.6) we want the terms of order $\frac{1}{\varepsilon}$ to vanish. For $(z, r, t) \in L^2(I) \times \mathcal{R} \times \mathbb{R}_+$ this leads to

$$\mathcal{B}(u_t)\psi(z, r) = 0. \tag{5.7}$$

That is to say, the application ψ does not depend on $r \in \mathcal{R}$ and is of the form

$$\psi(z, r) = \psi(z),$$

where $\psi : L^2(I) \to \mathbb{R}$ is of class $\mathcal{C}^2$. The generator is then of the following form, where we gather the terms of the same order in ε

$$\begin{aligned}\mathcal{G}(t)\phi(z, r, t) = &\frac{1}{\sqrt{\varepsilon}}\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[G_r(u_t) - F(u_t)] + \mathcal{B}(u_t)\gamma(z, r, t) + \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}u}(u_t)[z]\psi(z)\right)\\ &+ \frac{\mathrm{d}\psi}{\mathrm{d}z}(z)\left[\Delta z + \frac{\mathrm{d}G_r}{\mathrm{d}u}(u_t)[z]\right] + \frac{\mathrm{d}\gamma}{\mathrm{d}z}(z, r, t)[G_r(u_t) - F(u_t)] + \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}u}(u_t)[z]\gamma(z, r, t)\\ &+ \sqrt{\varepsilon}\left(\partial_t\gamma(z, r, t) + \frac{\mathrm{d}\gamma}{\mathrm{d}z}(z, r, t)\left[\Delta z + \frac{\mathrm{d}G_r}{\mathrm{d}u}(u_t)[z]\right]\right) + \mathrm{o}\left(\sqrt{\varepsilon}\right).\end{aligned}$$

We now want the terms of order $\frac{1}{\sqrt{\varepsilon}}$ to vanish, that is to say, or $(z, r, t) \in L^2(I) \times \mathcal{R} \times \mathbb{R}_+$

$$\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[G_r(u_t) - F(u_t)] + \mathcal{B}(u_t)\gamma(z, r, t) + \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}u}(u_t)[z]\psi(z) = 0.$$

Notice that $\mathcal{B}(u_t)1 = 0$ implies that for all $(z, t) \in L^2(I) \times \mathbb{R}_+$

$$\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}u}(u_t)[z]\psi(z) = 0$$

and we are left with the equation

$$\mathcal{B}(u_t)\gamma(z, r, t) = -\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[G_r(u_t) - F(u_t)]. \tag{5.8}$$

We look for γ of the form:

$$\gamma(z, r, t) = \frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[\Phi(r, u_t)],$$

where $\Phi : \mathcal{R} \times L^2(I) \to L^2(I)$ has to be identified. Inserting the above expression of γ in (5.8) we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[\mathcal{B}(u_t)\Phi(r, u_t)] = -\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[G_r(u_t) - F(u_t)].$$

Therefore, it is enough that for any $(u, r) \in L^2(I) \times \mathcal{R}$

$$\mathcal{B}(u)\Phi(r, u) = -(G_r(u) - F(u)). \tag{5.9}$$

To ensure uniqueness of the solution for equation (5.9) we impose moreover the condition

$$\int_{\mathcal{R}} \Phi(r, u)\mu(u)(\mathrm{d}r) = 0.$$

Then, from the definition of F we have $\int_{\mathcal{R}} (G_r(u) - F(u))\, \mu(u)(\mathrm{d}r) = 0$. Moreover, equation (5.9) has a unique solution Φ thanks to the Fredholm alternative.

Step 3. We have identified the terms of order 1 in ε of the generator of the process $(z^\varepsilon, r^\varepsilon)$. It remains to show that the terms of order 1 in ε correspond, after averaging, to the generator of the process z. For $(z, r, t) \in L^2(I) \times \mathcal{R} \times \mathbb{R}_+$ we define

$$\begin{aligned}\mathcal{G}^1(t,r)\psi(z) = &\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)\left[\Delta z + \frac{\mathrm{d}G_r}{\mathrm{d}u}(u_t)[z]\right] + \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}(z)[\Phi(r,u_t), G_r(u_t) - F(u_t)]\\ &+ \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}u}(u_t)[z]\frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[\Phi(r,u_t)]. \end{aligned} \tag{5.10}$$

Let us define also the following process

$$N^\varepsilon_\psi(t) = \psi(z^\varepsilon_t) - \int_0^t \mathcal{G}^1(s, r^\varepsilon_s)\psi(z^\varepsilon_s)\mathrm{d}s.$$

By construction we see that $\mathbb{E}(|M^\varepsilon_\phi(t) - N^\varepsilon_\psi(t)|^2) = \mathrm{O}(\varepsilon)$. When ε goes to 0, by the averaging result of Theorem 3.2, we see that the term $\int_0^t \mathcal{G}^1(s, r^\varepsilon_s)\psi(z^\varepsilon_s)ds$ should converge to

$$\int_0^t \int_{\mathcal{R}} \mathcal{G}^1(s,r)\psi(z_s)\mu(u_s)(\mathrm{d}r)\mathrm{d}s.$$

Therefore, we want to prove that, whenever z is an accumulation point of the family $(z^\varepsilon, \varepsilon \in]0,1])$, the process

$$\bar{N}_\psi(t) = \psi(z_t) - \int_0^t \bar{\mathcal{G}}^1(s)\psi(z_s)\mathrm{d}s,$$

is a martingale w.r.t. the natural filtration associated to the process $(z_t, t \geq 0)$ where

$$\bar{\mathcal{G}}^1(t)\psi(z) = \frac{\mathrm{d}\psi}{\mathrm{d}z}(z)[\Delta z + \frac{\mathrm{d}F}{\mathrm{d}u}(u_t)[z]] + \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}(z)\int_{\mathcal{R}}[\Phi(r,u_t), G_r(u_t) - F(u_t)]\mu(u_t)(\mathrm{d}r). \tag{5.11}$$

This is not straightforward since we have no information on the asymptotic behavior of the process $(z^\varepsilon, r^\varepsilon)$ when ε goes to 0.

