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OPTIMALLY CONVERGENT HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR FIFTH-ORDER KORTEWEG-DE VRIES TYPE EQUATIONS ⋆

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Abstract. We develop and analyze the first hybridizable discontinuous Galerkin (HDG) method for solving fifth-order Korteweg-de Vries (KdV) type equations. We show that the semi-discrete scheme is stable with proper choices of the stabilization functions in the numerical traces. For the linearized fifth-order equations, we prove that the approximations to the exact solution and its four spatial derivatives as well as its time derivative all have optimal convergence rates. The numerical experiments, demonstrating optimal convergence rates for both the linear and nonlinear equations, validate our theoretical findings.

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1. Introduction

Natural phenomena modeled by nonlinear partial differential equations are ubiquitous, appearing in numerous areas of science and engineering such as plasma physics, fluid dynamics, nonlinear optics, quantum mechanics, mathematical biology and chemical kinetics etc. Of particular interest to us is the class of nonlinear, dispersive partial differential equations called Korteweg-de Vries (KdV) equations. They model the evolution of long, one-dimensional waves such as the shallow-water waves with weakly nonlinear restoring forces and the long internal waves in a density-stratified ocean etc.

Due to their pervasiveness in applications and theoretical studies, it is critical to devise efficient numerical schemes for KdV equations that have mathematically provable stability and accuracy. This paper is a continuation of our systematic effort [6, 9] toward that end. Indeed, we focus on the following fifth-order KdV type equation:

$$u_t + \alpha u_{xxx} + \beta u_{xxxxx} + \frac{\partial}{\partial x} F(u, u_x, u_{xx}) = f \tag{1.1}$$

for $x \in \Omega := (0, L)$ and $t \in (0, T]$ with the initial condition

$$u(x, 0) = u_0(x)$$

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and periodic boundary condition. Here the coefficients α, β are real numbers with $\beta < 0$, $f \in L^2(\Omega)$ and F is a smooth function.

The fifth-order KdV equation (1.1) is the generic model for studying shallow water waves having surface tension and acoustic waves in plasma [1, 12, 13, 16, 17]. One characteristic feature is the existence of solitary wave solutions that do not vanish at infinity maintaining their shapes [2, 3]. The first fifth-order KdV type equation introduced in literature is the Kawahara equation [15, 16] for which $F(u, u_x, u_{xx}) = Ku^2$ where K is a nonzero constant. Later, other forms of F with more complicated nonlinearities appeared due to the need to capture solitary wave solutions. In applications, the form of F used most often is $F(u, u_x, u_{xx}) = Ku^{n+1}$ where $n \geq 1$ an integer. The scheme developed in this paper can be extended to general form of F in a trivial fashion. However, for simplicity we focus on the case where F is a function of u only.

The *acute challenge* in numerically resolving these equations are two-fold. First and foremost is the need for accurate long-time integration. To capture the structure of the characteristic soliton solutions over a long time-span, it is imperative for the scheme to have very low phase error which is a trademark of high-order methods. The second challenge originates from the demand for preservation of physical quantities such as mass and Hamiltonian. Discontinuous Galerkin (DG) method emerges as a natural choice for this scenario thanks to that it is easily implementable to be high order, its capability of handling nonuniform meshes (*e.g.* for a moving mesh following the soliton) with variable degrees, and its flexibility in design allowing for purposely constructing the numerical fluxes to achieve preservation of particular physical quantities [5]. Indeed, in 2002, the first local DG method was introduced for third- and fifth-order KdV equations [22]. It was further studied in [10, 11, 19, 20] and was shown to have approximate solutions with optimal convergence rates for linearized KdV equations [21]. In [7], another DG method was designed for KdV type equations based on repeated integration by parts. This method has optimal convergence rates for linearized KdV equations, but needs to use at least second-order polynomials for third-order equations and fourth-order polynomials for fifth-order equations. In [4, 5, 14], conservative DG methods were constructed for third-order KdV type equations to preserve quantities such as the mass and the L^2-norm of the solutions. These methods have optimal convergence rates when approximate solutions have even polynomial degrees and suboptimal convergence rates when the polynomial degrees are odd.

Traditional DG methods, despite their prominent features, were criticized for having too many degrees of freedom and for not being efficiently implementable. As a response to these criticisms, the hybridizable discontinuous Galerkin (HDG) methods were designed. Their unique feature is that the globally coupled degrees of freedom are only those on the element interfaces. Hence, they are more advantageous than traditional DG methods for solving stationary equations or time-dependent problems that require global solvers. In addition to the dimension reduction, they usually produce optimally convergent approximations for both the primal and dual variables, a feature notably lacking for the traditional DG formulation. The dimension reduction in the globally coupled degrees of freedom has a remarkable consequence for the one-dimensional setting of the KdV equations. In fact, *the size and bandwidth of the global system to be numerically solved are independent of the polynomial degrees in the finite element space.* This feature renders the HDG method particular attractive in the setting of this paper since the high-order accuracy is indispensable for accurate long time integration.

The idea of utilizing HDG method for KdV type equations is not new. In [6], we developed the first class of HDG methods for solving stationary third-order linear equations and proved super-convergence results. In [9], the HDG method is extended to the (nonlinear) third-order KdV type equations, attaining rigorous optimal convergence of approximate solutions to the primary variable and its two derivatives for the linearized equations. This paper amounts to *the first* attempt to design HDG and analyze it for the fifth-order KdV type equations to the best of our knowledge. Indeed, we design a semi-discrete HDG scheme for the general nonlinear equation (1.1) and prove its L^2-stability. For the fully discrete scheme, we adopt an implicit time-marching approach to mitigate the severe time-step restriction, and at each time step the HDG method to solve the resulting stationary fifth-order equation. To ensure compatibility at the first step and avoid loss of accuracy during the time-marching, the HDG solver for stationary fifth-order equations is also used to compute the initial approximations for the

time discretization. For stationary fifth-order linear equations, we prove that the HDG approximations of the primary variable with its four spatial derivatives superconverge to an HDG projection of the solutions with order $k+2$ and the numerical traces superconverge with the order $2k+1$ when the polynomial degree of the approximate solutions $k>0$. This is obtained by using a special duality argument and Green's functions. For the time-dependent case, we combine several energy identities and prove that the semi-discrete form has optimal convergence rates for the approximations to the primary variable as well as its four spatial derivatives and the time derivative.

The remainder of the paper is structured as follows. In Section 2, we introduce the central theme of this paper, a new HDG method for the KdV equation (1.1), and present the associated stability analysis. *A priori* error analyses for the linearized KdV equations are presented in Section 3. We omit some details of the proof and postpone them until the appendices. The numerical results collected in Section 4 corroborate our theoretical predictions and show optimal convergence of the scheme for both linear and nonlinear equations.

2. The HDG scheme and its stability

In this section, we define the HDG method before stating and proving the theoretical result on the stability of the scheme.

