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ASYMPTOTIC ANALYSIS OF OPTIMIZED SCHWARZ METHODS FOR MAXWELL'S EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. Discretized time harmonic Maxwell's equations are hard to solve by iterative methods, and the best currently available methods are based on domain decomposition and optimized transmission conditions. Optimized Schwarz methods were the first ones to use such transmission conditions, and this approach turned out to be so fundamentally important that it has been rediscovered over the last years under the name sweeping preconditioners, source transfer, single layer potential method and the method of polarized traces. We show here how one can optimize transmission conditions to take benefit from the jumps in the coefficients of the problem, when they are aligned with the subdomain interface, and obtain methods which converge for two subdomains in certain situations independently of the mesh size, which would not be possible without jumps in the coefficients.

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1. Introduction

Time harmonic wave propagation problems are well known to be challenging to solve by iterative methods, for an overview in the case of the Helmholtz equation, see [14]. The most promising iterative algorithms are based on domain decomposition methods. After the seminal work in the PhD thesis [5] of Deprés, who devised an algorithm with Robin transmission conditions and proved its convergence, substantial progress has been made, leading to the class of optimized Schwarz methods, see [3, 4, 15, 17, 18], and the more recent algorithms of sweeping type, source transfer, single layer potential and polarized traces use the same underlying ideas of optimized Schwarz methods, see [16] and references therein. Time harmonic Maxwell's equations inherit all these difficulties from the Helmholtz equation, and the unknown becomes in addition vector valued. Nevertheless, optimized Schwarz methods have been successfully also developed for Maxwell's equations, see [1,7,20,21] for the case without conductivity and [8, 12] for the case with conductivity, where one sees that the presence of a non-zero conductivity is beneficial for convergence. For discretizations of Maxwell's equations using Discontinuous Galerkin methods, results on optimized Schwarz solvers can be found in [6, 9], and for scattering problems and large scale applications, see [20, 21].

Keywords and phrases. Domain decomposition, Maxwell's equations, discontinuous coefficients, optimized Schwarz methods.

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We are interested here in Maxwell's equations in the presence of jumps in the coefficients. We study in particular the case where the jumps can be aligned with subdomain interfaces, and show that then jumps can actually help convergence of the optimized Schwarz method, if the transmission conditions are appropriately chosen. In the presence of coefficients jumps, it is natural to consider non-overlapping Schwarz methods, and we study here the particular case of the 2D Maxwell's equations for TMz and TEz modes. For a special case in 2D, a convergence result stated without proof in [10] showed that well chosen transmission conditions can lead to non-overlapping Schwarz algorithms that converge independently of the mesh parameter. In addition it was shown in [11] that complete results for the 2D transverse magnetic (TMz) and transverse electric (TEz) modes[1] will imply directly convergence results in 3D as well. The purpose of this manuscript is to prove the 2D results announced in [10] using asymptotic analysis. The analysis is very technical, and many cases need to be considered. We therefore give all the details only for the first case, and then only outline the most important technical steps for the remaining cases. We finally illustrate our results with numerical experiments.

2. Schwarz Methods for Maxwell's Equations

We consider in this paper the time-harmonic Maxwell equations with appropriate boundary conditions

$$-i\omega\varepsilon\mathbf{E} + \nabla\times\mathbf{H} = \mathbf{J}, \qquad i\omega\mu\mathbf{H} + \nabla\times\mathbf{E} = \mathbf{0}, \quad \text{in } \Omega, \tag{2.1}$$

and we study the heterogeneous case where the domain Ω consists of two non-overlapping sub-domains Ω_1 and Ω_2 with interface $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$, with piecewise constant electric permittivity ε_j and piecewise constant magnetic permeability μ_j in Ω_j, $j = 1, 2$. A general parallel Schwarz algorithm for these two non-overlapping subdomains Ω_1 and Ω_2 would start with some initial guess $\mathbf{E}^{j,0}$ and $\mathbf{H}^{j,0}$ on subdomain Ω_j and then compute for iteration index $n = 1, 2,$

$$\left\{\begin{array}{ll} -i\omega\varepsilon_1\mathbf{E}^{1,n} + \nabla\times\mathbf{H}^{1,n} = \mathbf{J} & \text{in } \Omega_1,\\ i\omega\mu_1\mathbf{H}^{1,n} + \nabla\times\mathbf{E}^{1,n} = \mathbf{0} & \text{in } \Omega_1,\\ (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{1,n}, \mathbf{H}^{1,n}) = (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{2,n-1}, \mathbf{H}^{2,n-1}) & \text{on } \Gamma,\\ -i\omega\varepsilon_2\mathbf{E}^{2,n} + \nabla\times\mathbf{H}^{2,n} = \mathbf{J} & \text{in } \Omega_2,\\ i\omega\mu_2\mathbf{H}^{2,n} + \nabla\times\mathbf{E}^{2,n} = \mathbf{0} & \text{in } \Omega_2,\\ (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{2,n}, \mathbf{H}^{2,n}) = (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{1,n-1}, \mathbf{H}^{1,n-1}) & \text{on } \Gamma, \end{array}\right. \tag{2.2}$$

where $\mathcal{S}_j$, $j = 1, 2$ are tangential operators which will be differential or pseudo-differential, and

$$\mathcal{B}_{\mathbf{n}_j}(\mathbf{E}^{j,n}, \mathbf{H}^{j,n}) = \frac{\mathbf{E}^{j,n}}{Z_j}\times\mathbf{n}_j + \mathbf{n}_j\times(\mathbf{H}^{j,n}\times\mathbf{n}_j), \qquad Z_j := \sqrt{\mu_j/\varepsilon_j}, \tag{2.3}$$

are the impedance conditions, and $\mathbf{n}_j$ is the unit outward normal for domain Ω_j, $j = 1, 2$. Different choices of $\mathcal{S}_j$, $j = 1, 2$ lead to different Schwarz methods. Since it was shown in [11] that it is sufficient to study the 2D TMz and TEz variants of (2.2) to get a complete understanding of (2.2), we focus in what follows on the 2D case.

2.1. Schwarz methods for the TMz and TEz modes

We now present algorithm (2.2) for the TMz and TEz modes in two spatial dimensions. Since our analysis will be performed on the error equations, we directly state the algorithms for the homogeneous problems. For

[1] For a detailed introduction to the TMz and TEz equations see [19].

the TMz case, we obtain from (2.2)

$$\left\{\begin{array}{rll} i\omega\varepsilon_1 E_z^{1,n} - \partial_x H_y^{1,n} + \partial_y H_x^{1,n} &= 0 & \text{in } \Omega_1,\\ i\omega\mu_1 H_x^{1,n} + \partial_y E_z^{1,n} &= 0 & \text{in } \Omega_1,\\ i\omega\mu_1 H_y^{1,n} - \partial_x E_z^{1,n} &= 0 & \text{in } \Omega_1,\\ (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(E_z^{1,n}, H_x^{1,n}, H_y^{1,n}) &= (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(E_z^{2,n-1}, H_x^{2,n-1}, H_y^{2,n-1}) & \text{on } \Gamma,\\ i\omega\varepsilon_2 E_z^{2,n} - \partial_x H_y^{2,n} + \partial_y H_x^{2,n} &= 0 & \text{in } \Omega_2,\\ i\omega\mu_2 H_x^{2,n} + \partial_y E_z^{2,n} &= 0 & \text{in } \Omega_2,\\ i\omega\mu_2 H_y^{2,n} - \partial_x E_z^{2,n} &= 0 & \text{in } \Omega_2,\\ (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(E_z^{2,n}, H_x^{2,n}, H_y^{2,n}) &= (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(E_z^{1,n-1}, H_x^{1,n-1}, H_y^{1,n-1}) & \text{on } \Gamma. \end{array}\right. \tag{2.4}$$

which was obtained from (2.3) by replacing $\mathbf{E}^{j,n} = (0, 0, E_z^{j,n})$ and $\mathbf{H}^{j,n} = (H_x^{j,n}, H_y^{j,n}, 0)$. For the TEz case, we get

$$\left\{\begin{array}{rll} -i\omega\varepsilon_1 E_x^{1,n} + \partial_y H_y^{1,n} &= 0 & \text{in } \Omega_1,\\ -i\omega\varepsilon_1 E_y^{1,n} - \partial_x H_y^{1,n} &= 0 & \text{in } \Omega_1,\\ i\omega\mu_1 H_z^{1,n} + \partial_x E_y^{1,n} - \partial_y E_x^{1,n} &= 0 & \text{in } \Omega_1,\\ (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(E_x^{1,n}, E_y^{1,n}, H_z^{1,n}) &= (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(E_x^{2,n-1}, E_y^{2,n-1}, H_z^{2,n-1}) & \text{on } \Gamma,\\ -i\omega\varepsilon_2 E_x^{2,n} + \partial_y H_y^{2,n} &= 0 & \text{in } \Omega_2,\\ -i\omega\varepsilon_2 E_y^{2,n} - \partial_x H_y^{2,n} &= 0 & \text{in } \Omega_2,\\ i\omega\mu_2 H_z^{2,n} + \partial_x E_y^{2,n} - \partial_y E_x^{2,n} &= 0 & \text{in } \Omega_2,\\ (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(E_x^{2,n}, E_y^{2,n}, H_z^{2,n}) &= (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(E_x^{1,n-1}, E_y^{1,n-1}, H_z^{1,n-1}) & \text{on } \Gamma. \end{array}\right. \tag{2.5}$$

which we obtained from (2.3) by replacing $\mathbf{E}^{j,n} = (E_x^{j,n}, E_y^{j,n}, 0)$ and $\mathbf{H}^{j,n} = (0, 0, H_z^{j,n})$.

Remark 2.1. From the TEz equations we see that the TEz mode is related to the TMz mode, we just have to exchange the variables H_z with $-E_z$, E_x with H_x and E_y with H_y, and we get the same algorithm, provided we also switch μ and ε. It therefore suffices to either analyze the TMz or the TEz case, the results for the other case then follow by switching the roles of μ and ε.

For the TMz case, we can obtain an optimal Schwarz algorithm that converges in two iterations by choosing $\mathcal{S}_1$ and $\mathcal{S}_2$ such that in the transmission conditions in (2.4), namely

$$\begin{array}{l} (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(E_z^{1,n+1}, \mathbf{H}^{1,n+1}) = (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1\mathcal{B}_{\mathbf{n}_2})(E_z^{2,n}, \mathbf{H}^{2,n}),\\ (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(E_z^{2,n+1}, \mathbf{H}^{2,n+1}) = (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2\mathcal{B}_{\mathbf{n}_1})(E_z^{1,n}, \mathbf{H}^{1,n}), \end{array} \tag{2.6}$$

the right hand side becomes homogeneous after the first iteration, which implies after a Fourier transform in the y direction with Fourier parameter k that

$$\widehat{\mathcal{S}}_1 = -\frac{\lambda_2 - i\omega_2 Z^{-1}}{\lambda_2 + i\omega_2}, \quad \widehat{\mathcal{S}}_2 = -\frac{\lambda_1 - i\omega_1 Z}{\lambda_1 + i\omega_1}, \tag{2.7}$$

with $\lambda_j := \sqrt{k^2 - \omega_j^2}$ and $\omega_j := \omega\sqrt{\varepsilon_j\mu_j}$, see Theorem 3.2 of [7] for more details in the constant coefficient case. The choice (2.7) corresponds to the so called transparent conditions that were pioneered by Engquist and Majda [13]. They are well defined for any value of the Fourier parameter k, but are computationally expensive to use because Fourier transforms have to be performed due to the square root terms.

2.2. Optimized convergence factor and min-max problems

As for absorbing boundary conditions [13], we propose to approximate the transparent conditions in (2.7) by

$$\widehat{\mathcal{S}}_1 \approx -\frac{s_2 - i\omega_2 Z^{-1}}{s_2 + i\omega_2}, \quad \widehat{\mathcal{S}}_2 \approx -\frac{s_1 - i\omega_1 Z}{s_1 + i\omega_1}, \tag{2.8}$$

with s_1 and s_2 two complex parameters. Using the approximations of $\widehat{\mathcal{S}}_1$ and $\widehat{\mathcal{S}}_2$ from (2.8) in the transmission conditions (2.6), we obtain after a by now standard computation, see [10] for details, the convergence factor

$$\rho_{\rm opt}(k,\omega_1,\omega_2,\mu_1,\mu_2,s_1,s_2) = \left(\frac{(\lambda_1 - s_1)(\lambda_2 - s_2)}{(\lambda_1 + s_2\mu_1/\mu_2)(\lambda_2 + s_1\mu_2/\mu_1)}\right)^{\frac{1}{2}}. \tag{2.9}$$

To get a fast algorithm, we have to chose the complex parameters s_1 and s_2 such that the convergence factor is minimized for all relevant numerical frequencies, *i.e.* we have to minimize $|\rho_{\rm opt}|$ for all $k \in K := [k_{\min}, k_{\max}]$, where $k_{\min}$ is the smallest relevant frequency ($k_{\min}$ depends on the geometry of the domain) and $k_{\max}$ is the largest frequency supported by the numerical grid (if the grid size h is constant we have $k_{\max} = \frac{c_{\max}}{h}$, where $c_{\max}$ is some constant, often estimated by π). This leads to the min-max problem

$$\min_{s_1,s_2\in\mathbb{C}} \max_{k\in K} \left|\rho_{\rm opt}(k,\omega_1,\omega_2,\mu_1,\mu_2,s_1,s_2)\right|. \tag{2.10}$$

Remark 2.2. For the TEz case we obtain the convergence factor

$$\rho_{\rm opt}(k,\omega_1,\omega_2,\varepsilon_1,\varepsilon_2,s_1,s_2) = \left(\frac{(\lambda_1 - s_1)(\lambda_2 - s_2)}{(\lambda_1 + s_2\varepsilon_1/\varepsilon_2)(\lambda_2 + s_1\varepsilon_2/\varepsilon_1)}\right)^{\frac{1}{2}} \tag{2.11}$$

and the equivalent min-max problem

$$\min_{s_1,s_2\in\mathbb{C}} \max_{k\in K} |\rho_{\rm opt}(k,\omega_1,\omega_2,\varepsilon_1,\varepsilon_2,s_1,s_2)|. \tag{2.12}$$

We assume in what follows that the parameters are of the form $s_1 = (1+i)C_1$, $s_2 = (1+i)C_2$, a choice that was justified for Helmholtz equations in [17] and Maxwell's equations in [2].