Proposition 5.6. *The process $(\bar{N}_\psi(t), t \geq 0)$ is a martingale w.r.t. the natural filtration associated to the process $(z_t, t \geq 0)$.*

Proof. Let $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq s \leq t$ be $k+2$ reals, with $k \geq 1$ an integer. For $i \in \{1, \cdots, k\}$, we take a measurable and bounded function g_i. In order to show that the process $(\bar{N}_\psi(t), t \geq 0)$ is a martingale for the natural filtration associated to the process $(z_t, t \geq 0)$ we will prove that

$$\mathbb{E}((\bar{N}_\psi(t) - \bar{N}_\psi(s))g_1(z_{t_1})\ldots g_k(z_{t_k})) = 0.$$

In order to not overload the proof with too many computations, we write Z_k for the random variable $g_1(z_{t_1})\ldots g_k(z_{t_k})$ and Z^ε_k for $g_1(z^\varepsilon_{t_1})\ldots g_k(z^\varepsilon_{t_k})$. Using elementary substitution and the fact that z^ε converges in law toward z when ε goes to 0 we have

$$\begin{aligned}\mathbb{E}((\bar{N}_\psi(t) - \bar{N}_\psi(s))Z_k) &= \mathbb{E}\left(\left(\psi(z_t) - \psi(z_s) - \int_s^t \bar{\mathcal{G}}^1(l)\psi(z_l)\mathrm{d}l\right)Z_k\right)\\ &= \mathbb{E}((\psi(z_t) - \psi(z_s))Z_k) - \mathbb{E}\left(\left(\int_s^t \bar{\mathcal{G}}^1(l)\psi(z_l)\mathrm{d}l\right)Z_k\right)\\ &= \lim_{\varepsilon\to0}\mathbb{E}\left((\psi(z^\varepsilon_t) - \psi(z^\varepsilon_s))Z^\varepsilon_k\right) - \mathbb{E}\left(\left(\int_s^t \bar{\mathcal{G}}^1(l)\psi(z_l)\mathrm{d}l\right)Z_k\right)\\ &= \lim_{\varepsilon\to0}\mathbb{E}\left(\left(N^\varepsilon_\psi(t) - N^\varepsilon_\psi(s)\right)Z^\varepsilon_k\right) + \lim_{\varepsilon\to0}\mathbb{E}\left(\left(\int_s^t \mathcal{G}^1\left(l, r^\varepsilon_l\right)\psi\left(z^\varepsilon_l\right)\mathrm{d}l\right)Z^\varepsilon_k\right)\\ &- \mathbb{E}\left(\left(\int_s^t \bar{\mathcal{G}}^1(l)\psi(z_l)\mathrm{d}l\right)Z_k\right).\end{aligned}$$

On one hand, from the definition of N^ε_ψ and the previous study of the different orders in ε of the martingale M^ε_ϕ we see that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\left(N^\varepsilon_\psi(t)-N^\varepsilon_\psi(s)\right)Z^\varepsilon_k\right)=\lim_{\varepsilon\to 0}\mathbb{E}\left(\left(N^\varepsilon_\psi(t)-N^\varepsilon_\psi(s)\right)Z^\varepsilon_k\right)-\mathbb{E}\left(\left(M^\varepsilon_\phi(t)-M^\varepsilon_\phi(s)\right)Z^\varepsilon_k\right).$$

From the previous study of the different orders in ε, the right hand side is $\mathrm{O}(\sqrt{\varepsilon})$ and therefore converges to 0 when ε goes to 0. On the other hand, for $\varepsilon^1>0$ which will be chosen later

$$\lim_{\varepsilon\to 0}\mathbb{E}\left(\left(\int_s^t\mathcal{G}^1(l,r^\varepsilon_l)\psi(z^\varepsilon_l)dl\right)Z^\varepsilon_k\right)-\mathbb{E}\left(\left(\int_s^t\bar{\mathcal{G}}^1(l)\psi(z_l)\mathrm{d}l\right)Z_k\right) \tag{5.12}$$

$$=\lim_{\varepsilon\to 0}\mathbb{E}\left(\left(\int_s^t\mathcal{G}^1(l,r^\varepsilon_l)\psi(z^\varepsilon_l)dl\right)Z^\varepsilon_k\right)-\mathbb{E}\left(\left(\int_s^t\mathcal{G}^1(l,r^{\varepsilon^1}_l)\psi(z^\varepsilon_l)\mathrm{d}l\right)Z^\varepsilon_k\right) \tag{5.13}$$

$$+\lim_{\varepsilon\to 0}\mathbb{E}\left(\left(\int_s^t\mathcal{G}^1(l,r^{\varepsilon^1}_l)\psi(z^\varepsilon_l)dl\right)Z^\varepsilon_k\right)-\mathbb{E}\left(\left(\int_s^t\mathcal{G}^1(l,r^{\varepsilon^1}_l)\psi(z_l)\mathrm{d}l\right)Z_k\right) \tag{5.14}$$

$$+\mathbb{E}\left(\left(\int_s^t\mathcal{G}^1(l,r^{\varepsilon^1}_l)\psi(z_l)dl\right)Z_k\right)-\mathbb{E}\left(\left(\int_s^t\bar{\mathcal{G}}^1(l)\psi(z_l)\mathrm{d}l\right)Z_k\right). \tag{5.15}$$

We know that the quantity corresponding to (5.15) can be made arbitrarily small by conditioning appropriately (as in the proof of Prop. 5.3 for example) for small enough ε^1. Then, since z^ε converges in law towards z when ε goes to 0, the quantity (5.14) converges to 0 when ε goes to 0. This shows finally that (5.12) converges to 0 when ε goes to 0 and therefore

$$\mathbb{E}((\bar{N}_\psi(t)-\bar{N}_\psi(s))g_1(z_{t_1})\cdots g_k(z_{t_k}))=0,$$

as announced. □

We can now conclude that the limit process z is solution of the following martingale problem: for any measurable, bounded and twice Fréchet differentiable function ψ, the process defined by

$$\bar{N}_\psi(t)=\psi(z_t)-\int_0^t\bar{\mathcal{G}}^1(s)\psi(z_s)\mathrm{d}s,$$

for $t\in[0,T]$ is a martingale, where $\bar{\mathcal{G}}^1$ is given by (5.11).