2.1. The HDG scheme

2.1.1. Notation

We first introduce a partition of the computational domain,

$$\mathcal{T}_h = \{I_i := (x_{i-1}, x_i) : i = 1, \dots, N\},$$

with $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = L$. We use $\partial\mathcal{T}_h := \{\partial I_i : i = 1, \dots, N\}$ to denote the boundaries of the intervals, and $\mathcal{E}_h := \{x_i\}_{i=0}^N$ to denote the set of all nodes. We set $h_i = x_i - x_{i-1}$ and $h := \max\{h_i,\ i = 1, \dots, N\}$.

For any function $\zeta \in L^2(\partial\mathcal{T}_h)$, we denote its values on $\partial I_i := \{x_{i-1}^+, x_i^-\}$ by $\zeta(x_{i-1}^+)$ (or simply ζ_{i-1}^+) and $\zeta(x_i^-)$ (or simply ζ_i^-). Note that $\zeta(x_i^+)$ does not have to be equal to $\zeta(x_i^-)$. On the other hand, for any function $\eta \in L^2(\mathcal{E}_h)$, its value at x_i, $\eta(x_i)$ (or simply η_i) is uniquely defined. We denote by $(\varphi, v)_{I_i}$ the integral of the product of φ and v on the interval I_i, and $\langle \varphi, vn\rangle_{\partial I_i}$ denotes that on the boundary which, in the one-dimensional case, simply becomes $\varphi(x_i^-)v(x_i^-)n(x_i^-) + \varphi(x_{i-1}^+)v(x_{i-1}^+)n(x_{i-1}^+)$. Here n denotes the outward unit normal to I_i, that is $n(x_{i-1}^+) := -1$ and $n(x_i^-) := 1$. Finally, we will need the finite element space which, when restricted to any particular element I_j, is the space of polynomials of degree at most k:

$$W_h^k = \{\omega \in L^2(\mathcal{T}_h) : \quad \omega|_{I_i} \in P_k(I_i) \quad \forall\, i = 1, \cdots, N\}.$$

2.1.2. Spatial discretization

To define the semi-discretization of the equation (1.1), we first rewrite the time-dependent fifth-order equation as the following first-order system:

$$q - u_x = 0, \quad p - q_x = 0, \quad r - p_x = 0, \quad s - r_x = 0, \quad u_t + \mathcal{F}(s, p, u)_x = f, \tag{2.1a}$$

where $\mathcal{F}(s, p, u) = \alpha p + \beta s + F(u)$, and the initial and boundary conditions are

$$u(x, 0) = u_0(x), \qquad \omega(0, t) = \omega(L, t) \quad \text{for } \omega = u, q, p, r, \mathcal{F}. \tag{2.1b}$$

Next, we characterize the exact solution of (2.1) as solutions to local problems on each element which we patch together to obtain the global solution through the so-called transmission conditions. HDG approximate solution is nothing but the discrete counterpart of this global solution constructed in a similar fashion.

Indeed, when F and f are smooth enough and $\{\widehat{u}_i\}_{i=0}^N, \{\widehat{q}_i\}_{i=0}^N$, and $\{\widehat{p}_i\}_{i=1}^N$ are given for $i=1,\ldots,N$, on each element I_i we can solve the local problem

$$Q - U_x = 0, \quad P - Q_x = 0, \quad R - P_x = 0, \quad S - R_x = 0, \quad U_t + \mathcal{F}(S,P,U)_x = f$$

with the intial and boundary conditions

$$\begin{aligned} &U = u_0 \quad \text{for } t = 0, \\ &U(x_{i-1}^+) = \widehat{u}_{i-1}, \quad U(x_i^-) = \widehat{u}_i, \quad Q(x_{i-1}^+) = \widehat{q}_{i-1}, \quad Q(x_i^-) = \widehat{q}_i, \quad P(x_i^-) = \widehat{p}_i. \end{aligned}$$

The solution (U,Q,P,R,S) of these local problems coincides with the solution (u,q,p,r,s) of (2.1) if and only if the transmission conditions

$$P(x_i^-) = P(x_i^+), \quad R(x_i^-) = R(x_i^+), \quad \mathcal{F}(S,P,U)(x_i^-) = \mathcal{F}(S,P,U)(x_i^+), \quad i = 1,\ldots,N-1$$

and the boundary conditions

$$\omega(0,t) = \omega(L,t) \quad \text{for } \omega = U, Q, P, R, \mathcal{F}.$$

are satisfied.

Our HDG scheme is inspired by this characterization and defined as its discrete counterpart. It provides approximations

$$(u_h, q_h, p_h, r_h, s_h, \widehat{u}_h, \widehat{q}_h, \widehat{p}_h, \widehat{r}_h, \widehat{s}_h) \in \left[W_h^k\right]^5 \times \left[L^2(\partial\mathcal{T}_h)\right]^5$$

to

$$(u|_\Omega, q|_\Omega, p|_\Omega, r|_\Omega, s|_\Omega, u|_{\mathcal{E}_h}, q|_{\mathcal{E}_h}, p|_{\mathcal{E}_h}, r|_{\mathcal{E}_h}, s|_{\mathcal{E}_h}).$$

Indeed, to determine these approximations, assuming that we are given the boundary values $\{\widehat{u}_{hi}\}_{i=0}^N$, $\{\widehat{q}_{hi}\}_{i=0}^N$, and $\{\widehat{p}_{hi}^-\}_{i=1}^N$ which are the only globally coupled unknowns, we solve the equation (2.1) locally on each element by adopting a Galerkin method. More specifically, on the element I_i, we give f and the boundary data $\widehat{u}_{h\,i-1}, \widehat{u}_{h\,i}$, $\widehat{q}_{h\,i-1}, \widehat{q}_{h\,i}$, and $\widehat{p}_{h\,i}^-$ and take $(u_h, q_h, p_h, r_h, s_h) \in [P_k(I_i)]^5$ to be the solution of the equations,

$$(q_h, v)_{I_i} + (u_h, v_x)_{I_i} - \langle \widehat{u}_h, vn\rangle_{\partial I_i} = 0, \tag{2.2a}$$
$$(p_h, z)_{I_i} + (q_h, z_x)_{I_i} - \langle \widehat{q}_h, zn\rangle_{\partial I_i} = 0, \tag{2.2b}$$
$$(r_h, w)_{I_i} + (p_h, w_x)_{I_i} - \langle \widehat{p}_h, wn\rangle_{\partial I_i} = 0, \tag{2.2c}$$
$$(s_h, \phi)_{I_i} + (r_h, \phi_x)_{I_i} - \langle \widehat{r}_h, \phi n\rangle_{\partial I_i} = 0, \tag{2.2d}$$
$$(u_{ht}, \psi)_{I_i} - (\mathcal{F}(s_h, p_h, u_h), \psi_x)_{I_i} + \langle \widehat{\mathcal{F}}_h, \psi n\rangle_{\partial I_i} = (f, \psi)_{I_i}, \tag{2.2e}$$

for all $(v,z,w,\phi,\psi) \in (P_k(I_i))^5$. We close the local system by defining the remaining numerical traces as follows