To study the min-max problem (2.10), we have to divide the frequency interval K into three sub intervals ($[0,\omega_{\min}]$, $[\omega_{\min},\omega_{\max}]$, $[\omega_{\max},k_{\max}]$), where $\omega_{\min} := \min\{\omega_1,\omega_2\}$ and $\omega_{\max} := \max\{\omega_1,\omega_2\}$. These three intervals are implied by the change of λ_1 and λ_2 from imaginary to real. We only consider the case $\omega_1 \le \omega_2$ because the other results follow by symmetry of (2.9). For $k \in [0,\omega_1]$, the convergence factor in (2.9) is equal to

$$\rho_{\rm opt}(k,\omega_1,\omega_2,\mu_1,\mu_2,C_1,C_2) = \frac{(i\tilde{\lambda}_1 - (1+i)C_1)(i\tilde{\lambda}_2 - (1+i)C_2)}{(i\tilde{\lambda}_1 + \frac{\mu_1}{\mu_2}(1+i)C_2)(i\tilde{\lambda}_2 + \frac{\mu_2}{\mu_1}(1+i)C_1)}, \tag{2.13}$$

with $\tilde{\lambda}_j := \sqrt{\omega_j^2 - k^2}$. For $k \in [\omega_1,\omega_2]$ we have

$$\rho_{\rm opt}(k,\omega_1,\omega_2,\mu_1,\mu_2,C_1,C_2) = \frac{(\lambda_1 - (1+i)C_1)(i\tilde{\lambda}_2 - (1+i)C_2)}{(\lambda_1 + \frac{\mu_1}{\mu_2}(1+i)C_2)(i\tilde{\lambda}_2 + \frac{\mu_2}{\mu_1}(1+i)C_1)}. \tag{2.14}$$

Finally, for $k \in [\omega_2, k_{\max}]$ we have

$$\rho_{\rm opt}(k,\omega_1,\omega_2,\mu_1,\mu_2,C_1,C_2) = \frac{(\lambda_1 - (1+i)C_1)(\lambda_2 - (1+i)C_2)}{(\lambda_1 + \frac{\mu_1}{\mu_2}(1+i)C_2)(\lambda_2 + \frac{\mu_2}{\mu_1}(1+i)C_1)}. \tag{2.15}$$

In nature the magnetic permeability μ is almost constant and the rate of change of the magnetic permeability μ for different materials can be neglected in comparison to the rate of change of the electric permittivity ε. We thus present here the case $\mu_1 = \mu_2$ and $\varepsilon_1 \neq \varepsilon_2$ both for the TMz and the TEz mode. Using Remark 2.1, one can then also read off the corresponding results for the physically less important case $\mu_1 \neq \mu_2$ and $\varepsilon_1 = \varepsilon_2$. The case where both coefficients have jumps can also be treated under an additional hypothesis, see Theorem 2.5.1 of [22].

3. Transmission conditions for the TMz mode

The condition $\varepsilon_1 \neq \varepsilon_2$ implies that $\omega_1 \neq \omega_2$. The case $\omega_1 = \omega_2$ is a resonance case that was studied in [22] and needs special treatment, like the particular resonance case where ε and μ are continuous, see [7]. The fact to have a jump in ε helps convergence, and Theorem 3.1 below was presented in a less general form in [10] without proof, and was the main building block to understand the 3D case in [11]. We give now the general result, and a detailed proof based on asymptotic analysis.

Theorem 3.1. *If $\mu_1 = \mu_2$, $\varepsilon_1 \neq \varepsilon_2$, $s_1 = (1+i)C_1$, $s_2 = (1+i)C_2$ and $r = \sqrt{|\omega_1^2 - \omega_2^2|}$, then the solution of the min-max problem (2.10) for h small is given by*

$$C_1^* = \left(\frac{r}{2}\right)^{\frac{1}{4}} \left(\frac{c_{\max}}{h}\right)^{\frac{3}{4}},\ C_2^* = \frac{1}{2}\left(\frac{r}{2}\right)^{\frac{3}{4}} \left(\frac{c_{\max}}{h}\right)^{\frac{1}{4}},$$
$$\rho^*_{opt} = 1 - \left(\frac{r}{2c_{\max}}\right)^{\frac{1}{4}} h^{\frac{1}{4}} + \mathcal{O}(h^{\frac{1}{2}}), \tag{3.1}$$

and the roles of s_1 and s_2 can also be reversed.

Proof. We set $C_1 := \frac{c_1}{h^\alpha}$ and $C_2 := \frac{c_2}{h^\beta}$, and determine the exponents and constants which solve the min-max problem (2.10). We divide the proof into three cases (Case I: $\beta < \alpha$, Case II: $\beta = \alpha$, and Case III: $\beta > \alpha$). In every case we will perform the following steps:

1. Search and classify the extrema for $k = c$ constant.
2. Verify that we do not have an extremum for k close to 0, which means for $k = c_m h^\gamma$, with $\gamma > 0$.
3. Search and classify the variable extrema for $k = \frac{c_m}{h^\gamma}$ with $0 \le \gamma \le 1$.
4. Compare the possible maxima for the values found in the previous steps and balance them to solve the min-max problem (2.10).

Remark 3.2. The constant γ can not be greater than 1 because $k_{\max} = \frac{c_{\max}}{h}$ and we are not interested in higher frequencies than $k_{\max}$.

Case I ($\beta < \alpha$):

1. To show asymptotically that we only have one local extremum for k constant, we have to study the convergence factor in the three intervals (2.13, 2.14, 2.15): if $k \in [0, \omega_1]$ we obtain for the modulus squared of the convergence factor in (2.13) as a function of C_1 and C_2

$$R_1(k,\omega_1,\omega_2,C_1,C_2) := \frac{(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_1 - \frac{c_1}{h^\alpha})^2)(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_2 - \frac{c_2}{h^\beta})^2)}{(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \frac{c_2}{h^\beta})^2)(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \frac{c_1}{h^\alpha})^2)}. \tag{3.2}$$

Taking a partial derivative with respect to k, we get

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k,\omega_1,\omega_2,C_1,C_2) = R_{11} + R_{12} + R_{13} + R_{14}, \tag{3.3}$$

where

$$\begin{aligned}
R_{11} &= \frac{-\frac{2k}{\tilde\lambda_1}(\tilde\lambda_1 - \frac{c_1}{h^\alpha})(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_2 - \frac{c_2}{h^\beta})^2)}{(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \frac{c_2}{h^\beta})^2)(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \frac{c_1}{h^\alpha})^2)},\\
R_{12} &= \frac{-\frac{2k}{\tilde\lambda_2}(\tilde\lambda_2 - \frac{c_2}{h^\beta})(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_1 - \frac{c_1}{h^\alpha})^2)}{(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \frac{c_2}{h^\beta})^2)(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \frac{c_1}{h^\alpha})^2)},\\
R_{13} &= \frac{\frac{2k}{\tilde\lambda_1}(\tilde\lambda_1 + \frac{c_2}{h^\beta})(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_1 - \frac{c_1}{h^\alpha})^2)(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_2 - \frac{c_2}{h^\beta})^2)}{(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \frac{c_2}{h^\beta})^2)^2(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \frac{c_1}{h^\alpha})^2)},\\
R_{14} &= \frac{\frac{2k}{\tilde\lambda_2}(\tilde\lambda_2 + \frac{c_1}{h^\alpha})(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_1 - \frac{c_1}{h^\alpha})^2)(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_2 - \frac{c_2}{h^\beta})^2)}{(\frac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \frac{c_2}{h^\beta})^2)(\frac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \frac{c_1}{h^\alpha})^2)^2}.
\end{aligned} \tag{3.4}$$

In order to evaluate R_1 in (3.2) and $\frac{\mathrm{d}R_1}{\mathrm{d}k}$ in (3.4) asymptotically, we need to expand all terms in (3.4), which leads to

$$\begin{aligned}
(\tilde\lambda_1 - \tfrac{c_1}{h^\alpha}) &\simeq -\tfrac{c_1}{h^\alpha}\left(1-\tfrac{\tilde\lambda_1}{c_1}h^\alpha\right),\\
(\tilde\lambda_2 - \tfrac{c_2}{h^\beta}) &\simeq -\tfrac{c_2}{h^\beta}\left(1-\tfrac{\tilde\lambda_2}{c_2}h^\beta\right),\\
(\tilde\lambda_1 + \tfrac{c_2}{h^\beta}) &\simeq \tfrac{c_2}{h^\beta}\left(1+\tfrac{\tilde\lambda_1}{c_2}h^\beta\right),\\
(\tilde\lambda_2 + \tfrac{c_1}{h^\alpha}) &\simeq \tfrac{c_1}{h^\alpha}\left(1+\tfrac{\tilde\lambda_2}{c_1}h^\alpha\right),\\
\left(\tfrac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_1 - \tfrac{c_1}{h^\alpha})^2\right) &\simeq \tfrac{2c_1^2}{h^{2\alpha}}\left(1-\tfrac{\tilde\lambda_1}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right),\\
\left(\tfrac{c_2^2}{h^{2\beta}} + (\tilde\lambda_2 - \tfrac{c_2}{h^\beta})^2\right) &\simeq \tfrac{2c_2^2}{h^{2\beta}}\left(1-\tfrac{\tilde\lambda_2}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right),\\
\left(\tfrac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \tfrac{c_2}{h^\beta})^2\right) &\simeq \tfrac{2c_2^2}{h^{2\beta}}\left(1+\tfrac{\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right),\\
\left(\tfrac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \tfrac{c_1}{h^\alpha})^2\right) &\simeq \tfrac{2c_1^2}{h^{2\alpha}}\left(1+\tfrac{\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right),\\
\left(\tfrac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \tfrac{c_2}{h^\beta})^2\right)^{-1} &\simeq \tfrac{h^{2\beta}}{2c_2^2}\left(1-\tfrac{\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right),\\
\left(\tfrac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \tfrac{c_1}{h^\alpha})^2\right)^{-1} &\simeq \tfrac{h^{2\alpha}}{2c_1^2}\left(1-\tfrac{\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right),\\
\left(\tfrac{c_2^2}{h^{2\beta}} + (\tilde\lambda_1 + \tfrac{c_2}{h^\beta})^2\right)^{-2} &\simeq \tfrac{h^{4\beta}}{4c_2^4}\left(1-\tfrac{2\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right),\\
\left(\tfrac{c_1^2}{h^{2\alpha}} + (\tilde\lambda_2 + \tfrac{c_1}{h^\alpha})^2\right)^{-2} &\simeq \tfrac{h^{4\alpha}}{4c_1^4}\left(1-\tfrac{2\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right).
\end{aligned}$$