In other word, the limit process z is solution to the martingale problem associated with the operator $\bar{\mathcal{G}}^1$. Then z is a solution of the SPDE (4.6) where the diffusion operator $C(u)$ for $u\in L^2(I)$ is identified thanks to the relation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}(z)\int_{\mathcal{R}}[\Phi(r,u),G_r(u)-F(u)]\mu(u)(\mathrm{d}r)=\mathrm{Tr}\,\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}(z)C(u) \tag{5.16}$$

for $(u,z)\in L^2(I)\times L^2(I)$. The uniqueness of z follows from the properties of the Laplacian operator, the reaction term $\frac{\mathrm{d}F}{\mathrm{d}u}$ and the operator $C(u)$. For more insight in the properties of the diffusion operator, see the following section.

5.3. The diffusion operator C

In this section, we give more details about the diffusion operator C. In particular, we make explicit the dependence of Φ in (5.16) w.r.t. the data of our problem.

Proposition 5.7 (first representation of the diffusion operator)**.** *For $u\in L^2(I)$ and $r\in\mathcal{R}$ we have:*

$$\Phi(r,u)=-\left(\mu^*(u)\mu(u)+\mathcal{B}^*(u)\mathcal{B}(u)\right)^{-1}\mathcal{B}^*(u)(G_\cdot(u)-F(u))(r).$$

That is, the function $\Phi(\cdot,u)$ is explicitly given as a function of the "fast jumping part" operator $\mathcal{B}(u)$ and the associated invariant measure $\mu(u)$.

Proof. The application Φ is defined by the two conditions

$$\begin{cases} \mathcal{B}(u)\Phi(r,u) & = -(G_r(u) - F(u)) \\ \int_{\mathcal{R}} \Phi(r,u)\mu(u)(\mathrm{d}r) = 0 \end{cases} \tag{5.17}$$

for $(u,r) \in L^2(I) \times \mathcal{R}$. Let $u \in L^2(I)$ be held fixed. Defining $D(u) = (\mu(u), \mathcal{B}(u))^T$ reduces (5.17) to

$$D(u)\Phi(\cdot,u) = -\begin{pmatrix} 0 \\ G_{\cdot}(u) - F(u) \end{pmatrix}.$$

Then

$$D^*(u)D(u)\Phi(\cdot,u) = -D^*(u)\begin{pmatrix} 0 \\ G_{\cdot}(u) - F(u) \end{pmatrix}.$$

It remains to prove that the operator $D^*(u)D(u)$ is invertible which is the key point to conclude. Indeed

$$D^*(u)D(u) = \mu(u)^*\mu(u) + \mathcal{B}^*(u)\mathcal{B}(u)$$

and the kernel of the two operators $\mu(u)^*\mu(u)$ and $\mathcal{B}^*(u)\mathcal{B}(u)$ are in direct sum and span the whole space $\mathbb{R}^{|\mathcal{R}|}$. Let $x \in \mathrm{Ker} D^*(u)D(u)$, then x can be written uniquely as $z + y$ with $z \in \mathrm{Ker}\mu(u)^*\mu(u)$ and y $\in \mathrm{Ker}\mathcal{B}^*(u)\mathcal{B}(u)$. We have

$$\mu(u)^*\mu(u)y + \mathcal{B}^*(u)\mathcal{B}(u)z = 0.$$

Since $\mathcal{B}(u)\mathbf{1} = 0$ (where $\mathbf{1} \in \mathbb{R}^{|\mathcal{R}|}$) and $\mu(u)\mathbf{1} = 1$, multiplying the above equation to the left by $\mathbf{1}^T$ we have

$$\mu(u)y = 0.$$

Since $y \in \mathrm{Ker}\mathcal{B}^*(u)\mathcal{B}(u) = \mathrm{Ker}\mathcal{B}(u) = \mathrm{span}\mathbf{1}$ here, we have $y = y\mathbf{1}$ with $y \in \mathbb{R}$ and

$$\mu(u)\mathrm{y} = \mu(u)y\mathbf{1} = y$$

and thus $y = 0$ and $y = 0$. Therefore $x = z \in \mathrm{Ker}\mu(u)^*\mu(u)$ and $\mathcal{B}^*(u)\mathcal{B}(u)z = 0$. Thus $z \in \mathrm{Ker}\mu(u)^*\mu(u) \cap \mathrm{Ker}\mathcal{B}^*(u)\mathcal{B}(u) = \{0\}$ and $x = z = 0$. The operator $D^*(u)D(u)$ is then invertible. $\square$

Proposition 5.8 (second representation of the diffusion operator)**.** *For any* $(u,r) \in L^2(I) \times \mathcal{R}$

$$\Phi(u,r) = \int_0^\infty \mathbb{E}_r(G_{r_s^u}(u) - F(u))\mathrm{d}s,$$

where for a given u, r^u *denotes a Markov chain on* $\mathcal{R}$ *with transition rates* $q_{r\tilde{r}}$ *(cf. (2.2)).*

Proof. The process

$$M_t = \Phi(u, r_t^u) - \Phi(u,r) - \int_0^t \mathcal{B}(u)\Phi(u, r_s^u)\mathrm{d}s,$$

is a martingale w.r.t. the natural filtration generated by the process r^u. Let us take expectation and remember that

$$\begin{cases} \mathcal{B}(u)\Phi(r,u) & = -(G_r(u) - F(u)) \\ \int_{\mathcal{R}} \Phi(r,u)\mu(u)(\mathrm{d}r) = 0, \end{cases} . \tag{5.18}$$

Then,

$$\mathbb{E}_r\left(\Phi(u, r_t^u)\right) = \Phi(u,r) - \int_0^t \mathbb{E}_r\left(G_{r_s^u}(u) - F(u)\right)\mathrm{d}s.$$

The desired result follows since:

$$\lim_{t\to\infty} \mathbb{E}_r\left(\Phi(u, r_t^u)\right) = \int_{\mathcal{R}} \Phi(r,u)\mu(u)(\mathrm{d}r) = 0. \qquad \square$$

Proposition 5.9. *The diffusion operator $C(u)$, for $u \in L^2(I)$, is positive in the sense that*

$$\mathrm{Tr}\ C(u) \geq 0.$$

Therefore the operator $\Gamma(u)$ such that $C(u) = \Gamma^*(u)\Gamma(u)$ is well defined.