$$\widehat{p}_h^+ = p_h^+ + \tau_{pu}^+ (\widehat{u}_h - u_h^+)\, n^+ + \tau_{pq}^+ (\widehat{q}_h - q_h^+)\, n^+ \qquad \text{at } x_{i-1}^+, \tag{2.3a}$$
$$\widehat{r}_h^+ = r_h^+ + \tau_{ru}^+ (\widehat{u}_h - u_h^+)\, n^+ + \tau_{rq}^+ (\widehat{q}_h - q_h^+)\, n^+ \qquad \text{at } x_{i-1}^+, \tag{2.3b}$$
$$\widehat{s}_h^+ = s_h^+ + \tau_{su}^+ (\widehat{u}_h - u_h^+)\, n^+ + \tau_{sq}^+ (\widehat{q}_h - q_h^+)\, n^+ \qquad \text{at } x_{i-1}^+, \tag{2.3c}$$
$$\widehat{r}_h^- = r_h^- + \tau_{ru}^- (\widehat{u}_h - u_h^-)\, n^- + \tau_{rq}^- (\widehat{q}_h - q_h^-)\, n^- + \tau_{rp}^- (\widehat{p}_h^- - p_h^-)\, n^- \qquad \text{at } x_i^-, \tag{2.3d}$$
$$\widehat{s}_h^- = s_h^- + \tau_{su}^- (\widehat{u}_h - u_h^-)\, n^- + \tau_{sq}^- (\widehat{q}_h - q_h^-)\, n^- + \tau_{sp}^- (\widehat{p}_h^- - p_h^-)\, n^- \qquad \text{at } x_i^-, \tag{2.3e}$$
$$\widehat{\mathcal{F}}_h = \mathcal{F}(\widehat{s}_h, \widehat{p}_h, \widehat{u}_h) - \tau_{\mathcal{F}}(\widehat{u}_h, u_h)(\widehat{u}_h - u_h)n \qquad \text{at } x_i^-, x_{i-1}^+. \tag{2.3f}$$

Obviously, (2.3a)–(2.3c) are defined at $x_0, \cdots, x_{N-1}$, while (2.3d)–(2.3e) are defined at $x_1, \cdots, x_N$. The stabilization functions τ_{pu}^+, τ_{pq}^+, $\tau_{ru}^\pm$, $\tau_{rq}^\pm$, τ_{rp}^-, $\tau_{su}^\pm$ $\tau_{sq}^\pm$, τ_{sp}^- are defined on $\partial\mathcal{T}_h$, and are usually piece-wise constant.

Due to the nonlinearity of $F(u)$, $\tau_{\mathcal{F}}(\cdot,\cdot)$ can be a nonlinear function of $\widehat{u}_h$ and u_h, and is taken to be 0 when $F \equiv 0$. These functions, when satisfing certain conditions to be specified later, ensure that the above problem has a unique solution.

It remains to impose the transmission conditions which allows us to solve for the globally coupled unknowns $\{\widehat{u}_{hi}\}_{i=0}^{N}$, $\{\widehat{q}_{hi}\}_{i=0}^{N}$, and $\{\widehat{p}_{hi}^{-}\}_{i=1}^{N}$:

$$[\![\widehat{p}_h]\!](x_i) = 0, \;\; [\![\widehat{r}_h]\!](x_i) = 0, \quad \text{and} \;\; [\![\widehat{\mathcal{F}}_h]\!](x_i) = 0 \qquad \text{for all } i = 1, \ldots, N, \tag{2.4}$$

where $[\![\omega]\!](x_i) := \omega(x_i^-) - \omega(x_i^+)$ for any function ω. Here, to honor the periodic boundary conditions, we define $[\![\widehat{\omega}_h]\!](x_N) = \widehat{\omega}_h(x_N^-) - \widehat{\omega}_h(x_0^+)$ for $\omega \in \{p, r, \mathcal{F}\}$. This completes the definition of the HDG method.

Remark.

1. In implementation, the globally coupled degrees of freedom of the HDG method are those of the numerical traces $(\widehat{u}_h, \widehat{q}_h, \widehat{p}_h^{-})$. They are determined by the transmission conditions (2.4) and then used in (2.2) to locally solve for the approximations $(u_h, q_h, p_h, r_h, s_h)$ on each element.

2. The choice of $(\widehat{u}_h, \widehat{q}_h, \widehat{p}_h^{-})$ as the globally coupled unknown for the HDG method is not unique. It corresponds to the choice of boundary data for the local problems in the characterization of the exact solution. Other characterizations of the exact solution are possible through imposing boundary conditions to the local problems in different ways, as long as they ensure the well-posedness of the local PDE problems. The corresponding HDG methods are then well-defined. It is known that HDG methods derived from different characterizations of the same exact solutions are equivalent to each other if the stabilization functions τ are finite and nonzero; see [6].

3. In (2.2) and (2.3), once the numerical traces $(\widehat{u}_h, \widehat{q}_h, \widehat{p}_h^{-})$ are chosen as the globally coupled unknowns, the corresponding local problems have enough boundary conditions. This means that the remaining numerical traces $\widehat{p}_h^{+}, \widehat{r}_h, \widehat{s}_h$ and $\widehat{\mathcal{F}}_h$ need to be defined in terms of the approximations from the interior of the elements and $(\widehat{u}_h, \widehat{q}_h, \widehat{p}_h^{-})$. The definition of numerical traces in (2.3) is standard for HDG methods. For maximum generality, they includes all possible stabilization terms. We note that, in actual implementation, most of the stabilization functions τ can be taken to be zero.

2.1.3. Time discretization

For time discretization of the KdV equation, we employ implicit time-marching schemes to mitigate the severe time-step restriction due to the fifth-order spatial derivative. One may use implicit time-marching schemes such as BDF or DIRK methods. Here for simplicity, we apply the following second-order midpoint rule [4, 5, 9] to discretize the time derivative.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$ and $\Delta t_j = t_{j+1} - t_j$. At the initial time $t = 0$, we choose the initial approximation to be the HDG approximate solution of the stationary fifth-order equation

$$\mathcal{D}(u) + \gamma u = \tilde{f}, \tag{2.5}$$

where $\mathcal{D}(u) := \alpha u_{xxx} + \beta u_{xxxxx} + F(u)_x$, $\gamma = 1$ and $\tilde{f} = \mathcal{D}(u_0) + \gamma u_0$. At later time $t = t_{j+1}$ for $j = 0, \cdots, N-1$, we let the approximation u^{j+1} to $u(\cdot, t_{j+1})$ be

$$u^{j+1} = 2u^{j,1} - u^j,$$

where $u^{j,1}$ is the solution of the equation

$$\frac{u^{j,1} - u^j}{\frac{1}{2}\Delta t_j} + \mathcal{D}(u^{j,1}) = f.$$

It is easy to see that the equation above for $u^{j,1}$ can be rewritten into the form of (2.5) with $\gamma = 2/\Delta t_j$ and $\tilde{f} = f + \gamma u^j$. Therefore, at each time step, we use the HDG method to solve the stationary fifth-order equation (2.5). To do that, we rewrite (2.5) into the following first-order system:

$$q - u_x = 0, \quad p - q_x = 0, \quad r - p_x = 0, \quad s - r_x = 0, \quad \gamma u + \mathcal{F}(s, p, u)_x = \tilde{f}. \tag{2.6}$$