Replacing these expressions into (3.4) and collecting leading order terms, we get

$$\begin{aligned}
R_{11} &\simeq \left(-\tfrac{2k}{\tilde\lambda_1}\right)\left(-\tfrac{c_1}{h^\alpha}\right)\left(1-\tfrac{\tilde\lambda_1}{c_1}h^\alpha\right)\left(\tfrac{2c_2^2}{h^{2\beta}}\right)\left(1-\tfrac{\tilde\lambda_2}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(\tfrac{h^{2\beta}}{2c_2^2}\right)\\
&\quad\left(1-\tfrac{\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(\tfrac{h^{2\alpha}}{2c_1^2}\right)\left(1-\tfrac{\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\\
&\simeq \left(\tfrac{k}{\tilde\lambda_1 c_1}h^\alpha\right)\left(1-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_2}h^\beta-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\beta})\right),\\
R_{12} &\simeq \left(-\tfrac{2k}{\tilde\lambda_2}\right)\left(-\tfrac{c_2}{h^\beta}\right)\left(1-\tfrac{\tilde\lambda_2}{c_2}h^\beta\right)\left(\tfrac{2c_1^2}{h^{2\alpha}}\right)\left(1-\tfrac{\tilde\lambda_1}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\left(\tfrac{h^{2\beta}}{2c_2^2}\right)\\
&\quad\left(1-\tfrac{\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(\tfrac{h^{2\alpha}}{2c_1^2}\right)\left(1-\tfrac{\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\\
&\simeq \left(\tfrac{k}{\tilde\lambda_2 c_2}h^\beta\right)\left(1-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_2}h^\beta-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\beta})\right),\\
R_{13} &\simeq \left(\tfrac{2k}{\tilde\lambda_1}\right)\left(\tfrac{c_2}{h^\beta}\right)\left(1+\tfrac{\tilde\lambda_1}{c_2}h^\beta\right)\left(\tfrac{2c_1^2}{h^{2\alpha}}\right)\left(1-\tfrac{\tilde\lambda_1}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\left(\tfrac{2c_2^2}{h^{2\beta}}\right)\\
&\quad\left(\tfrac{h^{4\beta}}{4c_2^4}\right)\left(\tfrac{h^{2\alpha}}{2c_1^2}\right)\left(1-\tfrac{\tilde\lambda_2}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(1-\tfrac{2\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\\
&\quad\left(1-\tfrac{\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\\
&\simeq \left(\tfrac{k}{\tilde\lambda_1 c_2}h^\beta\right)\left(1-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_2}h^\beta-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\beta})\right),\\
R_{14} &\simeq \left(\tfrac{2k}{\tilde\lambda_2}\right)\left(\tfrac{c_1}{h^\alpha}\right)\left(1+\tfrac{\tilde\lambda_2}{c_1}h^\alpha\right)\left(\tfrac{2c_1^2}{h^{2\alpha}}\right)\left(1-\tfrac{\tilde\lambda_1}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\left(\tfrac{2c_2^2}{h^{2\beta}}\right)\\
&\quad\left(\tfrac{h^{2\beta}}{2c_2^2}\right)\left(\tfrac{h^{4\alpha}}{4c_1^4}\right)\left(1-\tfrac{\tilde\lambda_2}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(1-\tfrac{\tilde\lambda_1}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\\
&\quad\left(1-\tfrac{2\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\\
&\simeq \left(\tfrac{k}{\tilde\lambda_2 c_1}h^\alpha\right)\left(1-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_2}h^\beta-\tfrac{\tilde\lambda_1+\tilde\lambda_2}{c_1}h^\alpha+\mathcal{O}(h^{2\beta})\right).
\end{aligned} \tag{3.5}$$

We thus obtain for the asymptotic behavior of $\frac{\mathrm{d}R_1}{\mathrm{d}k}$

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k,\omega_1,\omega_2,C_1,C_2) = k\left(\frac{1}{\tilde\lambda_1}+\frac{1}{\tilde\lambda_2}\right)\left[\frac{1}{c_2}h^\beta+\frac{1}{c_1}h^\alpha+\mathcal{O}(h^{2\beta})\right], \tag{3.6}$$

and hence there is a local extremum at $k = 0$. For $k \in (0, \omega_1)$ we have $\frac{\mathrm{d}R_1}{\mathrm{d}k} > 0$, because $\left(\frac{1}{\tilde{\lambda}_1} + \frac{1}{\tilde{\lambda}_2}\right) > 0$ for all $k \in (0, \omega_1)$, so there are no other local extrema in $(0, \omega_1)$ for k fixed and thus $k = 0$ is a minimum and $k = \omega_1$ is a maximum.

Performing a similar study for the interval $[\omega_1, \omega_2]$, for details see [22], we obtain

$$\frac{\mathrm{d}R_2}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = k\left(\frac{\lambda_1 - \tilde{\lambda}_2}{\lambda_1\tilde{\lambda}_2}\right)\left[\frac{1}{c_2}h^\beta + \frac{1}{c_1}h^\alpha + \mathcal{O}(h^{2\beta})\right]. \tag{3.7}$$

Here the leading order term can vanish if $\tilde{\lambda}_2 = \lambda_1$, *i.e.* when

$$k = k_1 := \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}}.$$

If $k < \sqrt{\frac{\omega_1^2+\omega_2^2}{2}}$ we have $\lambda_1 < \tilde{\lambda}_2$ and $\frac{\mathrm{d}R_2}{\mathrm{d}k} < 0$, and if $k > \sqrt{\frac{\omega_1^2+\omega_2^2}{2}}$ we have $\lambda_1 > \tilde{\lambda}_2$ and thus $\frac{\mathrm{d}R_2}{\mathrm{d}k} > 0$. Therefore k_1 is a local minimum, not a maximum. The maximum on the interval is thus either at $k = \omega_1$ or $k = \omega_2$.

Now we verify that we do not have extrema for k fixed in the third interval $[\omega_2, k_{\max}]$. With similar computations as before, see [22] for details, we obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = k\left(\frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}\right)\left[-\frac{1}{c_2}h^\beta - \frac{1}{c_1}h^\alpha + \mathcal{O}(h^{2\beta})\right]. \tag{3.8}$$

Since $\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2} > 0$ for $k \in [\omega_2, k_{\max}]$, it follows that R_3 does not have an extremum for fixed $k \in [\omega_2, k_{\max}]$, and the sign of the derivative shows that R_3 is decreasing for $k \geq \omega_2$.

2. Now we show that there is no variable extremum close to $k = 0$: we suppose that $k = c_m h^\gamma$, with $\gamma > 0$, and obtain with a similar approach as before (for details, see [22]), that

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = k\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\left[\frac{1}{c_2}h^\beta + \frac{1}{c_1}h^\alpha + \mathcal{O}(h^*)\right], \tag{3.9}$$

with $*$ denoting a number bigger than β. The leading order term can not vanish, and we thus can not have further extrema close to $k = 0$.

3. We now study possible extrema for $k = \frac{c_m}{h^\gamma}$, with $0 < \gamma \leq 1$. There are five sub-cases to consider: $\gamma < \beta < \alpha$, $\beta < \gamma < \alpha$, $\beta < \alpha < \gamma$ and also the two particular cases $\beta = \gamma < \alpha$ and $\beta < \gamma = \alpha$. For every case we have to do a similar calculation as we did for the case k constant in the interval $[0, \omega_1]$. For $\gamma < \beta < \alpha$ we get (for details see [22])

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = -\frac{2}{c_1}h^\beta - \frac{2}{c_2}h^\alpha + \mathcal{O}(h^{\min\{2\beta-\gamma, \beta+2\gamma\}}). \tag{3.10}$$

This shows that the leading order term of (3.10) can not vanish by an appropriate choice of γ and c_m, since it does not depend on them, and thus there is no local extremum for $\gamma < \beta < \alpha$. For $\beta < \alpha < \gamma$, we get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = \frac{4}{c_m^2}h^\gamma\left(c_1 h^{\gamma-\alpha} + c_2 h^{\gamma-\beta} + \mathcal{O}(h^{2\gamma-2\alpha})\right), \tag{3.11}$$

and again the leading order term can not vanish by an appropriate choice of γ and c_m; we can not have a local extremum either. For the case $k = c_m/h^\gamma$ and $\beta < \gamma < \alpha$, we get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = \frac{4c_2}{c_m^2}h^{2\gamma-\beta} - \frac{2}{c_1}h^\alpha + \mathcal{O}(h^{\min\{2\alpha-\gamma, \alpha+\gamma-\beta, 3\gamma-2\beta\}}). \tag{3.12}$$

Choosing $\gamma = (\alpha + \beta)/2$, we obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = \frac{4c_1c_2 - 2c_m^2}{c_m^2 c_1}h^\alpha + \mathcal{O}(h^{(3\alpha-\beta)/2}),$$

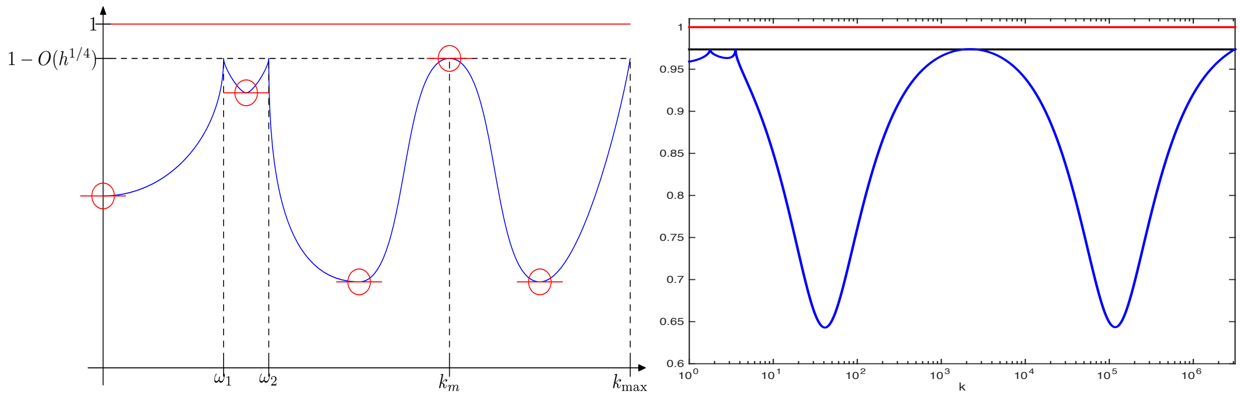

FIGURE 1. *Left*: drawing of the optimized convergence factor ρ_{opt} studied in Theorem 3.1. *Right*: actual plot of the optimized convergence factor from Theorem 3.1 for $\mu_1 = \mu_2 = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 4$, $\omega = \pi$ and $h = 10^{-6}$. The red line is at 1 and the black line is the asymptotic maximum $(1 - \mathcal{O}(h^{\frac{1}{4}})$ for this case).

and hence the leading order term vanishes if we choose for the constant $c_m = \sqrt{2c_1c_2}$. We thus have a local extremum, and it is a maximum after studying the signs of the expressions above. For $\beta = \gamma < \alpha$ we get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k,\omega_1,\omega_2,C_1,C_2) = \left(\frac{2c_2(c_m^2 - 2c_2^2)}{(c_2^2 + (c_m + c_2)^2)^2}\right) h^\beta + \mathcal{O}(h^{\min\{\alpha,2\alpha-\beta\}}),$$

and the leading order term vanishes if $c_m = \sqrt{2}c_2$. We thus have a local extremum, but it is a local minimum after a study of the signs. Finally, for $k = \frac{c_m}{h^\alpha}$ in the case $\beta < \alpha = \gamma$, we obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k,\omega_1,\omega_2,C_1,C_2) = \frac{4c_1(c_m^2 - 2c_1^2)}{(c_1^2 + (c_m + c_1)^2)^2} h^\alpha + \mathcal{O}(h^{2\alpha-\beta}),$$

and the leading order term vanishes for $c_m = \sqrt{2}c_1$, which is however also a minimum after a study of the signs.

We therefore know now that asymptotically the possible maxima of the convergence factor are at $k = \omega_1, \omega_2, k_m$ and $k_{\max}$, as illustrated in Figure 1 by a drawing and an actually computed example.

4. First we show that $R_1(\omega_1,\omega_1,\omega_2,C_1,C_2) \simeq R_2(\omega_2,\omega_1,\omega_2,C_1,C_2)$. To compute the asymptotic expansions, we use (3.2) and a similar expression for R_2, see [22] for details. For $k = \omega_1$ we have $\tilde{\lambda}_1 = 0$ and $\tilde{\lambda}_2 = \sqrt{\omega_2^2 - \omega_1^2} =: r$. In order to simplify the notation we denote $R_1(k,\omega_1,\omega_2,C_1,C_2)$ by $R_1(k)$ unless we have to specify the other parameters. The asymptotic expansion of $R_1(k)$ for $k = \omega_1$ gives

$$\begin{aligned} R_1(\omega_1) &= \left(\tfrac{2c_1^2}{h^{2\alpha}}\right)\left(1 + \mathcal{O}(h^{2\alpha})\right)\left(\tfrac{2c_2^2}{h^{2\beta}}\right)\left(1 - \tfrac{r}{c_2}h^\beta + \mathcal{O}(h^{2\beta})\right) \\ &\quad\times \left(\tfrac{h^{2\beta}}{2c_2^2}\right)\left(1 + \mathcal{O}(h^{2\beta})\right)\left(\tfrac{h^{2\alpha}}{2c_1^2}\right)\left(1 - \tfrac{r}{c_1}h^\alpha + \mathcal{O}(h^{2\alpha})\right) \\ &= 1 - \tfrac{r}{c_2}h^\beta - \tfrac{r}{c_1}h^\alpha + \mathcal{O}(h^{2\beta}). \end{aligned} \tag{3.13}$$

For $k = \omega_2$ we have $\lambda_1 = \sqrt{\omega_2^2 - \omega_1^2} = r$ and $\tilde{\lambda}_2 = 0$, and we get

$$\begin{aligned} R_2(\omega_2) &= \left(\tfrac{2c_1^2}{h^{2\alpha}}\right)\left(1 - \tfrac{r}{c_1}h^\alpha + \mathcal{O}(h^{2\alpha})\right)\left(\tfrac{2c_2^2}{h^{2\beta}}\right)\left(1 + \mathcal{O}(h^{2\beta})\right) \\ &\quad\times \left(\tfrac{h^{2\beta}}{2c_2^2}\right)\left(1 - \tfrac{r}{c_2}h^\beta + \mathcal{O}(h^{2\beta})\right)\left(\tfrac{h^{2\alpha}}{2c_1^2}\right)\left(1 + \mathcal{O}(h^{2\alpha})\right) \\ &= 1 - \tfrac{r}{c_2}h^\beta - \tfrac{r}{c_1}h^\alpha + \mathcal{O}(h^{2\beta}). \end{aligned} \tag{3.14}$$

Hence (3.13) and (3.14) are asymptotically equal, and we only need to consider one of them to perform the optimization; we will use $R_1(\omega_1)$.