Proof. For $u \in L^2(I)$ we have:

$$\begin{aligned}\mathrm{Tr}\ C(u) &= \sum_{k\geq 1} \int_{\mathcal{R}} (G_r(u) - F(u), f_k)_{L^2(I)} (\Phi(r,u), f_k)_{L^2(I)} \mu(u)(\mathrm{d}r) \\ &= -\sum_{k\geq 1} \int_{\mathcal{R}} (\mathcal{B}(u)\Phi(r,u), f_k)_{L^2(I)} (\Phi(r,u), f_k)_{L^2(I)} \mu(u)(\mathrm{d}r).\end{aligned}$$

We conclude that $\mathrm{Tr}\ C(u) \geq 0$ because all the eigenvalues of the operator $\mathcal{B}(u)$ are non positive. □

5.4. Langevin approximation

We are interested in this section by the Langevin approximation of the averaged model. We start with the proof of Propositions 4.3 and 4.4. As for the Central Limit Theorem we detail the proof only in the all-fast case.

Proposition 5.10. *The following estimate holds,*

$$\mathrm{Tr} \int_0^t \mathrm{e}^{\Delta(t-s)} C(u_s) \mathrm{e}^{\Delta(t-s)} \mathrm{d}s \leq \sum_{k\geq 1} \int_0^t (\alpha \|u_s\|^2_{L^2(I)} + \beta \|u_s\|_{L^2(I)} + \gamma) \mathrm{e}^{-2(k\pi)^2(t-s)} \mathrm{d}s$$

for all $t \in [0,T]$ and all functions $u \in \mathcal{C}([0,T], L^2(I))$. The trace is taken in the $L^2(I)$-sense and α, β, γ are three constants.

Proof. This is a direct consequence of Proposition 5.8. Indeed, Proposition 5.8 implies that

$$\left|(\Phi(u,r), f_k)_{L^2(I)}\right| \leq \frac{c_1}{N} \sum_{i=1}^N \left|(\phi_{z_i}, f_k)_{L^2(I)}\right| \left(1 + \|u_s\|_{L^2(I)}\right)$$

for a constant c_1 and $\{f_k, k \geq 1\}$ a Hilbert basis of $L^2(I)$. Since each ϕ_{z_i} is in $L^2(I)$ we obtain

$$\left|(\Phi(u,r), f_k)_{L^2(I)}\right| \leq c_1 \left(1 + \|u_s\|_{L^2(I)}\right)$$

for another constant c_1. Let us write, in the same way as in the proof of Proposition 5.9,

$$\mathrm{Tr} \int_0^t \mathrm{e}^{\Delta(t-s)} C(u_s) \mathrm{e}^{\Delta(t-s)} \mathrm{d}s \leq \sum_{k\geq 1} \mathrm{e}^{-(k\pi)^2(t-s)} \int_{\mathcal{R}} (G_r(u) - F(u), f_k)(\Phi(r,u), f_k)\mu(u)(\mathrm{d}r). \tag{5.19}$$

Using the explicit expression of $G_r(u) - F(u)$, it is not difficult to show that there exists a constant c_2 such that

$$|(G_r(u) - F(u), f_k)| \leq c_2 \left(1 + \|u_s\|_{L^2(I)}\right).$$

Plugging the latter inequality in (5.19) leads to the result. An explicit computation of Tr C is presented in Section 6. □

In particular, the operator Q_t defined by

$$Q_t : \psi \in \mapsto \int_0^t \mathrm{e}^{\Delta(t-s)} C(u_s) \mathrm{e}^{\Delta(t-s)} \psi \mathrm{d}s$$

with $(j,u) \in \{1, \cdots, l\} \times \mathcal{C}([0,T], L^2(I))$ is of trace class in $L^2(I)$. The Langevin approximation of u is then well defined as stated in the following proposition. We recall that in the all-fast case

$$F(u) = \frac{1}{N} \sum_{i=1}^{N} \sum_{\xi \in E} c_\xi \mu(\bar{u}_i)(\xi)(v_\xi - \bar{u}_i) \phi_{z_i}$$

for $u \in L^2(I)$.

Proposition 5.11. *Let $\varepsilon > 0$. The SPDE*

$$\mathrm{d}\tilde{u}^\varepsilon = [\Delta \tilde{u}^\varepsilon + F(\tilde{u}^\varepsilon)]\mathrm{d}t + \sqrt{\varepsilon} \Gamma(\tilde{u}^\varepsilon) \mathrm{d}W_t \tag{5.20}$$

with initial condition u_0 and zero Dirichlet boundary condition has a unique solution with sample paths in $\mathcal{C}([0,T], L^2(I))$. Moreover the quantity

$$\sup_{t \in [0,T]} \mathbb{E}(\|\tilde{u}_t^\varepsilon\|_{L^2(I)}^2) < \infty. \tag{5.21}$$

Proof. Thanks to the properties of the laplacian operator, the local Lipschitz continuity of F and Proposition 4.3, we can apply classical results on SPDE, see for example ([11], Chap. 7, Thm. 7.4) to prove existence and uniqueness of solution to (5.20) in $\mathcal{C}([0,T], L^2(I))$. The bound (5.21) is obtained using similar arguments to those used in the proof of the following theorem. □

We proceed to the Proof of Theorem 4.5.