In order to write the HDG formulation in a compact form, we use the notation

$$(\varphi, v) := \sum_{i=1}^{N} (\phi, v)_{I_i}, \quad \langle \varphi, vn \rangle := \sum_{i=1}^{N} \langle \varphi, vn \rangle_{\partial I_i}.$$

The HDG approximations $(u_h, q_h, p_h, r_h, s_h, \widehat{u}_h, \widehat{q}_h, \widehat{p}_h^{-}) \in \left[W_h^k\right]^5 \times \left[L^2(\mathscr{E}_h)\right]^3$ for (2.6) satisfy

$$\begin{aligned}
(q_h, v) + (u_h, v_x) - \langle \widehat{u}_h, vn \rangle &= 0, \\
(p_h, z) + (q_h, z_x) - \langle \widehat{q}_h, zn \rangle &= 0, \\
(r_h, w) + (p_h, w_x - \langle \widehat{p}_h, wn \rangle &= 0, \\
(s_h, \phi) + (r_h, \phi_x) - \langle \widehat{r}_h, \phi n \rangle &= 0, \\
(\gamma u_h, \psi) - (\mathcal{F}(s_h, p_h, u_h), \psi_x) + \langle \widehat{\mathcal{F}}_h, \psi n \rangle &= (\tilde{f}, \psi),
\end{aligned} \tag{2.7}$$

for all $(v, z, w, \phi, \psi) \in \left[W_h^k\right]^5$. Here the numerical traces $(\widehat{p}_h^{+}, \widehat{r}_h, \widehat{s}_h, \widehat{\mathcal{F}}_h)$ are defined in the same way as (2.3). The globally coupled degrees of freedom are those associated with $\{\widehat{u}_{hi}\}_{i=0}^{N}$, $\{\widehat{q}_{hi}\}_{i=0}^{N}$, and $\{\widehat{p}_{hi}^{-}\}_{i=1}^{N}$ and they are determined by the transmission conditions (2.4).

Note that the initial approximations at $t=0$ are obtained by using (2.7) with $\tilde{f} = \mathcal{D}(u_0) + \gamma u_0$. This way, they satisfy the discrete equations (2.2a)–(2.2d) and the transmission conditions (2.4) of the time-dependent problem exactly. We emphasize that the main reason for adopting this initial approximation is for the ease of analysis. Other choices of initial approximations may still produce optimal convergence numerically.

2.2. Stability of the semi-discrete scheme

Theorem 2.1. *The semi-discrete scheme (2.2) for the fifth-order KdV equation is L^2-stable if the stabilization functions satisfy the following conditions*

$$\begin{cases}
\tau_{\mathcal{F}} \geq \tilde{\tau} := \frac{1}{(\widehat{u}_h - u_h)^2} \int_{\widehat{u}_h}^{u_h} (F(\widehat{u}_h) - F(s)) \cdot n \mathrm{d}s, \\
\tau_{su}^{+} \geq -\frac{\alpha}{\beta}\tau_{pu}^{+} + \frac{1}{2}{\tau_{pu}^{+}}^2 \quad or \quad \tau_{rq}^{+} \leq \frac{\alpha}{2\beta} - \frac{1}{2}{\tau_{pq}^{+}}^2, \\
\left(\tau_{su}^{+} + \frac{\alpha}{\beta}\tau_{pu}^{+} - \frac{1}{2}{\tau_{pu}^{+}}^2\right)\left(-\tau_{rq}^{+} + \frac{\alpha}{2\beta} - \frac{1}{2}{\tau_{pq}^{+}}^2\right) \geq \frac{1}{4}\left(\tau_{sq}^{+} - \tau_{ru}^{+} + \frac{\alpha}{\beta}\tau_{pq}^{+} - \tau_{pu}^{+}\tau_{pq}^{+}\right)^2, \\
\tau_{su}^{-} \geq \frac{1}{2}(\tau_{sp}^{-} + \frac{\alpha}{\beta})^2, \\
\tau_{rq}^{-} \leq -\frac{\alpha}{2\beta} - \frac{1}{2}{\tau_{rp}^{-}}^2, \\
-\tau_{su}^{-}\left(\tau_{rq}^{-} + \frac{\alpha}{2\beta}\right) \geq \frac{1}{4}(\tau_{sq}^{-} - \tau_{ru}^{-})^2.
\end{cases}$$

Remark 2.2. The conditions in Theorem 2.1 are only technical conditions required by our analysis. They are sufficient but not necessary for the stability of the method. As complicated as they may appear, it is nevertheless easy to identify stabilization functions to satisfy them. For example, we can take

$$\begin{aligned}
&\tau_{pu}^{+} = \tau_{pq}^{+} = \tau_{sq}^{+} = \tau_{ru}^{+} = 0, \quad \tau_{su}^{+} \geq 0, \quad \tau_{rq}^{+} \leq \frac{\alpha}{2\beta}, \\
&\tau_{sp}^{-} = \tau_{rp}^{-} = \tau_{sq}^{-} = \tau_{ru}^{-} = 0, \quad \tau_{su}^{-} \geq \frac{1}{2}(\frac{\alpha}{\beta})^2, \quad \tau_{rq}^{-} \leq -\frac{\alpha}{2\beta}
\end{aligned} \tag{2.8}$$

given that $\beta < 0$. Then the numerical traces in (2.3a)–(2.3e) have the simple form

$$\begin{aligned}
\widehat{p}_h^{+} &= p_h^{+} && \text{at } x_{i-1}^{+}, \\
\widehat{r}_h &= r_h + \tau_{rq}(\widehat{q}_h - q_h)\, n && \text{at } x_{i-1}^{+} \text{ and } x_i^{-}, \\
\widehat{s}_h &= s_h + \tau_{su}\,(\widehat{u}_h - u_h)\, n && \text{at } x_{i-1}^{+} \text{ and } x_i^{-}.
\end{aligned}$$

To have $\tau_{\mathcal{F}} \geq \tilde{\tau}$, we take $\tau_{\mathcal{F}} = 0$ if $F(u) \equiv 0$. If F is nonzero, we can take

$$\tau_{\mathcal{F}} \geq \frac{1}{2} \sup_{s \in J(u_h, \widehat{u}_h)} |F'(s)|,$$

where $J(u_h, \widehat{u}_h) = [\min\{u_h, \widehat{u}_h\}, \max\{u_h, \widehat{u}_h\}]$, because

$$\tilde{\tau} = \frac{1}{(u_h - \widehat{u}_h)^2} \int_{\widehat{u}_h}^{u_h} F'(\xi)(s - \widehat{u}_h) n \, \mathrm{d}s \leq \frac{1}{2} \sup_{s \in J(u_h, \widehat{u}_h)} |F'(s)|.$$

There are also other choices of $\tau_{\mathcal{F}}$ that satisfy the condition $\tau_{\mathcal{F}} \geq \tilde{\tau}$; see [18].