Now the possible maxima are $R_1(\omega_1,\omega_1,\omega_2,C_1,C_2)$, $R_3(k_m,\omega_1,\omega_2,C_1,C_2)$ and $R_3(k_{\max},\omega_1,\omega_2,C_1,C_2)$. For $R_3(k_m,\omega_1,\omega_2,C_1,C_2)$ we obtain with $k=k_m:=\frac{\sqrt{2c_1c_2}}{h^{\alpha/2+\beta/2}}$ the asymptotic expansion

$$\begin{aligned}(3.15)\quad R_3(k_m) &= \left(\tfrac{c_1^2}{h^{2\alpha}}\right)\left(1-\sqrt{\tfrac{2c_2}{c_1}}h^{\frac{\alpha}{2}-\frac{\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)\left(\tfrac{h^{\alpha+\beta}}{2c_1c_2}\right)\left(1-\sqrt{\tfrac{2c_2}{c_1}}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)\\ &\quad\times\left(1-\sqrt{\tfrac{2c_2}{c_1}}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)\left(\tfrac{h^{2\alpha}}{c_1^2}\right)\left(\tfrac{2c_1c_2}{h^{\alpha+\beta}}\right)\left(1-\sqrt{\tfrac{2c_2}{c_1}}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)\\ &= 1-4\sqrt{2}\sqrt{\tfrac{c_2}{c_1}}h^{\frac{\alpha}{2}-\frac{\beta}{2}}+\mathcal{O}(h^{\alpha-\beta}).\end{aligned}$$

For $R_3(k_{\max},\omega_1,\omega_2,C_1,C_2)$ we obtain with $k=\frac{c_{\max}}{h}$

$$\begin{aligned}(3.16)\quad R_3(k_{\max}) &= \left(\tfrac{c_m^2}{h^2}\right)\left(1-\tfrac{2c_1}{c_{\max}}h^{1-\alpha}+\mathcal{O}(h^{2-2\alpha})\right)\left(\tfrac{c_{\max}^2}{h^2}\right)\left(1-\tfrac{2c_2}{c_{\max}}h^{1-\beta}+\mathcal{O}(h^{2-2\beta})\right)\\ &\quad\times\left(\tfrac{h^{2\gamma}}{c_{\max}^2}\right)\left(1-\tfrac{2c_2}{c_{\max}}h^{1-\beta}+\mathcal{O}(h^{2-2\beta})\right)\left(\tfrac{h^2}{c_{\max}^2}\right)\left(1-\tfrac{2c_1}{c_{\max}}h^{1-\alpha}+\mathcal{O}(h^{2-2\alpha}\right)\\ &= 1-\tfrac{4c_1}{c_{\max}}h^{1-\alpha}+\mathcal{O}(h^{\min\{1-\beta,2-2\alpha\}}).\end{aligned}$$

We thus need to choose α and β to minimize the maximum of (3.13), (3.15) and (3.16), *i.e.* the maximum of

$$\begin{aligned}(3.17)\quad R_1(\omega_1,\omega_1,\omega_2,C_1,C_2) &= 1-\frac{r}{c_2}h^\beta+\mathcal{O}(h^{\min\{\alpha,2\beta\}}),\\ R_3(k_m,\omega_1,\omega_2,C_1,C_2) &= 1-4\sqrt{2}\sqrt{\frac{c_2}{c_1}}h^{\alpha/2-\beta/2}+\mathcal{O}(h^{\alpha-\beta}),\\ R_3(k_{\max},\omega_1,\omega_2,C_1,C_2) &= 1-\frac{4c_1}{c_{\max}}h^{1-\alpha}+\mathcal{O}(h^{\min\{1-\beta,2-2\alpha\}}).\end{aligned}$$

To make $R_1(\omega_1)$ small we need β small, and to make $R_3(k_m)$ small we need β large, which implies that equioscillation gives the minimum,

$$\beta=\alpha/2-\beta/2 \quad\Longleftrightarrow\quad 3\beta=\alpha.$$

Now to make $R_3(k_{\max})$ small we need α large and to make $R_3(k_m)$ small we need α small, which implies again equioscillation for the minimum,

$$1-\alpha=\alpha/2-\beta/2 \quad\Leftrightarrow\quad 3\alpha-\beta=2.$$

The two equations thus imply $\alpha=3/4$ and $\beta=1/4$. We show an example of the three functions whose maximum we minimize in Figure 2, where the minimizing point is clearly visible. Using the same argument for equioscillation, we can also determine the constants, and find

$$c_1^*=\left(\frac{r}{2}\right)^{\frac{1}{4}}c_{\max}^{\frac{3}{4}},\quad c_2^*=\frac{1}{2}\left(\frac{r}{2}\right)^{\frac{3}{4}}c_{\max}^{\frac{1}{4}}.$$

Hence the asymptotic solution of the min-max problem (2.10) for $\alpha>\beta$ is

$$\begin{aligned}C_1^* &= \left(\frac{r}{2}\right)^{\frac{1}{4}}\left(\frac{c_{\max}}{h}\right)^{\frac{3}{4}},\ C_2^*=\frac{1}{2}\left(\frac{r}{2}\right)^{\frac{3}{4}}\left(\frac{c_{\max}}{h}\right)^{\frac{1}{4}},\\ \rho_{\text{opt}}^* &= 1-\left(\frac{r}{2c_{\max}}\right)^{\frac{1}{4}}h^{\frac{1}{4}}+\mathcal{O}(h^{\frac{1}{2}}).\end{aligned}$$

Case II ($\beta=\alpha$): In this case we have $C_1=\frac{c_1}{h^\alpha}$ and $C_2=\frac{c_2}{h^\alpha}$ and we follow the same steps as in the case $\alpha>\beta$.

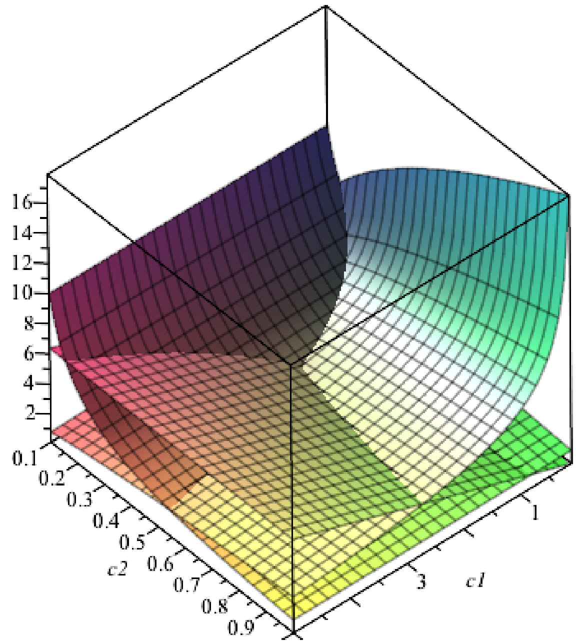

FIGURE 2. The three functions $R_1(\omega_1)$, $R_3(k_m)$ and $R_3(k_{\max})$ with the minimizing point in the middle.

1. For the first interval $[0, \omega_1]$ we obtain from (3.6) for $\alpha = \beta$

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = k\left[\left(\frac{1}{c_1} + \frac{1}{c_2}\right)\left(\frac{1}{\tilde{\lambda}_1} + \frac{1}{\tilde{\lambda}_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha})\right],$$

which shows that the only extremum at $k = 0$ is a minimum. For the second interval $[\omega_1, \omega_2]$ we get from (3.7) for $\alpha = \beta$

$$\frac{\mathrm{d}R_2}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = k\left[\left(\frac{1}{c_1} + \frac{1}{c_2}\right)\left(\frac{\lambda_1 - \tilde{\lambda}_2}{\lambda_1\tilde{\lambda}_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha})\right].$$

This shows that we have a local extremum if $\lambda_1 = \tilde{\lambda}_2$, *i.e.* $k = k_1 := \frac{\omega_1^2+\omega_2^2}{2}$, and a further study of the signs reveals that it is a local minimum. For the third interval $[\omega_2, k_{\max}]$ we get from (3.8)

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = -k\left[\left(\frac{1}{c_1} + \frac{1}{c_2}\right)\left(\frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha})\right],$$

which shows that R_3 is decreasing in the third interval.

2. For extrema close to 0 we set $k = c_m h^\gamma$, with $\gamma > 0$, and get from (3.9)

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = c_m\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\left(\frac{1}{c_1} + \frac{1}{c_2}\right)h^{\alpha-\gamma} + \mathcal{O}(h^{\min\{\alpha+\gamma, 2\alpha\}}),$$

which shows that R_1 is asymptotically increasing close to 0.

3. Now we classify the extrema of the form $k = c_m/h^\gamma$, with $0 < \gamma \leq 1$. Here, we only have 3 cases ($\gamma < \alpha$, $\gamma = \alpha$ and $\alpha < \gamma$). For $\gamma < \alpha$, we obtain from (3.10)

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = -2\left(\frac{1}{c_1} + \frac{1}{c_2}\right)h^\alpha + \mathcal{O}(h^{\min\{2\alpha-\gamma, \alpha+2\gamma\}}),$$

and thus R_3 is decreasing. For $\alpha < \gamma$, using (3.11) we obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k\omega_1, \omega_2, C_1, C_2) = \left(\frac{4}{c_m^2}\right)(c_1 + c_2)h^{\gamma-\alpha} + \mathcal{O}(h^{3\gamma-2\alpha}),$$

and therefore R_3 is increasing. Finally for $\alpha = \gamma$, we perform the calculations as in the case $[0, \omega_1]$ for $\alpha > \beta$ (for details see [22]), and obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = \frac{4(c_1+c_2)(c_m^2 - 2c_1c_2)(c_m^4 - 2c_m^2(c_1-c_2)^2 + 4c_1^2c_2^2)}{(c_m^2 + 2c_mc_1 + 2c_1^2)^2(c_m^2 + 2c_mc_2 + 2c_2^2)^2}h^\alpha + \mathcal{O}(h^*), \tag{3.18}$$

with $*$ a term greater than α. The leading order term of (3.18) thus vanishes for three positive values of c_m, namely

$$\begin{aligned} c_{m_1} &= \sqrt{(c_1-c_2)^2 - \sqrt{(c_1^2+c_2^2)(c_1^2 - 4c_1c_2 + c_2^2)}},\\ c_{m_2} &= \sqrt{2c_1c_2},\\ c_{m_3} &= \sqrt{(c_1-c_2)^2 + \sqrt{(c_1^2+c_2^2)(c_1^2 - 4c_1c_2 + c_2^2)}}. \end{aligned}$$

One can verify that these solutions are in increasing order if they are real (which means that $c_1^2 - 4c_1c_2 + c_2^2 \geq 0$). A sign study of (3.18) implies that we have a maximum at $k = k_m := \frac{c_{m_2}}{h^\alpha} = \frac{\sqrt{2c_1c_2}}{h^\alpha}$ when c_{m_1} and c_{m_3} are real and only a minimum at $k = k_m$ when c_{m_1} and c_{m_3} are complex.