Proof of Theorem 4.5. Since there is no ambiguity in the proof, we write simply $\|\cdot\|$ for the $L^2(I)$ -norm $\|\cdot\|_{L^2(I)}$. First we notice that we have

$$\tilde{u}_t^\varepsilon - u_t = \int_0^t \mathrm{e}^{\Delta(t-s)}(F(\tilde{u}_s^\varepsilon) - F(u_s))\mathrm{d}s + \sqrt{\varepsilon} \int_0^t \mathrm{e}^{\Delta(t-s)} \Gamma(\tilde{u}_s^\varepsilon) \mathrm{d}W_s.$$

Remember that, for any $u \in L^2(I)$,

$$F(u) = \frac{1}{N} \sum_{\xi \in E} \sum_{i=1}^{N} c_\xi \mu(\bar{u}_i)(\xi)(v_\xi - \bar{u}_i) \phi_{z_i} = \frac{1}{N} \sum_{i=1}^{N} f(\bar{u}_i) \phi_{z_i},$$

and if moreover $\tilde{u} \in L^2(I)$ we have

$$|f(\bar{\tilde{u}}_i) - f(\bar{u}_i)| \le \max_{\xi \in E} c_\xi \left(1 + \max_{\xi \in E} |v_\xi| + \|u\|_{L^2(I)}\right) \|\tilde{u} - u\|_{L^2(I)}. \tag{5.22}$$

By the Parseval identity we have

$$\begin{aligned} &\left\| \int_0^t \mathrm{e}^{\Delta(t-s)}(F(\tilde{u}_s^\varepsilon) - F(u_s))\mathrm{d}s \right\|^2 \\ &= \left\| \sum_{k \ge 1} \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left(f\left(\bar{\tilde{u}}_{s,i}\right) - f\left(\bar{u}_{s,i}\right)\right) \mathrm{e}^{-(k\pi)^2(t-s)} \mathrm{d}s (\phi_{z_i}, f_k)_{L^2(I)} f_k \right\|^2. \end{aligned}$$

Then, using the fact $(f_k, k \geq 1)$ is an orthonormal basis, $\sup_{t\in[0,T]} \|u_t\|_{L^2(I)}$ is finite a.s. and (5.22) we obtain

$$\begin{aligned}\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}\left(F\left(\tilde{u}_s^\varepsilon\right) - F(u_s)\right)\mathrm{d}s\right\|^2 &\leq \frac{C_1}{N^2}\sum_{i=1}^N\sum_{k\geq 1}(\phi_{z_i}, f_k)^2_{L^2(I)}\left(\int_0^t \max_{\xi\in E} c_\xi\left(1+\max_{\xi\in E}|v_\xi|+\|u_s\|\right)\|\tilde{u}_s^\varepsilon - u_s\|\mathrm{d}s\right)^2\\ &\leq \frac{C_2}{N^2}\sum_{i=1}^N\sum_{k\geq 1}(\phi_{z_i}, f_k)^2_{L^2(I)}\left(\int_0^t \|\tilde{u}_s^\varepsilon - u_s\|\mathrm{d}s\right)^2,\end{aligned}$$

where C_1 and C_2 are two deterministic constants depending only on T. Since each ϕ_{z_i} is in $L^2(I)$, there exists a constant C_3 depending only on T such that

$$\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}(F(\tilde{u}_s^\varepsilon) - F(u_s))\mathrm{d}s\right\| \leq C_3\int_0^t \|\tilde{u}_s^\varepsilon - u_s\|\mathrm{d}s.$$

We prove now that there exists a constant C_4 such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}\Gamma(\tilde{u}_s^\varepsilon)\mathrm{d}W_s\right\|^2\right) \leq C_4. \tag{5.23}$$

Using Proposition 1.3 of [19], (analogous to the Burkholder–Davis–Gundy inequality but for stochastic convolutions rather than martingales) to control the supremum on $[0,T]$ by the value at T, we obtain that there exists a constant c_4 independent of ε such that

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}\Gamma(\tilde{u}_s^\varepsilon)\mathrm{d}W_s\right\|^2\right) \leq c_4\mathbb{E}\left(\mathrm{Tr}\int_0^T \mathrm{e}^{\Delta s}C(\tilde{u}_{t-s}^\varepsilon)\mathrm{e}^{\Delta s}\mathrm{d}s\right).$$

According to Proposition 4.3 and Proposition 4.4, the latter term is upper bounded by a constant C_4 depending only on T. From the above inequalities we obtain that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\|\tilde{u}_t^\varepsilon - u_t\|^2\right) &= \mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}(F(\tilde{u}_s^\varepsilon) - F(u_s))\mathrm{d}s + \sqrt{\varepsilon}\int_0^t \mathrm{e}^{\Delta(t-s)}\Gamma(\tilde{u}_s^\varepsilon)\mathrm{d}W_s\right\|^2\right)\\ &\leq 2\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}(F(\tilde{u}_s^\varepsilon) - F(u_s))\mathrm{d}s\right\|^2\right) + 2\varepsilon\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t \mathrm{e}^{\Delta(t-s)}\Gamma(\tilde{u}_s^\varepsilon)\mathrm{d}W_s\right\|^2\right)\\ &\leq 2\mathbb{E}\left(\left(C_3\int_0^T \sup_{t\in[0,s]}\|\tilde{u}_t^\varepsilon - u_t\|\mathrm{d}s\right)^2\right) + 2\varepsilon C_4\\ &\leq 2C_3^2 T\int_0^T \mathbb{E}\left(\sup_{t\in[0,s]}\|\tilde{u}_t^\varepsilon - u_t\|^2\right)\mathrm{d}s + 2\varepsilon C_4.\end{aligned}$$

A standard application of Gronwall's lemma leads to the result stated in Theorem 4.5. □