Now let us prove the stability result in Theorem 2.1, and the proof is similar to that in [9,18].

Proof. Taking $\psi = u_h$, $v = -(\alpha p_h + \beta s_h)$, $\phi = \beta q_h$, $z = \alpha q_h + \beta r_h$, and $w = -\beta p_h$ in (2.2a)–(2.2e), adding the five resulting equations together, and performing trivial algebraic manipulations, we have

$$\begin{aligned}
(f, u_h) = {} & (u_{ht}, u_h) - (F(u_h), u_{hx}) + \langle \widehat{\mathcal{F}}_h, u_h \cdot n \rangle \\
& - (\alpha p_h + \beta s_h, u_{hx}) - (u_h, (\alpha p_h + \beta s_h)_x) + \langle \widehat{u}_h, (\alpha p_h + \beta s_h) n \rangle \\
& + (r_h, \beta q_{hx}) - \langle \widehat{r}_h, \beta q_h \cdot n \rangle - (p_h, \beta p_{hx}) + \langle \widehat{p}_h, \beta p_h \cdot n \rangle \\
& + (q_h, (\alpha q_h + \beta r_h)_x) - \langle \widehat{q}_h, (\alpha q_h + \beta r_h) \cdot n \rangle.
\end{aligned}$$

Using integration by parts, the fact that $\widehat{u}_h, \widehat{q}_h, \widehat{p}_h, \widehat{r}_h, \widehat{\mathcal{F}}_h$ are single-valued, and the periodic boundary condition, we obtain

$$(f, u_h) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{T}_h} u_h^2 \mathrm{d}x + \Phi_1 + \Phi_2 + \Phi_3,$$

where

$$\begin{aligned}
\Phi_1 = {} & -\langle \alpha(\widehat{p}_h - p_h) + \beta(\widehat{s}_h - s_h), (\widehat{u}_h - u_h) \cdot n \rangle + \langle \widehat{r}_h - r_h, \beta(\widehat{q}_h - q_h) \cdot n \rangle \\
& + \frac{\alpha}{2} \langle (\widehat{q}_h - q_h)^2, n \rangle - \frac{\beta}{2} \langle (\widehat{p}_h - p_h)^2, n \rangle, \\
\Phi_2 = {} & -(F(u_h), u_{hx}) - \langle F(\widehat{u}_h) - \tau_{\mathcal{F}}(\widehat{u}_h - u_h) \cdot n, (\widehat{u}_h - u_h) \cdot n \rangle, \\
\Phi_3 = {} & \frac{\beta}{2} \langle \widehat{p}_h^2, n \rangle.
\end{aligned}$$

We immediately realize that $\Phi_3 = 0$ due to the periodicity. Next, we plug in the definition of the numerical fluxes (2.3) to simplify Φ_1 and Φ_2. Indeed, Φ_1 can be rewritten as $\Phi_1 = \Phi_1^+ + \Phi_1^-$, where

$$\Phi_1^+ = \langle a_{uu}, (\widehat{u}_h - u_h)^2 \rangle_{\mathscr{E}_h^+} + \langle a_{qq}, (\widehat{q}_h - q_h)^2 \rangle_{\mathscr{E}_h^+} + \langle a_{uq}, (\widehat{u}_h - u_h)(\widehat{q}_h - q_h) \rangle_{\mathscr{E}_h^+}$$

and

$$\begin{aligned}
\Phi_1^- = {} & \langle b_{uu}, (\widehat{u}_h - u_h)^2 \rangle_{\mathscr{E}_h^-} + \langle b_{qq}, (\widehat{q}_h - q_h)^2 \rangle_{\mathscr{E}_h^-} + \langle b_{pp}, (\widehat{p}_h - p_h)^2 \rangle_{\mathscr{E}_h^-} \\
& + \langle b_{uq}, (\widehat{u}_h - u_h)(\widehat{q}_h - q_h) \rangle_{\mathscr{E}_h^-} + \langle b_{up}, (\widehat{u}_h - u_h)(\widehat{p}_h - p_h) \rangle_{\mathscr{E}_h^-} \\
& + \langle b_{qp}, (\widehat{p}_h - p_h)(\widehat{q}_h - q_h) \rangle_{\mathscr{E}_h^-}.
\end{aligned}$$

Here

$$\begin{aligned}
a_{uu} &= -\beta\tau_{su}^{+} - \alpha\tau_{pu}^{+} + \frac{\beta}{2}{\tau_{pu}^{+}}^2, \qquad a_{qq} = \beta\tau_{rq}^{+} - \frac{\alpha}{2} + \frac{\beta}{2}{\tau_{pq}^{+}}^2,\\
a_{uq} &= -\beta\tau_{sq}^{+} + \beta\tau_{ru}^{+} - \alpha\tau_{pq}^{+} + \beta\tau_{pu}^{+}\tau_{pq}^{+},\\
b_{uu} &= -\beta\tau_{su}^{-}, \qquad b_{qq} = \beta\tau_{rq}^{-} + \frac{\alpha}{2}, \qquad b_{pp} = -\frac{\beta}{2},\\
b_{uq} &= -\beta\tau_{sq}^{-} + \beta\tau_{ru}^{-}, \qquad b_{up} = -\beta\tau_{sp}^{-} - \alpha, \qquad b_{qp} = \beta\tau_{rp}^{-}.
\end{aligned}$$

To simplify Φ_2, we introduce $G(s)$, the antiderivative of $f(s)$. We have

$$\begin{aligned}
\Phi_2 &= -\langle G(u_h), n\rangle + \langle F(\widehat{u}_h), (\widehat{u}_h - u_h)\cdot n\rangle + \langle \tau_{\mathcal{F}}, (\widehat{u}_h - u_h)^2\rangle\\
&= -\langle \int_{\widehat{u}_h}^{u_h} F(s)\mathrm{d}s, n\rangle + \langle \int_{\widehat{u}_h}^{u_h} F(\widehat{u}_h)\mathrm{d}s, n\rangle + \langle \tau_{\mathcal{F}}, (\widehat{u}_h - u_h)^2\rangle\\
&= \langle \tau_{\mathcal{F}} - \tilde{\tau}, (\widehat{u}_h - u_h)^2\rangle,
\end{aligned}$$

where

$$\tilde{\tau} = \frac{1}{(\widehat{u}_h - u_h)^2}\int_{\widehat{u}_h}^{u_h} (F(\widehat{u}_h) - F(s))\cdot n\mathrm{d}s.$$

A sufficient condition for the L^2-stability is then $\Phi_2 \geq 0$, $\Phi_1^{+} \geq 0$, and $\Phi_1^{-} \geq 0$. That is to ask

$$\begin{cases}
\tau_{\mathcal{F}} \geq \tilde{\tau},\\
a_{uu} \geq 0, \quad a_{qq} \geq 0, \quad 4a_{uu}a_{qq} \geq a_{uq}^2,\\
b_{uu} \geq 0, \quad b_{qq} \geq 0, \quad b_{pp} \geq 0, \quad 4b_{uu}b_{qq} \geq b_{uq}^2, \quad 4b_{uu}b_{pp} \geq b_{up}^2, \quad 4b_{pp}a_{qq} \geq b_{pq}^2.
\end{cases}$$