4. From the previous analysis we know that the candidates for the optimization are $k = \omega_1$, $k = k_m = \frac{\sqrt{2c_1c_2}}{h^\alpha}$ and $k = k_{\max} = \frac{c_{\max}}{h}$. From (3.13) for the case $\alpha = \beta$ we get

$$R_1(\omega_1, \omega_1, \omega_2, C_1, C_2) = 1 - r\left(\frac{1}{c_1} + \frac{1}{c_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha}). \tag{3.19}$$

Similarly from (3.16), we have

$$R_3(k_{\max}, \omega_1, \omega_2, C_1, C_2) = 1 - \frac{4}{c_{\max}}(c_1 + c_2)h^{1-\alpha} + \mathcal{O}(h^{2-2\alpha}). \tag{3.20}$$

For $k_m = \frac{\sqrt{2c_1c_2}}{h^\alpha}$, we use the asymptotic expansions we computed, and obtain after simplifying

$$R_3(k_m, \omega_1, \omega_2, C_1, C_2) = \left(\frac{(c_1 - \sqrt{2c_1c_2} + c_2)^2}{(c_1 + \sqrt{2c_1c_2} + c_2)^2}\right)(1 + \mathcal{O}(h^{2\alpha})). \tag{3.21}$$

The leading order term of (3.21) does not dependent on h, and it is not difficult to verify that it is smaller than 1 for any $c_1, c_2 > 0$. Then the only candidates for the optimization are (3.19) and (3.20). Comparing equations (3.19) and (3.20) we see that α and $1-\alpha$ are in competition and we thus have to equilibrate them (*i.e.* $\alpha = 1-\alpha$ which implies $\alpha = 1/2$). This gives the order of the convergence factor $\rho^* = 1 - \mathcal{O}(h^{\frac{1}{2}})$ which is worse than Case I ($\beta < \alpha$) studied before. So we can exclude this case as solution of the optimization problem.

Case III ($\beta > \alpha$): This case is completely symmetric to $\beta < \alpha$, just the roles of c_1 and c_2 and the roles of α and β are exchanged. The solution then gives the second part of the theorem. □

4. Transmission conditions for the TEz mode

We now study transmission conditions for the TEz mode for the physically important case $\mu_1 = \mu_2$ and $\varepsilon_1 \neq \varepsilon_2$. Using Remark 2.1, one can also obtain an equivalent result for the TEz mode for the case $\mu_1 \neq \mu_2$ and $\varepsilon_1 = \varepsilon_2$, which was announced without proof and in less general form in [10], and used in [11] to obtain results for the 3D case.

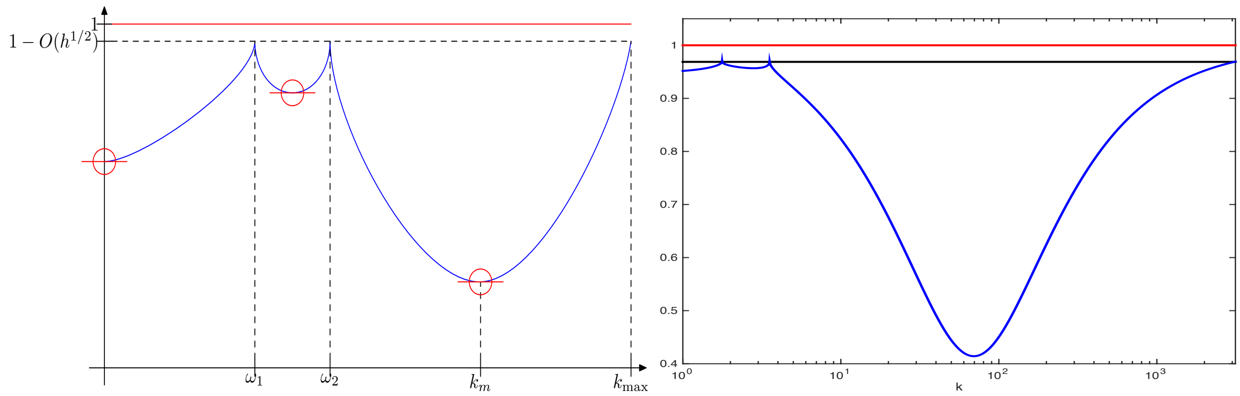

FIGURE 3. *Left*: drawing of the optimized convergence factor ρ_{opt} studied in Theorem 3.1 for the case $\alpha = \beta$. *Right*: actual plot of the optimized convergence factor from Theorem 3.1 for $\mu_1 = \mu_2 = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 4$, $\omega = \pi$ and $h = 10^{-3}$. The red line is at 1 and the black line is the asymptotic maximum $(1 - \mathcal{O}(h^{\frac{1}{2}})$ for this case).

Theorem 4.1. *If $\mu_1 = \mu_2$, $\varepsilon_1 \neq \varepsilon_2$, $s_1 = (1+i)C_1$, $s_2 = (1+i)C_2$, $r = \sqrt{|\omega_2^2 - \omega_1^2|}$, $\varepsilon = \sqrt{\varepsilon_1/\varepsilon_2}$ and $c_{\max}$ is given by the relationship $k_{\max} = \frac{c_{\max}}{h}$, we have the following results:*

- *If $0 < \varepsilon^2 \leq \frac{1}{2}$, then the asymptotic solution of the min-max problem (2.12) for h small is given by*

$$C_1^* \geq \frac{c_{\max}\varepsilon\left(\sqrt{3+4\varepsilon}-1-2\varepsilon\right)}{2h(1-2\varepsilon^2)}, \quad C_2^* = r, \quad \rho_{opt}^* = \sqrt[4]{\frac{1}{2}} + \mathcal{O}(h). \tag{4.1}$$

- *If $\frac{1}{2} < \varepsilon^2 < 1$, then the asymptotic solution of the min-max problem (2.12) for h small is given by*

$$C_1^* \geq \frac{c_{\max}(1-\varepsilon)}{2h}, \quad \frac{r}{1+\sqrt{2\varepsilon^2-1}} \leq C_2^* \leq \frac{r}{1-\sqrt{2\varepsilon^2-1}}, \quad \rho_{opt}^* = \sqrt{\varepsilon} + \mathcal{O}(h). \tag{4.2}$$

- *If $1 < \varepsilon^2 \leq 2$, then the asymptotic solution of the min-max problem (2.12) for h small is given by*

$$\frac{r\varepsilon}{\varepsilon+\sqrt{2-\varepsilon^2}} \leq C_1^* \leq \frac{r\varepsilon}{\varepsilon-\sqrt{2-\varepsilon^2}}, \quad C_2^* \geq \frac{c_{\max}(\varepsilon-1)}{2h\varepsilon}, \quad \rho_{opt}^* = \frac{1}{\sqrt{\varepsilon}} + \mathcal{O}(h). \tag{4.3}$$

- *If $\varepsilon^2 \geq 2$, then the asymptotic solution of the min-max problem (2.12) for h small is given by*

$$C_1^* = r, \quad C_2^* \geq \frac{c_{\max}}{2h}\frac{\sqrt{3\varepsilon^2+4\varepsilon}-\varepsilon-2}{\varepsilon^2-2}, \quad \rho_{opt}^* = \sqrt[4]{\frac{1}{2}} + \mathcal{O}(h). \tag{4.4}$$

Remark 4.2. A more general form of Theorem 4.1 is presented in [22] for the TMz case with $\mu_1 \neq \mu_2$ and $\omega_1 \neq \omega_2$ which guarantees convergence independently of the mesh size h. Similarly, there is a more general form of Theorem 4.1 for $\varepsilon_1 \neq \varepsilon_2$ and $\omega_1 \neq \omega_2$ that guarantees also convergence independently of the mesh size h. Combining both results we have (for $\mu_1 \neq \mu_2$, $\varepsilon_1 \neq \varepsilon_2$ and $\omega_1 \neq \omega_2$) a non-overlapping Optimized Schwarz Method applied to the complete Maxwell system in 3D that converges independently of the mesh size h. The condition $\mu_1 \neq \mu_2$, $\varepsilon_1 \neq \varepsilon_2$ and $\omega_1 \neq \omega_2$ is usually verified if we consider two different materials, nevertheless for the case $\mu_1 = \mu_2$ and $\varepsilon_1 \neq \varepsilon_2$ the complete Maxwell system in 3D has a contraction factor of $1 - \mathcal{O}(h^{\frac{1}{4}})$, as in the continuous case studied in [7].

Proof. We proceed as in the proof of Theorem 3.1: we use the ansatz $C_1 := \frac{c_1}{h^\alpha}$ and $C_2 := \frac{c_2}{h^\beta}$, divide the proof into Case I: $\beta < \alpha$, Case II: $\beta = \alpha$, and Case III: $\beta > \alpha$, and perform the four steps to identify and balance maxima.

Case I ($\beta < \alpha$):

1. First we show asymptotically that there is only one local extremum for $k = c$ constant, proceeding as in Theorem 3.1 for the case $\alpha > \beta$, for details see [22]. We obtain for the derivative for $k \in (0, \omega_1)$

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k, \omega_1, \omega_2, \varepsilon, C_1, C_2) = k\left(\frac{1}{\tilde{\lambda}_1} + \frac{\varepsilon}{\tilde{\lambda}_2}\right)\left[\frac{1}{\varepsilon c_2}h^\beta + \frac{1}{c_1}h^\alpha + \mathcal{O}(h^{2\beta})\right]. \tag{4.5}$$

We thus have a local extremum for $k = 0$, and $\frac{\mathrm{d}R_1}{\mathrm{d}k} > 0$, $\forall k \in (0, \omega_1)$. So there are no other local extrema for fixed k, and R_1 is increasing in $(0, \omega_1)$, which implies that $k = 0$ is a minimum, and the maximum is at $k = \omega_1$. For $k \in (\omega_1, \omega_2)$ we get

$$\frac{\mathrm{d}R_2}{\mathrm{d}k}(k, \omega_1, \omega_2, C_1, C_2) = k\left(\frac{\lambda_1\varepsilon - \tilde{\lambda}_2}{\lambda_1\tilde{\lambda}_2}\right)\left[\frac{1}{\varepsilon c_2}h^\beta + \frac{1}{c_1}h^\alpha + \mathcal{O}(h^{2\beta})\right]. \tag{4.6}$$

Hence the leading order term vanishes if $\tilde{\lambda}_2 = \lambda_1\varepsilon$, which means $k = \sqrt{\frac{\omega_1^2\varepsilon^2+\omega_2^2}{1+\varepsilon^2}}$. If $k < \sqrt{\frac{\omega_1^2\varepsilon^2+\omega_2^2}{1+\varepsilon^2}}$ we have $\lambda_1\varepsilon < \tilde{\lambda}_2$ and thus $\frac{\mathrm{d}R_2}{\mathrm{d}k} < 0$. If $k > \sqrt{\frac{\omega_1^2\varepsilon^2+\omega_2^2}{1+\varepsilon^2}}$ we have $\lambda_1\varepsilon > \tilde{\lambda}_2$ which gives $\frac{\mathrm{d}R_2}{\mathrm{d}k} > 0$. These three conditions imply that we have a local minimum. The maximum of the interval is therefore necessarily either at $k = \omega_1$ or $k = \omega_2$. For $k \in (\omega_2, k_{\max})$, we finally get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k, \omega_1, \omega_2, \varepsilon, C_1, C_2) = -k\left(\frac{\varepsilon\lambda_1 + \lambda_2}{\lambda_1\lambda_2}\right)\left[\frac{1}{\varepsilon c_2}h^\beta + \frac{1}{c_1}h^\alpha + \mathcal{O}(h^{2\beta})\right]. \tag{4.7}$$

Since $\frac{\lambda_1\varepsilon+\lambda_2}{\lambda_1\lambda_2} > 0$ for $k \in (\omega_2, k_{\max})$ we deduce that R_3 does not have extrema for fixed k in $[\omega_2, k_{\max}]$, and the sign of the derivative shows that R_3 is decreasing for $k > \omega_2$. Therefore, for fixed k in $[0, k_{\max}]$, the maxima of ρ_{opt} are at $k = \omega_1$ and $k = \omega_2$, for an illustration, see Figures 4 and 5.

2. Now we show that there is no other extremum close to $k = 0$. Proceeding as in the proof of Theorem 3.1, we set $k = c_m h^\gamma$, with $\gamma > 0$ and obtain after some computations (for details see [22])

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(c_m h^\gamma, \omega_1, \omega_2, \varepsilon, C_1, C_2) = \left(\frac{c_m(\omega_1 + \varepsilon\omega_2)}{\omega_1\omega_2}h^\gamma\right)\left[\frac{h^\beta}{\varepsilon c_2} + \frac{h^\alpha}{c_1} + \mathcal{O}(h^*)\right], \tag{4.8}$$

with $*$ a term bigger than γ. Clearly the leading order term can not vanish for $c_m > 0$ and $\gamma > 0$, so we do not have an extremum close to $k = 0$.