6. Example

We consider in this section a spatially extended stochastic Morris–Lecar model. Since the seminal work [25], the deterministic Morris–Lecar model is considered as one of the classical mathematical models for investigating neuronal behavior. At first, this model was designed to describe the voltage dynamic of the barnacle giant muscle fiber (see [25] for a complete description of the deterministic Morris–Lecar model). To take into account the intrinsic variability of the ion channels dynamic, a stochastic interpretation of this class of models has been introduced (see [8] and [33], Chap. 5, Sect. 3) in which ion channels are modeled by jump Markov processes. The model then falls into the class of stochastic generalized Hodgkin–Huxley models considered in the present paper. Let us proceed to the mathematical description of the spatially extended stochastic Morris–Lecar model. In this model, the total current $G_{r^{\mathrm{K}},r^{\mathrm{Ca}}}(u)$ is given by

$$\frac{1}{C}\left[\frac{1}{N_{\mathrm{K}}}\sum_{i=1}^{N_{\mathrm{K}}} 1_1\left(r^{\mathrm{K}}(i)\right) c_{\mathrm{K}}\left(v_{\mathrm{K}}-\bar{u}_i\right)\phi_{z_i}+\frac{1}{N_{\mathrm{Ca}}}\sum_{i=1}^{N_{\mathrm{Ca}}} 1_1\left(r^{\mathrm{Ca}}(i)\right) c_{\mathrm{Ca}}\left(v_{\mathrm{Ca}}-\bar{u}_i\right)\phi_{z_i}+I\right]$$

and the evolution equation for the transmembrane potential

$$\partial_t u_t=\frac{a}{2RC}\Delta u_t+G_{r_t^{\mathrm{K}},r_t^{\mathrm{Ca}}}(u_t),$$

on $[0,T]\times[0,1]$ and with zero Dirichlet boundary condition. The total current is the sum of the potassium K current, the calcium Ca current and the impulse I. The positive constants a,R,C are relative to the bio-physical properties of the membrane. When the voltage is held fixed, for any $1\le i\le N_{\mathrm{q}}$ where q is equal to K or Ca, $r^q(i)$ is a continuous time Markov chain with only two states 0 for *closed* and 1 for *open*. The jump rate from 1 to 0 is given by $\beta_q(\bar{u}_i)$ and from 0 to 1 by $\alpha_q(\bar{u}_i)$. All the jump rates are bounded below and above by positive constants. We will assume that the potassium ion channels communicate at fast rates of order $\frac{1}{\varepsilon}$ for a small $\varepsilon>0$. The calcium rates are of order 1. The invariant measure associated to each channel $1\le i\le N_{\mathrm{K}}$ is given by

$$\mu_i^{\mathrm{K}}(\bar{u}_i)=\left(\frac{\beta_{\mathrm{K}}(\bar{u}_i)}{\alpha_{\mathrm{K}}(\bar{u}_i)+\beta_{\mathrm{K}}(\bar{u}_i)},\frac{\alpha_{\mathrm{K}}(\bar{u}_i)}{\alpha_{\mathrm{K}}(u(z_i))+\beta_{\mathrm{K}}(\bar{u}_i)}\right).$$

Therefore the averaged applied current is

$$\begin{aligned}F_{r^{\mathrm{Ca}}}(u)=\frac{1}{C}\Bigg[&\frac{1}{N_{\mathrm{K}}}\sum_{i=1}^{N_{\mathrm{K}}}\frac{\alpha_{\mathrm{K}}(\bar{u}_i)}{\alpha_{\mathrm{K}}(\bar{u}_i)+\beta_{\mathrm{K}}(\bar{u}_i)}c_{\mathrm{K}}\left(v_{\mathrm{K}}-\bar{u}_i\right)\phi_{z_i}\\&+\frac{1}{N_{\mathrm{Ca}}}\sum_{i=1}^{N_{\mathrm{Ca}}} 1_1\left(r^{\mathrm{Ca}}(i)\right) c_{\mathrm{Ca}}\left(v_{\mathrm{Ca}}-\bar{u}_i\right)\phi_{z_i}+I\Bigg].\end{aligned}$$

In this case the application Φ of Theorem 4.2 should read as follows for a model with Dirac mass. For $(u,r)\in L^2(I)\times\mathcal{R}_{\mathrm{K}}$, $\Phi(u,r)$ is given by

$$\frac{1}{C}\frac{1}{N_{\mathrm{K}}}\sum_{i=1}^{N_{\mathrm{K}}} c_{\mathrm{K}}(v_{\mathrm{K}}-\bar{u}_i)\phi_{z_i}\int_0^\infty \mathbb{E}_r\left(1_1(r_s^{\mathrm{K},u}(i))-\frac{\alpha_{\mathrm{K}}(\bar{u}_i)}{\alpha_{\mathrm{K}}(\bar{u}_i)+\beta_{\mathrm{K}}(\bar{u}_i)}\right)\mathrm{d}s,$$

where, for u held fixed, $r_s^{\mathrm{K},u}(i)$ is a Markov chain on $\{0,1\}$ with jump rate from 1 to 0 is given by $\beta_{\mathrm{K}}(\bar{u}_i)$ and from 0 to 1 by $\alpha_{\mathrm{K}}(\bar{u}_i)$. Of course, in this case, the law of $(r_s^{\mathrm{K},u}(i),s\ge 0)$ can be fully explicited. After some algebra one obtains that $\Phi(u,r)$ is given by

$$\frac{1}{C}\frac{1}{N_{\mathrm{K}}}\sum_{i=1}^{N_{\mathrm{K}}} c_{\mathrm{K}}\frac{v_{\mathrm{K}}-\bar{u}_i}{\alpha_{\mathrm{K}}(\bar{u}_i)+\beta_{\mathrm{K}}(\bar{u}_i)}\left(1_1(r(i))-\frac{\alpha_{\mathrm{K}}(\bar{u}_i)}{\alpha_{\mathrm{K}}(\bar{u}_i)+\beta_{\mathrm{K}}(\bar{u}_i)}\right)\phi_{z_i}.$$