Plugging in the definitions of a_*'s and b_*'s above, we have

$$\begin{cases}
-\beta\tau_{su}^{+} - \alpha\tau_{pu}^{+} + \frac{\beta}{2}{\tau_{pu}^{+}}^2 \geq 0,\\
\beta\tau_{rq}^{+} - \frac{\alpha}{2} + \frac{\beta}{2}{\tau_{pq}^{+}}^2 \geq 0,\\
(-\beta\tau_{su}^{+} - \alpha\tau_{pu}^{+} + \frac{\beta}{2}{\tau_{pu}^{+}}^2)(\beta\tau_{rq}^{+} - \frac{\alpha}{2} + \frac{\beta}{2}{\tau_{pq}^{+}}^2) \geq \frac{1}{4}\left(-\beta\tau_{sq}^{+} + \beta\tau_{ru}^{+} - \alpha\tau_{pq}^{+} + \beta\tau_{pu}^{+}\tau_{pq}^{+}\right)^2,
\end{cases}$$

and

$$\begin{cases}
\tau_{su}^{-} \geq 0, \quad \tau_{rq}^{-} \leq -\frac{\alpha}{2\beta},\\
-4\beta\tau_{su}^{-}(\beta\tau_{rq}^{-} + \frac{\alpha}{2}) \geq \left(\beta\tau_{sq}^{-} + \beta\tau_{ru}^{-}\right)^2,\\
2\beta^2\tau_{su}^{-} \geq \left(\alpha + \beta\tau_{sp}^{-}\right)^2,\\
-2\beta(\frac{\alpha}{2} + \beta\tau_{rq}^{-}) \geq \beta^2{\tau_{rp}^{-}}^2.
\end{cases}$$

It is easy to see that these conditions are equivalent to those listed in the theorem. We thus complete the proof. □

Remark 2.3. With little to no change to the proof, we can show that the scheme is also L^2-stable for the Dirichlet-type boundary condition such as the following

$$u|_{0,L} = 0, \quad q|_{0,L} = 0, \quad p|_{L} = 0. \tag{2.9}$$

3. *A PRIORI* ERROR ANALYSIS

The goal of this section is to establish the optimal accuracy of our scheme for the linear case, that is, when we assume $F \equiv 0$. Toward that end, we define the errors of the approximations ω_h and of the numerical fluxes $\widehat{\omega}_h$ as

$$e_\omega = \omega - \omega_h, \quad \widehat{e}_\omega = \omega - \widehat{\omega}_h \quad \text{for } \omega \in \{u, q, p, r, s\}.$$

The analysis is projection-based, see also [6,8,9]. We first define a projection

$$\Pi : \left[L^2(\mathcal{T}_h)\right]^5 \longrightarrow \left[W_h^k\right]^5$$

that is inspired by the scheme, in particular the definition of the numerical fluxes (2.3). Then we prove the optimal approximation property of this projection error

$$\delta_\omega = \omega - \Pi\omega \quad \text{for } \omega \in \{u, q, p, r, s\},$$

and that the projections of the error, defined as

$$\varepsilon_\omega = \Pi\omega - \omega_h \quad \text{for } \omega \in \{u, q, p, r, s\},$$

also converge optimally. We can then conclude that the errors e_ω converge optimally due to that $e_\omega = \delta_\omega + \varepsilon_\omega$ and the triangle inequality.

Let us first define the projection $\Pi : \left[L^2(\mathcal{T}_h)\right]^5 \longrightarrow \left[W_h^k\right]^5$. For $(u, q, p, r, s) \in \left[L^2(\mathcal{T}_h)\right]^5$, we find in $\left[W_h^k\right]^5$ the element $(\Pi u, \Pi q, \Pi p, \Pi r, \Pi s)$ whose restriction on element $I_i = (x_{i-1}, x_i)$ satisfies the following condition

$$\begin{cases} (\delta u, v)_{I_i} = 0, \quad (\delta q, z)_{I_i} = 0, \quad (\delta p, w)_{I_i} = 0, & (\delta r, \phi)_{I_i} = 0, \quad (\delta s, \psi)_{I_i} = 0, \\ \delta_p - \tau_{pu}^+ \delta_u n - \tau_{pq}^+ \delta_q n = 0 & \text{at } x_{i-1}^+ \\ \delta_r - \tau_{ru}^+ \delta_u n - \tau_{rq}^+ \delta_q n = 0 & \text{at } x_{i-1}^+ \\ \delta_s - \tau_{su}^+ \delta_u n - \tau_{sq}^+ \delta_q n = 0 & \text{at } x_{i-1}^+ \\ \delta_r - \tau_{ru}^- \delta_u n - \tau_{rq}^- \delta_q n - \tau_{rp}^- \delta_p n = 0 & \text{at } x_i^- \\ \delta_s - \tau_{su}^- \delta_u n - \tau_{sq}^- \delta_q n - \tau_{sp}^- \delta_p n = 0 & \text{at } x_i^- \end{cases} \tag{3.1}$$

for any $(v, z, w, \phi, \psi) \in [P_{k-1}(K)]^5$, where we have used the notation $\delta_\omega := \omega - \Pi\omega$ for any $\omega \in L^2(\mathcal{T}_h)$.

To state our error estimate, we use the following notation. The $H^s(D)$-norm is denoted by $\|\cdot\|_{s,D}$. We drop the first subindex if $s = 0$, and the second one if $D = \Omega$ or $D = \mathcal{T}_h$. We are now ready to state our error estimate which is given as an upper bound for the following quantity

$$\|\varepsilon\|^2 := \|\varepsilon_u\|^2 + \|\varepsilon_q\|^2 + \|\varepsilon_p\|^2 + \|\varepsilon_r\|^2 + \|\varepsilon_s\|^2 + \|\varepsilon_{u_t}\|^2. \tag{3.2}$$

Theorem 3.1. *Suppose that u is the solution to the fifth-order Korteweg-de Vries equation (1.1) with $F \equiv 0$, $\alpha = 0$ and $\beta = -1$, and u_h is the approximate solution given by the HDG scheme (2.2)–(2.4) with the stabilization functions $\tau_{su}^\pm = 1, \tau_{rq}^\pm = -1$ and all others being zero. Then for $k > 0$, the projection of the error (3.2) satisfies*

$$\|\varepsilon(t)\| \le C h^{k+1}.$$

This Theorem is for linear equations. The error analysis for the nonlinear case is more delicate requiring special treatment of the nonlinear terms, and is the subject of an upcoming paper.

To prove Theorem 3.1, we need the following three lemmas. They are the energy equalities, the approximation property of the projection, and the requirement on the initial condition.