3. Proceeding as in the proof of Theorem 3.1, we now study possible extrema for $k = c_m/h^\gamma$, with $0 < \gamma \leq 1$. We have to consider five sub-cases: $\gamma < \beta < \alpha$, $\beta < \gamma < \alpha$, $\beta < \alpha < \gamma$, and the two particular cases $\beta = \gamma < \alpha$ and $\beta < \gamma = \alpha$. In order to simplify the notation we use again $R_3(k)$ to denote $R_3(k, \omega_1, \omega_2, \varepsilon, C_1, C_2)$. For the case $\gamma < \beta < \alpha$, we obtain for the derivative

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = -\frac{(1+\varepsilon)}{\varepsilon c_2}h^\beta - \frac{(1+\varepsilon)}{c_1}h^\alpha + \mathcal{O}(h^{\min\{\beta+2\gamma,\, 2\beta-\gamma,\}}), \tag{4.9}$$

which shows that the leading order term can not vanish for a particular choice of γ and c_m, since they are not present in the leading order term. Hence there is no local extremum for $\gamma < \beta < \alpha$. For the case $\beta < \alpha < \gamma$, we get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = \left(\frac{2(1+\varepsilon)}{c_m^2}h^\gamma\right)\left(\frac{c_1}{\varepsilon}h^{\gamma-\alpha} + c_2 h^{\gamma-\beta} + \mathcal{O}(h^{2\gamma-2\alpha})\right). \tag{4.10}$$

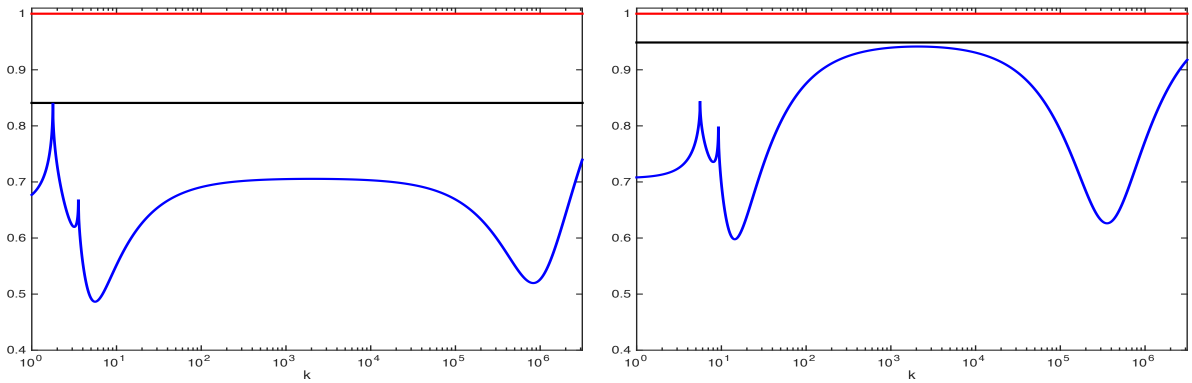

FIGURE 4. Convergence factor ρ_{opt} from the first and the second case of Theorem 4.1 ($\varepsilon_1 < \varepsilon_2$). The red line denotes 1, the black line is the asymptotic maximum: $\sqrt[4]{\frac{1}{2}}$ for the left case with $\mu_1 = \varepsilon_1 = \mu_2 = 1$, $\varepsilon_2 = 2$, $\omega = \pi$ and $h = 10^{-6}$, and ε for the right case with $\varepsilon_1 = 9$, $\varepsilon_2 = \mu_1 = \mu_2 = 10$, $\omega = \pi$ and $h = 10^{-6}$. Convergence does not depend on the mesh size h.

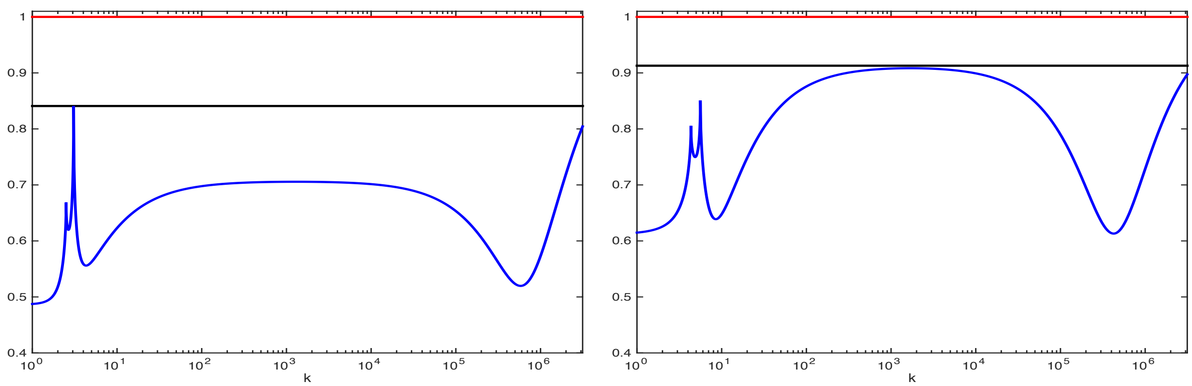

FIGURE 5. Convergence factor ρ_{opt} from the third and fourth case of Theorem 4.1 ($\varepsilon_2 < \varepsilon_1$). The red line is 1, the black line is the asymptotic maximum: $\sqrt[4]{\frac{1}{2}}$ for the left case with $\mu_1 = \varepsilon_2 = \mu_2 = 1$, $\varepsilon_1 = 2$, $\omega = \pi$ and $h = 10^{-6}$ and ε for the right case with $\varepsilon_1 = 6$, $\varepsilon_2 = 5$, $\mu_1 = 1$, $\mu_2 = 1$, $\omega = \pi$ and $h = 10^{-6}$. Convergence does not depend on the mesh size h.

Again the leading order term can not vanish, and hence there is no local extremum for k of the form $k = c_m/h^\gamma$ and $\beta < \alpha < \gamma$. We further see that R_3 is increasing in this case. For the case $\beta < \gamma < \alpha$, we find

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = \frac{2\varepsilon^2(1+\varepsilon)c_2}{c_m^2}h^{2\gamma-\beta} - \frac{\varepsilon^2(1+\varepsilon)}{c_1}h^\alpha + \mathcal{O}(h^*), \tag{4.11}$$

with $* = \min\{2\alpha - \gamma, \alpha + \gamma - \beta, 3\gamma - 2\beta\}$. The leading order term can thus vanish if we have $\gamma = (\alpha + \beta)/2$, and rewriting (4.11) for this choice, we obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = \frac{\varepsilon^2(1+\varepsilon)(2c_1c_2 - c_m^2)}{c_m^2 c_1}h^\alpha + \mathcal{O}(h^{(3\alpha-\beta)/2}), \tag{4.12}$$

which shows that the leading order term will be zero if $c_m = \sqrt{2c_1c_2}$. We thus have a local extremum for $\beta < \gamma < \alpha$ and $k = \frac{c_m}{h^\gamma}$ when $\gamma = (\alpha+\beta)/2$ and $c_m = \sqrt{2c_1c_2}$, and it is a maximum after a further study of the signs in (4.12). For $\beta = \gamma < \alpha$, we get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k)=\left(\frac{-2\varepsilon^2 c_2(c_m^2-2(c_m-c_2)(c_m+\varepsilon c_2))}{(\varepsilon^2c_2^2+(c_m+\varepsilon c_2)^2)^2}\right)h^\beta-\left(\frac{\varepsilon^2(1+\varepsilon)(c_2^2+(c_m-c_2)^2)}{c_1(\varepsilon^2c_2^2+(c_m+\varepsilon c_2)^2)}\right)h^\alpha+\mathcal{O}(h^{2\alpha-\beta}), \tag{4.13}$$

and the leading order term vanishes for $c_m = (1-\varepsilon+\sqrt{1+\varepsilon^2})c_2$, which leads however to a local minimum asymptotically at $k = \frac{(1-\varepsilon+\sqrt{1+\varepsilon^2})c_2}{h^\beta}$ after a further sign study. Finally we look for extrema when $k = \frac{c_m}{h^\alpha}$, which leads for the case $\beta < \alpha = \gamma$ to

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = \left(\frac{2\varepsilon^2c_1(1+\varepsilon)(\varepsilon c_m^2+2c_1(1-\varepsilon)c_m-2c_1^2)}{(c_1^2+(\varepsilon c_m+c_1)^2)^2}\right)h^\alpha + \mathcal{O}(h^{2\alpha-\beta}). \tag{4.14}$$

The leading order term vanishes for $c_m = \frac{1-\varepsilon+\sqrt{1+\varepsilon^2}}{\varepsilon}c_1$, we thus have asymptotically an extremum for $k = \frac{(1-\varepsilon+\sqrt{1+\varepsilon^2})c_1}{\varepsilon h^\alpha}$, which turns out however also to be a minimum.

4. We have again identified four candidates for the maximum. We start with the asymptotic expansions of $R_1(\omega_1,\omega_1,\omega_2,\varepsilon,C_1,C_2)$ and $R_2(\omega_2,\omega_1,\omega_2,\varepsilon,C_1,C_2)$, and we classify these two expressions as functions of the value of ε. We have

$$\begin{aligned} R_1(\omega_1) &= \frac{\left(\frac{2c_1^2}{h^{2\alpha}}\right)\left(1-\frac{r}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\left(\frac{2c_2^2}{h^{2\beta}}\right)\left(1-\frac{r}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)}{\left(\frac{2\varepsilon^2c_2^2}{h^{2\beta}}\right)\left(1+\frac{r}{\varepsilon c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(\frac{2c_1^2}{\varepsilon^2h^{2\alpha}}\right)\left(1+\frac{\varepsilon r}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)} \\ &= 1-\frac{r}{c_2}h^\beta-\frac{\varepsilon r}{c_1}h^\alpha+\mathcal{O}(h^{2\beta}), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} R_2(\omega_2) &= \frac{\left(\frac{2c_1^2}{h^{2\alpha}}\right)\left(1-\frac{r}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)\left(\frac{2c_2^2}{h^{2\beta}}\right)\left(1-\frac{r}{c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)}{\left(\frac{2\varepsilon^2c_2^2}{h^{2\beta}}\right)\left(1+\frac{r}{\varepsilon c_2}h^\beta+\mathcal{O}(h^{2\beta})\right)\left(\frac{2c_1^2}{\varepsilon^2h^{2\alpha}}\right)\left(1+\frac{r\varepsilon}{c_1}h^\alpha+\mathcal{O}(h^{2\alpha})\right)} \\ &= 1-\frac{r}{\varepsilon c_2}h^\beta-\frac{r}{c_1}h^\alpha+\mathcal{O}(h^{2\beta}), \end{aligned} \tag{4.16}$$

which shows that $R_1(\omega_1) < R_2(\omega_2)$ if $\varepsilon > 1$ and $R_1(\omega_1) > R_2(\omega_2)$ if $\varepsilon < 1$. Note also that $R_1(\omega_1)$ and $R_2(\omega_2)$ depend asymptotically on c_2, and not on c_1. We next derive the asymptotic expansions of $R_3(k_m,\omega_1,\omega_2,\varepsilon,C_1,C_2)$ and $R_3(k_{\max},\omega_1,\omega_2,\varepsilon,C_1,C_2)$, and obtain

$$\begin{aligned} R_3(k_{\max}) &= \frac{\left(\frac{c_{\max}^2}{h^2}\right)\left(1-\frac{2c_1}{c_{\max}}h^{1-\alpha}+\mathcal{O}(h^{2-2\alpha})\right)\left(\frac{c_{\max}^2}{h^2}\right)\left(1-\frac{2c_2}{c_{\max}}h^{1-\beta}+\mathcal{O}(h^{2-2\beta})\right)}{\left(\frac{c_{\max}^2}{h^2}\right)\left(1+\frac{2\varepsilon c_2}{c_{\max}}h^{1-\beta}+\mathcal{O}(h^{2-2\beta})\right)\left(\frac{c_{\max}^2}{h^2}\right)\left(1+\frac{2c_1}{\varepsilon c_{\max}}h^{1-\alpha}+\mathcal{O}(h^{2-2\alpha})\right)} \\ &= 1-\frac{2c_1}{\varepsilon c_{\max}}(1+\varepsilon)h^{1-\alpha}-\frac{2\varepsilon c_2}{c_{\max}}(1+\varepsilon)h^{1-\beta}+\mathcal{O}(h^{2-2\alpha}), \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} R_3(k_m) &= \frac{\left(\frac{c_1^2}{h^{2\alpha}}\right)\left(1-\frac{2c_m}{c_1}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)\left(\frac{c_m^2}{h^{2\gamma}}\right)\left(1-\frac{2c_2}{c_m}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)}{\left(\frac{c_m^2}{h^{2\gamma}}\right)\left(1+\frac{2\varepsilon c_2}{c_m}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)\left(\frac{c_1^2}{\varepsilon^2h^{2\alpha}}\right)\left(1+\frac{2\varepsilon c_m}{c_1}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right)} \\ &= \varepsilon^2\left(1-3\sqrt{2}(1+\varepsilon)\sqrt{\frac{c_2}{c_1}}h^{\frac{\alpha-\beta}{2}}+\mathcal{O}(h^{\alpha-\beta})\right). \end{aligned} \tag{4.18}$$

We observe from (4.18) that the leading order term of $R_3(k_m)$ does not depend on α, β, c_1 and c_2, and can thus not be optimized in this case, and additionally $R_3(k_m) < 1$ if and only if $\varepsilon < 1$. We thus consider for the rest of the case $\beta < \alpha$ that we have $\varepsilon < 1$ (the case $\varepsilon > 1$ will be considered when we study the case $(\alpha < \beta)$).