Then the diffusion operator $(C^{\mathrm{K}}(u)\phi,\psi)_{L^2(I)}$ is given for $u\in L^2(I)$ by

$$\frac{1}{N_{\mathrm{K}}^2}\sum_{i=1}^{N_{\mathrm{K}}} c_{\mathrm{K}}^2(v_{\mathrm{K}}-\bar{u}_i)^2\frac{a_{\mathrm{K}}(\bar{u}_i)b_{\mathrm{K}}(\bar{u}_i)}{(\alpha_{\mathrm{K}}(\bar{u}_i)+\beta_{\mathrm{K}}(\bar{u}_i))^3}\bar{\phi}_i\bar{\psi}_i$$

for $(\phi,\psi)\in L^2(I)\times L^2(I)$. From the above expression, we see that for any $u\in L^2(I)$, C^{K} is of trace class in $L^2(I)$. Let us consider, for $t\in[0,T]$

$$Q_t:\phi\in L^2(I)\mapsto\int_0^t \mathrm{e}^{\Delta(t-s)}C^{\mathrm{K}}(u_s)\mathrm{e}^{\Delta(t-s)}\phi\mathrm{d}s,$$

where $(u_s, s\in[0,T])$ is the averaged limit. From the expression of C^{K} we see that in the $L^2(I)$-sense, Tr Q_t is finite for any $t>0$.

We present in Figure 1 numerical simulations of the slow fast Morris–Lecar model with no Calcium current for various $\varepsilon>0$. The averaged model (denoted by $\varepsilon=0$) and the trace of the diffusion operator are also plotted. We set the calcium current equals to zero in our simulations to emphasize the convergence of the slow-fast spatially extended Morris–Lecar model towards the associated averaged model. See [25] Figure 2 for simulations of the deterministic finite dimensional Morris–Lecar system with no calcium current. We observe in Figure 1 that averaging affects the model in several ways. As ε goes to zero, the averaged number of spikes on a fixed time duration increases until finally form a front wave in the averaged model ($\varepsilon=0$). In the same time the intensity of the spikes decreases. Let us also mention the fact that the trace of the diffusion operator is higher in the neighborhood of a spike in accordance to [33], Chapter 5, Section 3, where the same phenomenon has been observed for the finite dimensional stochastic Morris–Lecar model.

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Appendix A. Basic facts on the spaces $L^2(I)$, $H^1_0(I)$ and Fréchet derivatives

Let $I=[0,1]$. $L^2(I)$ is the space of measurable and squared integrable functions. It is a Hilbert space endowed with the usual scalar product

$$(f,g)_{L^2(I)}=\int_I f(x)g(x)\mathrm{d}x$$

and norm $\|\cdot\|^2_{L^2(I)}=(\cdot,\cdot)_{L^2(I)}$. $H=H^1_0(I)$ denotes the completion of the set of $\mathcal{C}^\infty$ functions with compact support on I with respect to the norm $\|\cdot\|_H$ defined by

$$\|f\|_H=\sqrt{\int_I (f(x))^2+(f'(x))^2\mathrm{d}x}.$$

H is also a Hilbert space and we denote its scalar product simply by $(\cdot,\cdot)$. A Hilbert basis of H (resp. $L^2(I)$) is given by the following functions on I

$$e_k(\cdot)=\frac{\sqrt{2}}{\sqrt{1+(k\pi)^2}}\sin(k\pi\cdot),\quad (\text{resp. } f_k(\cdot)=\sqrt{2}\sin(k\pi\cdot)).$$

for $k\geq 1$. The dual space of H which is H^{-1} is denoted by H^*. $\langle\cdot,\cdot\rangle$ is the duality pairing between H and H^*. The triple of Banach spaces $H\subset L^2(I)\subset H^*$ is an evolution triple or Gelfand triple. The embeddings in between these three spaces are continuous and dense. For any $h\in L^2(I)$ and any $u\in H$: $\langle h,u\rangle=(h,u)_{L^2(I)}$ and for any $x\in I$, $k\geq 1$

$$\langle\delta_x,e_k\rangle=(1+(k\pi)^2)e_k(x).$$

The embedding $H\subset\mathcal{C}(I,\mathbb{R})$ also holds and we denote by C_P the constant such that, for all $u\in H$

$$\sup_I|u|\leq C_P\|u\|_H.$$

We refer the reader to [20], Chapter 1, Section 1.3 for more details.