Lemma 3.2 (Energy identities). *We have that*

$$0=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_u\|^2+S_1+T_1, \tag{3.3a}$$

$$0=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_q\|^2+S_2+T_2, \tag{3.3b}$$

$$0=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_p\|^2+S_3+T_3, \tag{3.3c}$$

$$0=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_r\|^2+S_4+T_4, \tag{3.3d}$$

$$0=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_s\|^2+S_5+T_5, \tag{3.3e}$$

$$0=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_{u_t}\|^2+S_6+T_6. \tag{3.3f}$$

Moreover,

$$T:=\sum_{k=1}^{6}T_k\geq 0.$$

Here

$$\begin{aligned}
S_1&=(\delta_{ut},\varepsilon_u)-(\delta_q,\alpha\varepsilon_p+\beta\varepsilon_s)+(\delta_p,\alpha\varepsilon_q+\beta\varepsilon_r)-(\delta_r,\beta\varepsilon_p)+(\delta_s,\beta\varepsilon_q)\\
T_1&=-\langle\alpha(\widehat{\varepsilon}_p-\varepsilon_p)+\beta(\widehat{\varepsilon}_s-\varepsilon_s),(\widehat{\varepsilon}_u-\varepsilon_u)\cdot n\rangle+\beta\langle(\widehat{\varepsilon}_r-\varepsilon_r),(\widehat{\varepsilon}_q-\varepsilon_q)\cdot n\rangle\\
&\quad+\frac{\alpha}{2}\langle(\widehat{\varepsilon}_q-\varepsilon_q)^2,n\rangle-\frac{\beta}{2}\langle(\widehat{\varepsilon}_p-\varepsilon_p)^2,n\rangle\\
S_2&=(\delta_{qt},\varepsilon_q)+(\delta_p,\varepsilon_{ut})+(\varepsilon_p,\varepsilon_{ut})\\
T_2&=\langle(\widehat{\varepsilon}_q-\varepsilon_q),(\widehat{\varepsilon}_{ut}-\varepsilon_{ut})\cdot n\rangle\\
S_3&=(\delta_{pt},\varepsilon_p)+(\delta_r,\varepsilon_{qt})-(\delta_{qt},\varepsilon_r)-(\delta_s,\varepsilon_{ut})-(\varepsilon_s,\varepsilon_{ut})\\
T_3&=\langle(\widehat{\varepsilon}_p-\varepsilon_p),(\widehat{\varepsilon}_{qt}-\varepsilon_{qt})\cdot n\rangle-\langle(\widehat{\varepsilon}_r-\varepsilon_r),(\widehat{\varepsilon}_{ut}-\varepsilon_{ut})\cdot n\rangle\\
S_4&=(\delta_{rt},\varepsilon_r)+(\delta_s,\varepsilon_{pt})-(\delta_{pt},\varepsilon_s+\frac{\alpha}{\beta}\varepsilon_p)+\frac{1}{\beta}(\delta_{ut},\varepsilon_{qt})-\frac{1}{\beta}(\delta_{qt},\varepsilon_{ut})\\
T_4&=\langle(\widehat{\varepsilon}_r-\varepsilon_r),(\widehat{\varepsilon}_{pt}-\varepsilon_{pt})\cdot n\rangle-\langle(\widehat{\varepsilon}_{qt}-\varepsilon_{qt}),(\widehat{\varepsilon}_s-\varepsilon_s)\cdot n\rangle\\
&\quad-\frac{1}{2\beta}\langle(\widehat{\varepsilon}_{ut}-\varepsilon_{ut})^2,n\rangle-\frac{\alpha}{\beta}\langle(\widehat{\varepsilon}_p-\varepsilon_p),(\widehat{\varepsilon}_{qt}-\varepsilon_{qt})\cdot n\rangle\\
S_5&=(\delta_{st},\varepsilon_s)-\frac{1}{\beta}(\delta_{ut},\varepsilon_{rt})+\frac{1}{\beta}(\delta_{rt},\varepsilon_{ut})+\frac{1}{\beta}(\delta_{qt},\varepsilon_{pt})-\frac{1}{\beta}(\delta_{pt},\varepsilon_{qt})-\frac{\alpha}{\beta}(\delta_r,\varepsilon_{rt})\\
T_5&=\langle(\widehat{\varepsilon}_{rt}-\varepsilon_{rt}),(\widehat{\varepsilon}_s-\varepsilon_s)\cdot n\rangle+\frac{1}{\beta}\langle(\widehat{\varepsilon}_{pt}-\varepsilon_{pt}),(\widehat{\varepsilon}_{ut}-\varepsilon_{ut})\cdot n\rangle-\frac{1}{2\beta}\langle(\widehat{\varepsilon}_{qt}-\varepsilon_{qt})^2,n\rangle\\
S_6&=(\delta_{u_{tt}},\varepsilon_{ut})-(\delta_{qt},\alpha\varepsilon_{pt}+\beta\varepsilon_{st})+(\delta_{pt},\alpha\varepsilon_{qt}+\beta\varepsilon_{rt})+(\delta_{st},\beta\varepsilon_{qt})-(\delta_{rt},\beta\varepsilon_{pt})\\
T_6&=-\langle\beta(\widehat{\varepsilon}_{st}-\varepsilon_{st})+\alpha(\widehat{\varepsilon}_{pt}-\varepsilon_{pt}),(\widehat{\varepsilon}_{ut}-\varepsilon_{ut})\cdot n\rangle+\frac{\alpha}{2}\langle(\widehat{\varepsilon}_{qt}-\varepsilon_{qt})^2,n\rangle\\
&\quad+\beta\langle(\widehat{\varepsilon}_{rt}-\varepsilon_{rt}),(\widehat{\varepsilon}_{qt}-\varepsilon_{qt})\cdot n\rangle-\frac{\beta}{2}\langle(\widehat{\varepsilon}_{pt}-\varepsilon_{pt})^2,n\rangle.
\end{aligned}$$

The proof of this lemma is long and technical, thus given in Appendix A.

The next lemma states that the projection Π is optimal, that is, the projection error is of order $k+1$. To describe it, we define the auxiliary penalization functions

$$\theta_\omega^\alpha:=\tau_{\alpha\omega}^-+\tau_{\alpha\omega}^+-\tau_{\alpha p}^-\tau_{p\omega}^+ \text{ for } \alpha\in\{s,r\} \text{ and } \omega\in\{u,q\}.$$

Lemma 3.3 (Approximation properties of the projection.). *When the penalization functions satisfy* $\theta_q^s\theta_u^r - \theta_u^s\theta_q^r \neq 0$, *we have that, for any* $(u,q,p,r,s) \in \left[H^{k+1}(K)\right]^5$,

$$\|\delta_u\| + \|\delta_q\| + \|\delta_p\| + \|\delta_r\| + \|\delta_s\| \leq Ch^{k+1}.$$

The proof of this lemma is given in Appendix B.

The last lemma is about the optimal convergence of the initial approximations. It is usual to require that the projection of the error at the initial step $\epsilon(0)$ satisfies certain condition for the solution at later time to have optimal convergence, see also [9,21].