From (4.15) and (4.17) we observe that the leading order terms are not dependent on the same variables (*i.e.* $1-\mathcal{O}(h^\beta)$ and $1-\mathcal{O}(h^{1-\alpha})$). This suggest to set $\alpha = 1$ and $\beta = 0$ to obtain terms independent of the mesh size h, since all the other choices will lead to mesh dependence. We thus recompute for $\alpha = 1$ and $\beta = 0$ the expansions of $R_1(\omega_1)$, $R_3(k_m)$, and $R_3(k_{\max})$, and obtain

$$\begin{aligned} R_1(\omega_1) &= \left(\tfrac{1}{2}\tfrac{2c_2^2-2rc_2+r^2}{c_2^2}\right)\left(1-\tfrac{\varepsilon r}{c_1}h+\mathcal{O}(h^2)\right),\\ R_3(k_m) &= \varepsilon^2\left(1-3\tfrac{\sqrt{c_1c_2}(1+\varepsilon)\sqrt{h}}{c_1}+\mathcal{O}(h)\right),\\ R_3(k_{\max}) &= \left(\tfrac{\varepsilon^2(c_{\max}^2-2c_1c_{\max}+2c_1^2)}{\varepsilon^2c_{\max}^2+2c_1c_{\max}\varepsilon+2c_1^2}\right)\left(1-\tfrac{2c_2(\varepsilon+1)}{c_{\max}}h+\mathcal{O}(h^2)\right). \end{aligned} \tag{4.19}$$

Note that the recomputation is really necessary, because the earlier expansions do not hold for $\beta = 0$ and $\alpha = 1$. Similarly we also check that with $\beta = 0$ and $\alpha = 1$ we still only have the same candidates for the maximum points as with $0 < \beta < \alpha < 1$, for more details, see [22].

We see from (4.19) that $R_3(k_m) \sim \varepsilon^2$, which can not be influenced any further with the remaining constants c_1 and c_2 we can choose. We thus try to minimize the leading order terms of $R_1(\omega_1)$ and $R_3(k_{\max})$ using c_2 and c_1. To do so, we check the asymptotic derivatives in c_1 and c_2,

$$\begin{aligned} \frac{\mathrm{d}R_1(\omega_1)}{dc_2} &\sim \frac{r(c_2-r)}{c_2^3},\\ \frac{\mathrm{d}R_3(k_{\max})}{dc_1} &\sim -2\frac{\varepsilon^2c_{\max}(\varepsilon+1)(c_{\max}^2\varepsilon-2c_1c_{\max}\varepsilon+2c_1c_{\max}-2c_1^2)}{(\varepsilon^2c_{\max}^2+2c_1c_{\max}\varepsilon+2c_1^2)^2}. \end{aligned}$$

The unique solution of $\frac{\mathrm{d}R_1(\omega_1)}{\mathrm{d}c_2} \sim 0$ is $\tilde{c}_2 = r$ and this is a minimum because $\frac{\mathrm{d}^2R_1}{\mathrm{d}c_2^2}(\omega_1, \tilde{c}_2) \sim \frac{1}{r^2}$. We have in this case

$$R_1(\omega_1, \omega_1, \omega_2, \varepsilon, C_1, (1+i)r) = \frac{1}{2} + \mathcal{O}(h).$$

Similarly the unique positive solution of $\frac{\mathrm{d}R_3(k_{\max})}{\mathrm{d}c_1} \sim 0$ is $\tilde{c}_1 := \frac{c_{\max}}{2}(1-\varepsilon+\sqrt{1+\varepsilon^2})$, which is also a minimum because second derivative is

$$\frac{\mathrm{d}^2R_3}{\mathrm{d}c_1^2}(k_{\max}, \tilde{c}_1) \sim 4\frac{\varepsilon^2(\varepsilon+1)\left(\sqrt{\varepsilon^2+1}\varepsilon^2+\varepsilon^2+\sqrt{\varepsilon^2+1}+1\right)}{c_{\max}^2\left(\varepsilon^2+\sqrt{\varepsilon^2+1}+1\right)^3} > 0.$$

At this minimum we have

$$R_3(k_{\max}, \omega_1, \omega_2, \varepsilon, (1+i)\tilde{c}_1/h, C_2) = \frac{\varepsilon^2(\sqrt{1+\varepsilon^2}-\varepsilon)}{1+\sqrt{1+\varepsilon^2}} + \mathcal{O}(h).$$

We now verify that for $0 < \varepsilon < 1$ we have asymptotically $R_3(k_{\max}, (1+i)\tilde{c}_1/h) < \frac{1}{2}$, which leads for c_1 to an entire interval in which it can be chosen such that $R_3(k_{\max}) < R_1(\omega_1)$, and thus the solution is not unique. To see why $R_3(k_{\max}, (1+i)\tilde{c}_1/h) < \frac{1}{2}$, we first check that the second factor in the numerator satisfies for $0 < \varepsilon < 1$ that $0 < \sqrt{1+\varepsilon^2} - \varepsilon < 1$, which can be obtained by noting that $\sqrt{1+\varepsilon^2} - \varepsilon$ is a decreasing function $((\sqrt{1+\varepsilon^2}-\varepsilon)' = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} - 1 < 0)$ and its maximum is thus attained at $\varepsilon = 0$. We then estimate for the denominator in $R_3(k_{\max}, (1+i)\tilde{c}_1/h)$ that $\frac{1}{1+\sqrt{2}} < \frac{1}{1+\sqrt{1+\varepsilon^2}} < \frac{1}{2}$, which implies $0 < R_3(k_{\max}) < \frac{1}{2}$ as claimed.

Since $R_3(k_m) \sim \varepsilon^2$ which can not be influenced, we can have two possible situations: either $\varepsilon^2 < 1/2$ and the convergence speed is limited by $R_1(\omega_1)$ at $\rho_{\mathrm{opt}}^4 = \frac{1}{2}$, or $\varepsilon^2 \geq 1/2$ and the convergence speed is limited by $\rho_{\mathrm{opt}}^4 = \varepsilon^2$ from $R_3(k_m)$. In the first case, the asymptotic solutions of the min-max problem (2.12) for h small are given by (4.1) in the statement of the theorem.

Now in the second case, $\varepsilon^2 \geq 1/2$, there exist two intervals, one for c_1 such that $R_3(k_{\max}, \omega_1, \omega_2, \varepsilon, C_1, C_2) < \varepsilon^2$ and one for c_2 such that $R_1(\omega_1, \omega_1, \omega_2, \varepsilon, C_1, C_2) < \varepsilon^2$. For $R_3(k_{\max}) \leq \varepsilon^2$ asymptotically, c_1 has to satisfy

$$\frac{\varepsilon^2(c_{\max}^2-2c_1c_{\max}+2c_1^2)}{\varepsilon^2c_{\max}^2+2c_1c_{\max}\varepsilon+2c_1^2} \leq \varepsilon^2 \quad \Longleftrightarrow \quad c_1 \geq \frac{c_{\max}(1-\varepsilon)}{2},$$

and for $R_1(\omega_1) \leq \varepsilon^2$ asymptotically, c_2 has to satisfy

$$\frac{2c_2^2 - 2rc_2 + r^2}{2c_2^2} \leq \varepsilon^2 \quad \Longleftrightarrow \quad \frac{r}{1+\sqrt{2\varepsilon^2-1}} \leq c_2 \leq \frac{r}{1-\sqrt{2\varepsilon^2-1}},$$

which leads to (4.2) in the statement of the theorem.

We now study the **Case** $\alpha = \beta$**:**

1. We proceed as in the case $\beta < \alpha$: for k constant, independent of h we can adapt the results from (4.5) to obtain

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(k,\omega_1,\omega_2,\varepsilon,C_1,C_2) = k\left(\frac{1}{\tilde{\lambda}_1} + \frac{\varepsilon}{\tilde{\lambda}_2}\right)\left(\frac{1}{c_1} + \frac{1}{\varepsilon c_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha}).$$

We see that $R_1(k)$ is an increasing function in $(0,\omega_1)$, and it thus has a minimum at $k=0$, and a maximum at $k=\omega_1$. Next we adapt the results from (4.6) for the interval $[\omega_1,\omega_2]$ to obtain

$$\frac{\mathrm{d}R_2}{\mathrm{d}k}(k,\omega_1,\omega_2,\varepsilon,C_1,C_2) = k\left(\frac{\lambda_1\varepsilon - \tilde{\lambda}_2}{\lambda_1\tilde{\lambda}_2}\right)\left(\frac{1}{c_1} + \frac{1}{\varepsilon c_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha}).$$

This function vanishes for $\lambda_1\varepsilon = \tilde{\lambda}_2$, which means for $k = \sqrt{\frac{\omega_1^2\varepsilon^2+\omega_2^2}{1+\varepsilon^2}}$, which turns out to be a local minimum after a sign study. The possible maximum of the interval must hence be at either $k=\omega_1$ or $k=\omega_2$. For the third interval $[\omega_2,k_{\max}]$ we can adapt the results in (4.7) to obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k,\omega_1,\omega_2,\varepsilon,C_1,C_2) = -k\left(\frac{1}{\lambda_1} + \frac{\varepsilon}{\lambda_2}\right)\left(\frac{1}{c_1} + \frac{1}{\varepsilon c_2}\right)h^\alpha + \mathcal{O}(h^{2\alpha}).$$

This shows that R_3 can not have a local extremum asymptotically for fixed k independent of h and $k \geq \omega_2$.

2. Now we consider $k = c_m h^\gamma$ with $\gamma > 0$, and adapting (4.8) leads to

$$\frac{\mathrm{d}R_1}{\mathrm{d}k}(c_m h^\gamma,\omega_1,\omega_2,\varepsilon,C_1,C_2) = c_m\left(\frac{1}{\omega_2} + \frac{\varepsilon}{\omega_1}\right)\left(\frac{1}{c_1} + \frac{1}{\varepsilon c_2}\right)h^{\alpha+\gamma} + \mathcal{O}(h^*),$$

with $* = \min\{2\alpha+\gamma, 3\gamma+\alpha\}$. The leading term can thus not vanish for any choice of c_m or γ either, and hence we do not have extrema dependent on h close to 0.

3. Now we have to study the situation when $k = \frac{c_m}{h^\gamma}$. For $\alpha = \beta$ we only have three cases to consider: $\gamma < \alpha$, $\gamma = \alpha$ and $\gamma > \alpha$. For $\gamma < \alpha$ we can adapt (4.9) to get

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k,\omega_1,\omega_2,\varepsilon,C_1,C_2) = -(1+\varepsilon)\left(\frac{1}{c_1} + \frac{1}{\varepsilon c_2}\right)h^\alpha + \mathcal{O}(h^{\min\{\alpha+2\gamma,2\gamma-\alpha\}}).$$

This shows that $R_3(k)$ is a decreasing function for $\gamma < \alpha$ and we do not have a local extremum. For $\alpha < \gamma$ we can also adapt (4.10) to obtain

$$\frac{\mathrm{d}R_3}{\mathrm{d}k}(k,\omega_1,\omega_2,\varepsilon,C_1,C_2) = \left(\frac{2(1+\varepsilon)}{c_m^2}\right)\left(\frac{c_1}{\varepsilon} + c_2\right)h^{2\gamma-\alpha} + \mathcal{O}(h^{3\gamma-2\alpha}).$$

Hence $R_3(k)$ is now an increasing function and we do not have an extremum either. The interesting case is $\gamma = \alpha$: here we have to recompute the asymptotic terms as we did in Theorem 3.1 in the case $\alpha > \beta$, see [22] for details, and obtain

$$\begin{aligned}\frac{\mathrm{d}R_3}{\mathrm{d}k}(k) = &\left[\frac{2\varepsilon^2 c_m(1+\varepsilon)(c_1+\varepsilon c_2)(c_m^2-2c_1c_2)}{(\varepsilon^2c_2^2+(c_m+\varepsilon c_2)^2)(c_1^2+(\varepsilon c_m+c_1)^2)}\right]\\ &\times\left[\varepsilon c_m^4 - 2(c_1+c_2)(c_1+\varepsilon^2c_2)c_m^3 + [-2(\varepsilon-1)(c_1-\varepsilon c_2)+\right.\\ &\left.+8\varepsilon c_1c_2]c_m^2 - 4c_1c_2(\varepsilon-1)(c_1-\varepsilon c_2)c_m + 4\varepsilon c_1^2c_2^2\right] + \mathcal{O}(h^{2\alpha}).\end{aligned} \tag{4.20}$$

The positive solutions for a vanishing leading order term are

$$\begin{aligned} c_{m_2} &= \sqrt{2c_1c_2}, \\ c_{m_{1,3}} &= \tfrac{1}{2\varepsilon}\Big[(\varepsilon-1)(c_1-\varepsilon c_2)+\sqrt{(\varepsilon^2+1)(c_1^2+\varepsilon^2c_2^2)}\pm \\ &\pm\Big(\Big((\varepsilon-1)(c_1-\varepsilon c_2)+\sqrt{(\varepsilon^2+1)(c_1^2+\varepsilon^2c_2^2)}\Big)^2-8\varepsilon^2c_1c_2\Big)^{\frac{1}{2}}\Big]. \end{aligned} \tag{4.21}$$