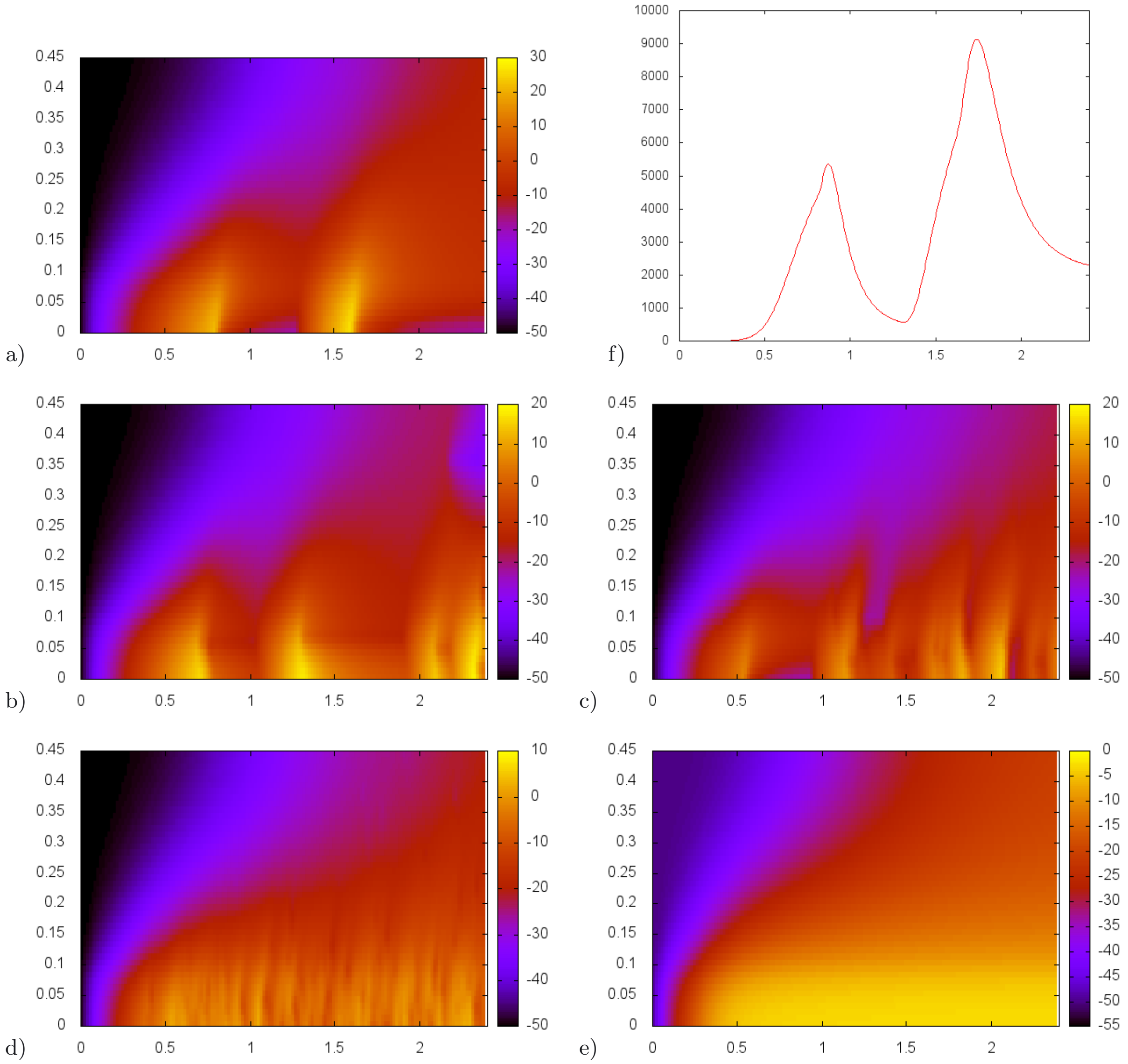

FIGURE 1. Simulations of the spatially extended Morris–Lecar model with no Calcium current for ε equals successively to a) $\varepsilon = 1$, b) $\varepsilon = 0.1$, c) $\varepsilon = 0.01$, d) $\varepsilon = 0.001$, e) $\varepsilon = 0$, that is for the averaged model. The plotted curve f) is related to the simulation of the Morris Lecar model on its left side a): it is the plot of the function $t \mapsto \mathrm{Tr}\, Q_t$. A stimulus is exciting the membrane during all the time duration of the simulation on the portion $[0, 0.1]$ of the fiber.

In $L^2(I)$, the Laplacian with zero Dirichlet boundary conditions has the following spectral decomposition

$$\Delta u = -\sum_{k\geq 1}(k\pi)^2(u, f_k)f_k$$

for u in the domain $\mathcal{D}(\Delta) = \{u \in L^2(I); \sum_{k\geq 1} k^4 (u, f_k)^2_{L^2(I)} < \infty\}$. It generates the semi-group of operators $\{e^{\Delta t}, t \geq 0\}$ defined for $u \in L^2(I)$ by

$$e^{\Delta t} u = \sum_{k\geq 1} e^{-(k\pi)^2 t} (u, f_k)_{L^2(I)} f_k.$$

We say that a function $f : L^2(I) \mapsto \mathbb{R}$ has a Fréchet derivative in $u \in L^2(I)$ if there exists a bounded linear operator $T_u : L^2(I) \mapsto \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(u+h) - f(u) - T_u(h)}{\|h\|_{L^2(I)}} = 0.$$

We then write $\frac{\mathrm{d}f}{\mathrm{d}u}(u)$ for the operator T_u. For example, the square of the $\|\cdot\|_{L^2(I)}$-norm is Fréchet differentiable on $L^2(I)$. For all $u \in L^2(I)$

$$\frac{\mathrm{d}\|\cdot\|^2_{L^2(I)}}{\mathrm{d}u}(u)[h] = 2(u, h)_{L^2(I)}$$

for all $h \in L^2(I)$. In the same way, we can define the Fréchet derivative of order 2. The second Fréchet derivative of a twice Fréchet differentiable function $f : L^2(I) \to \mathbb{R}$ is denoted by $\frac{\mathrm{d}^2 f}{\mathrm{d}u^2}(u)$. It can be considered as a bilinear form on $L^2(I) \times L^2(I)$. For instance

$$\frac{\mathrm{d}^2\|\cdot\|^2_{L^2(I)}}{\mathrm{d}u^2}(u)[h, k] = 2(h, k)_{L^2(I)},$$

for all $(h, k) \in L^2(I) \times L^2(I)$. Fréchet differentiation is stable by summation and multiplication.

Appendix B. Numerical data for the simulations

Here are the numerical data used for the simulations of the Morris Lecar model

$$\begin{aligned} C = 1, \quad & c_{\mathrm{K}} = 32, \ v_{\mathrm{K}} = -70, \\ a = 1, \quad & c_{\mathrm{Ca}} = 0, \quad v_{\mathrm{Ca}} = 0, \\ R = 0.5, \ & N_{\mathrm{K}} = 50, \ N_{\mathrm{Ca}} = 0. \end{aligned}$$

The length of the fiber is $l = 0.5$ and the time duration is $T = 2.4$. The impulse I is of the form

$$I(x, t) = \lambda 1_{[0,0.1]}(x)$$

with $\lambda = 300$. The data for the internal resistance R and the capacitance C are arbitraly chosen for the purpose of the simulations. The values for the other parameters correspond to [25].

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