To show that the first solution $c_{m_2}=\sqrt{2c_1c_2}$ lies between c_{m_1} and c_{m_3} if they are real, we define

$$\begin{aligned} a &:= (\varepsilon-1)(c_1-\varepsilon c_2)+\sqrt{(\varepsilon^2+1)(c_1^2+\varepsilon^2c_2^2)}, \\ b &:= \sqrt{2c_1c_2}. \end{aligned}$$

To have c_{m_1} and c_{m_3} real we need $a\geq 2\varepsilon b$. We then rewrite the inequality $c_{m_1}\leq c_{m_2}\leq c_{m_3}$ as

$$\frac{a-\sqrt{a^2-4\varepsilon^2b^2}}{2\varepsilon}\leq b\leq \frac{a+\sqrt{a^2-4\varepsilon^2b^2}}{2\varepsilon}.$$

The first inequality follows from $\frac{a-\sqrt{a^2-4\varepsilon^2b^2}}{2\varepsilon}=\frac{4\varepsilon^2b^2}{2\varepsilon(a+\sqrt{a^2-4\varepsilon^2b^2})}$ and using that $a\geq 2\varepsilon b$, and the second inequality holds because $2\varepsilon b-a\leq 0\leq\sqrt{a^2-4\varepsilon^2b^2}$. Hence $c_m=\sqrt{2c_1c_2}$ is a maximum and the other two are minima in the case when they are real. We denote by $k_m:=\sqrt{\frac{2c_1c_2}{h}}$ this maximum point. This leads to the asymptotic value

$$\begin{aligned} R_3(k_m,\omega_1,\omega_2,\varepsilon,C_1,C_2) &= \left(\tfrac{c_1^2+(c_m-c_1)^2}{h^{2\alpha}}\right)\left(\tfrac{h^{2\alpha}}{\varepsilon^2c_2^2+(c_m+\varepsilon c_2)^2}\right) \\ &\quad\left(\tfrac{c_2^2+(c_m-c_2)^2}{h^{2\alpha}}\right)\left(\tfrac{\varepsilon^2h^{2\alpha}}{c_1^2+(\varepsilon c_m+c_1)^2}\right)+\mathcal{O}(h^{2\alpha}) \\ &= \left(\tfrac{\varepsilon^2(c_1^2+(c_m-c_1)^2)(c_2^2+(c_m-c_2)^2)}{(\varepsilon^2c_2^2+(c_m+\varepsilon c_2)^2))(c_1^2+(\varepsilon c_m+c_1)^2)}\right)+\mathcal{O}(h^{2\alpha}). \end{aligned} \tag{4.22}$$

We have therefore identified the possible maxima: $R_1(\omega_1)$, $R_2(\omega_2)$, $R_3(k_m)$ and $R_3(k_{\max})$, which are asymptotically given by

$$\begin{aligned} R_1(\omega_1,\omega_1,\omega_2,\varepsilon,C_1,C_2) &= 1-r\left(\tfrac{1}{c_2}+\tfrac{\varepsilon}{c_1}\right)h^{\alpha}+\mathcal{O}(h^{2\alpha}), \\ R_2(\omega_2,\omega_1,\omega_2,\varepsilon,C_1,C_2) &= 1-\tfrac{r}{\varepsilon}\left(\tfrac{1}{c_2}+\tfrac{\varepsilon}{c_1}\right)h^{\alpha}+\mathcal{O}(h^{2\alpha}), \\ R_3(k_{\max},\omega_1,\omega_2,\varepsilon,C_1,C_2) &= 1-\tfrac{2(1+\varepsilon)}{c_m}\left(\tfrac{c_1}{\varepsilon}+c_2\right)h^{1-\alpha}+\mathcal{O}(h^{2-2\alpha}), \\ R_3(k_m,\omega_1,\omega_2,\varepsilon,C_1,C_2) &= \left(\tfrac{\varepsilon^2(c_1^2+(c_m-c_1)^2)(c_2^2+(c_m-c_2)^2)}{(\varepsilon^2c_2^2+(c_m+\varepsilon c_2)^2))(c_1^2+(\varepsilon c_m+c_1)^2)}\right)+\mathcal{O}(h^{2\alpha}). \end{aligned} \tag{4.23}$$

We see that $R_1(\omega_1)$ and $R_2(\omega_2)$ are in competition with $R_3(k_{\max})$, and we thus have by the equioscillation principle $\alpha=1-\alpha$ and $\alpha=1/2$. Then

$$R_1(\omega_1,\omega_1,\omega_2,\varepsilon,C_1,C_2)\simeq 1-\mathcal{O}(h^{1/2})\simeq R_3(k_{\max},\omega_1,\omega_2,\varepsilon,C_1,C_2).$$

This is however asymptotically worse than the solution we found for the case $\alpha>\beta$ which was independent of h, and hence $\alpha=\beta$ can not lead to the optimal choice asymptotically.

We finally treat the **Case** $\alpha<\beta$: here we can use the symmetry of (2.9); we just have to note that if we consider $\alpha<\beta$ this is equivalent to exchange the constants c_1 and c_2 and we have to replace ε with ε^{-1}. The asymptotic calculation performed for the case $\beta<\alpha$ can then be transformed into this case, which concludes the proof.

□

5. Numerical Experiments

To illustrate our theoretical results, we perform now numerical experiments on the rectangular domain $\Omega = (-1,1)\times(0,1)$ decomposed into the two sub-domains $\Omega_1 = (-1,0)\times(0,1)$ and $\Omega_2 = (0,1)\times(0,1)$. We consider constant coefficients ε_j, μ_j with $j = 1,2$. We discretize the Maxwell equation for $\omega = 2\pi$ using a finite volume scheme with mesh size $h = \frac{1}{64}$ based on the classical Yee scheme, see *e.g.* [23]. We impose on the outer boundaries the impedance condition $\frac{\mathbf{E}}{Z_j}\times\mathbf{n}_j + \mathbf{n}_j\times(\mathbf{H}\times\mathbf{n}_j) = 0$, with $j = 1,2$. We then use as initial guess each of the Fourier modes $\sin(k\pi y)$ at the interface to see if the discretized algorithm on the bounded domain behaves in a comparable way to our analysis of the convergence factor. We see in Figure 6 a comparison between the theoretical and numerical convergence factors, and also an asymptotic performance study. In the top row, and at the bottom left, the plots show that the numerical convergence factor of the discretized problem on a bounded domain is well predicted by the theoretical convergence factor in Theorems 3.1 and 4.1. On the bottom right,

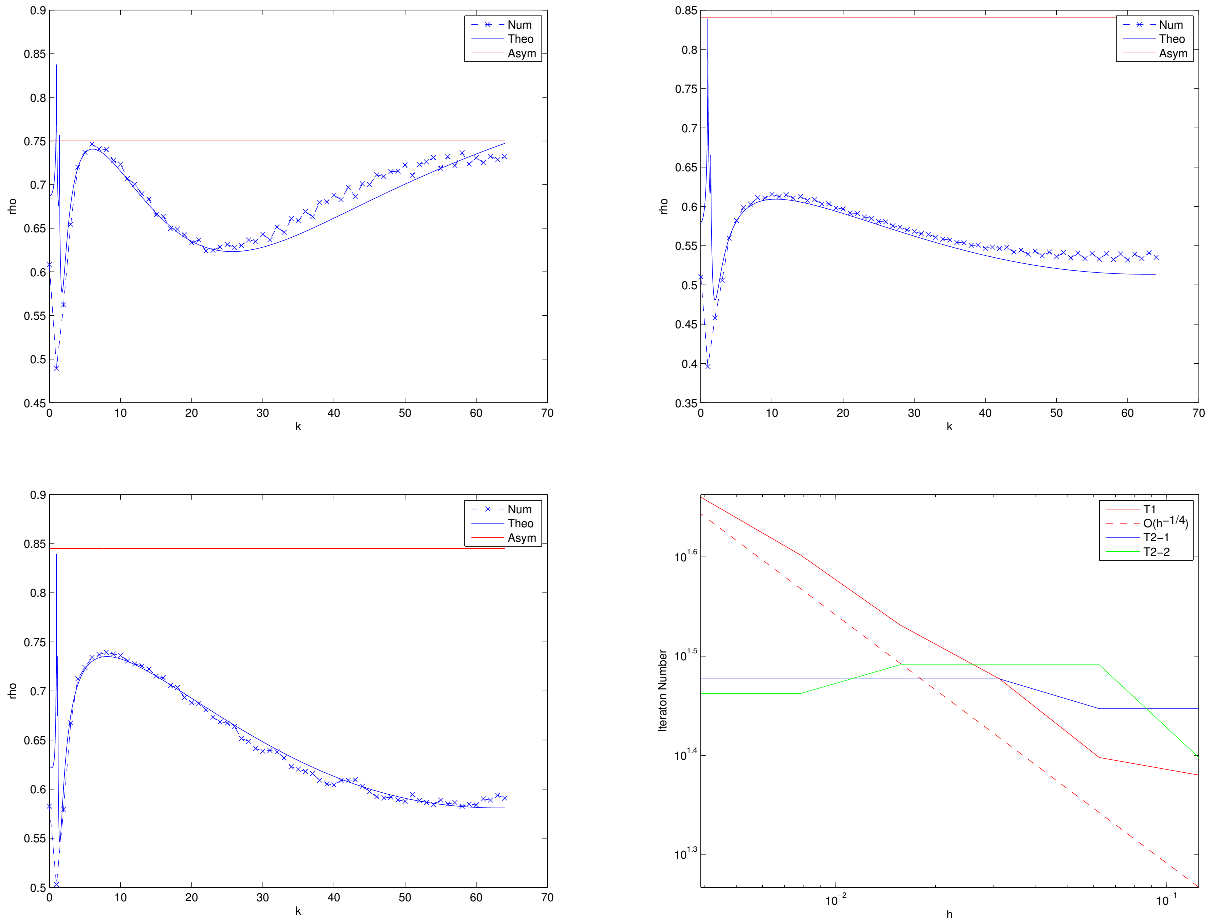

Figure 6. Comparison between the theoretical and numerical convergence factors as a function of the Fourier frequency k, where we set every Fourier mode $\sin(k\pi y)$ as initial guess and computed the numerical convergence factor for four iterations. *Top left*: case of Theorem 3.1 with $\varepsilon_1 = \mu_1 = \mu_2 = 1$, $\varepsilon = 2$ and $w = \pi$. *Top right*: first case of Theorem 4.1 with $\varepsilon_1 = \mu_1 = \mu_2 = 1$, $\varepsilon = 2$ and $w = \pi$. *Bottom left*: second case of Theorem 4.1 with $\varepsilon_1 = \mu_1 = \mu_2 = 1$, $\varepsilon = 1.4$ and $w = \pi$. *Bottom right*: Number of iterations when the mesh size h is refined.

TABLE 1. Number of iterations required using the optimized Schwarz methods to obtain an error of 10^{-6} for a random initial guess and different mesh sizes h.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Theorem 3.1	24	25	30	34	40	46
Theorem 4.1 (Case 1)	28	28	30	30	30	30
Theorem 4.1 (Case 2)	25	31	31	31	29	29

TABLE 2. Number of iterations required to obtain a relative residual reduction of 10^{-6} for different mesh sizes h, using the optimized Schwarz method as a preconditioner for GMRES.

h	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Theorem 3.1	11	17	21	26	30	35
Theorem 4.1 (Case 1)	11	16	19	21	23	24
Theorem 4.1 (Case 2)	11	18	23	26	30	34

we use a random initial guess and iterate until the error is reduced to $1e-6$ in the L^∞ norm at the interfaces for various mesh sizes h. We see that indeed when the mesh is refined and h becomes small, the optimized Schwarz methods corresponding to Theorem 4.1 have an iteration number that is independent of the mesh size, whereas the optimized Schwarz methods based on Theorem 3.1 slowly deteriorate when the mesh is refined, at the predicted rate in Theorem 3.1. These results are also shown in Table 1, where one can clearly see that Theorems 3.1 and 4.1 obtained at the continuous level on the unbounded domain predict well the behavior of the discretized optimized Schwarz method on the bounded domain. We finally show in Table 2 the number of GMRES iterations needed when solving Maxwell's equations with a right hand side equal to one, starting with a zero initial guess and reducing the relative residual to 10^{-6}. For more numerical experiments, see [10, 22].

6. Conclusion

We have determined the best choice of parameters in the transmission conditions of optimized Schwarz methods for Maxwell's equations in the presence of discontinuous coefficients, where the discontinuities are aligned with the subdomain interfaces. Using asymptotic analysis, we obtained closed form formulas for these parameters which can easily be used in implementations. Our results showed that with the specific transmission conditions in optimized Schwarz methods which take the physics of the underlying problem into account, one can not just obtain robustness in terms of the jumps in the coefficients, but even benefit from them, obtaining non-overlapping optimized Schwarz methods that converge independent of the mesh parameter h, which is not possible if the coefficients do not have jumps.

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