Lemma 3.4 (On the initial approximations). *Under the assumption in Theorem 3.1, we have the following estimates for the errors at* $t=0$ *by using the HDG method to solve the stationary fifth-order equation* (2.5)

$$\|\varepsilon_\omega(0)\| \leq Ch^{k+2}, \quad ||\widehat{e}_\omega(0)||_{\infty,\mathscr{E}_h} \leq Ch^{2k+1} \qquad \text{for } \omega = u,q,p,r,s$$

and

$$\|\varepsilon_{u_t}(0)\| \leq Ch^{k+1}.$$

Lemma 3.4 shows that our HDG method for the stationary fifth-order linear equation produces superconvergent approximations for all variables. This result does not hold for the time-dependent case; we observe in numerical tests that HDG approximations to some variables lose superconvergence rates as time evolves. However, they still maintain optimal convergence theoretically and numerically.

The proof of Lemma 3.4 involves the error analysis of the HDG method for stationary fifth-order equations. It is lengthy, thus we give a detailed sketch in Appendix C, and proceed here to prove the main theorem in the remaining of this section.

Proof of Theorem 3.1. For simplicity, we assume $\alpha = 0$. We first add the six energy identities (3.3) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon(t)\|^2 + S + T = 0 \quad \text{with } S = \sum_{i=1}^{6} S_i, \quad T = \sum_{i=1}^{6} T_i.$$

Next, we integrate it from 0 to t and apply the result $T \geq 0$ from Lemma 3.2. We have

$$\|\varepsilon(t)\|^2 \leq \|\varepsilon(0)\|^2 + \left|\int_0^t S(s)\mathrm{d}s\right|.$$

To estimate $\int_0^t S(s)\mathrm{d}s$, we rewrite S as $S = I_1 + I_2 + I_3$ with

$$\begin{aligned}
I_1 &= (\delta_{ut},\varepsilon_u) - (\delta_q,\beta\varepsilon_s) + (\delta_p,\beta\varepsilon_r) - (\delta_r,\beta\varepsilon_p) + (\delta_s,\beta\varepsilon_q)\\
&\quad + (\delta_{qt},\varepsilon_q) + (\delta_p,\varepsilon_{ut}) + (\delta_{pt},\varepsilon_p) - (\delta_{qt},\varepsilon_r) - (\delta_s,\varepsilon_{ut})\\
&\quad + (\delta_{rt},\varepsilon_r) - (\delta_{pt},\varepsilon_s) - \frac{1}{\beta}(\delta_{qt},\varepsilon_{ut}) + (\delta_{st},\varepsilon_s) + \frac{1}{\beta}(\delta_{rt},\varepsilon_{ut}) + (\delta_{u_{tt}},\varepsilon_{ut}),\\
I_2 &= (\delta_r,\varepsilon_{qt}) + (\delta_s,\varepsilon_{pt}) + \frac{1}{\beta}(\delta_{ut},\varepsilon_{qt}) - \frac{1}{\beta}(\delta_{ut},\varepsilon_{rt}) + \frac{1}{\beta}(\delta_{qt},\varepsilon_{pt}) - \frac{1}{\beta}(\delta_{pt},\varepsilon_{qt})\\
&\quad - (\delta_{qt},\beta\varepsilon_{st}) + (\delta_{pt},\beta\varepsilon_{rt}) + (\delta_{st},\beta\varepsilon_{qt}) - (\delta_{rt},\beta\varepsilon_{pt}),\\
I_3 &= (\varepsilon_p,\varepsilon_{ut}) + (\varepsilon_s,\varepsilon_{ut}).
\end{aligned}$$

Using the Cauchy inequality and the approximation property of the projection Lemma 3.3, we know $|I_1| \leq Ch^{k+1}\|\varepsilon\|$ which means that

$$\int_0^t |I_1|\,\mathrm{d}t \leq Ch^{k+1}\int_0^t \|\varepsilon(s)\|\mathrm{d}s \leq \frac{C}{\epsilon}Th^{2k+2} + C\epsilon\int_0^t \|\varepsilon\|^2\mathrm{d}t.$$

TABLE 1. Setup for the first two problems.

	Problem 1	Problem 2
Domain	$[0, 2\pi]$	$[0, 2\pi]$
(α, β)	$(0, -1)$	$(1, -1)$
F	0	$u + u^2 + u^3$
$u(x, t)$	$\sin(x + t)$	$\sin(x + t)$

For I_2, we have

$$\int_0^t I_2 \mathrm{d}t \le \left[(\delta_r, \varepsilon_q) + (\delta_s, \varepsilon_p) + \frac{1}{\beta}(\delta_{ut}, \varepsilon_q) - \frac{1}{\beta}(\delta_{ut}, \varepsilon_r) + \frac{1}{\beta}(\delta_{qt}, \varepsilon_p) - \frac{1}{\beta}(\delta_{pt}, \varepsilon_q)\right]_0^t + \int_0^t \left[-(\delta_{rt}, \varepsilon_q) - (\delta_{st}, \varepsilon_p) - \frac{1}{\beta}(\delta_{utt}, \varepsilon_q) + \frac{1}{\beta}(\delta_{utt}, \varepsilon_r) - \frac{1}{\beta}(\delta_{qtt}, \varepsilon_p) + \frac{1}{\beta}(\delta_{ptt}, \varepsilon_q)\right] \mathrm{d}t,$$

which means

$$\left|\int_0^t I_2 \mathrm{d}t\right| \le Ch^{k+1}\left(\|\varepsilon(0)\| + \|\varepsilon(t)\|\right) + Ch^{k+1}\int_0^t \|\varepsilon(s)\| \mathrm{d}s.$$

As to I_3, we have $|I_3| \le (\|\varepsilon_p\| + \|\varepsilon_s\|)\|\varepsilon_{ut}\| \le \|\varepsilon\|^2$ which means that

$$\int_0^t |I_3| \mathrm{d}t \le \int_0^t \|\varepsilon\|^2 \mathrm{d}t.$$

Putting these three inequalities together, we have

$$\begin{aligned}\left|\int_0^t S \mathrm{d}t\right| &\le \int_0^t |I_1| \mathrm{d}t + \left|\int_0^t I_2 \mathrm{d}t\right| + \int_0^t |I_3| \mathrm{d}t \\ &\le C(1+t)h^{2k+2} + C\|\varepsilon(0)\|^2 + \frac{1}{2}\|\varepsilon(t)\|^2 + C\int_0^t \|\varepsilon(t)\|^2 \mathrm{d}t.\end{aligned}$$

Hence, we have

$$\|\varepsilon(t)\|^2 \le C(1+t)h^{2k+2} + C\|\varepsilon(0)\|^2 + C\int_0^t \|\varepsilon(t)\|^2 \mathrm{d}t.$$

As a final step, we apply $\|\varepsilon(0)\| \le Ch^{k+1}$ from Lemma 3.4 and the Gronwall's inequality to obtain $\|\varepsilon(t)\|^2 \le Ch^{2k+2}$ which finishes the proof of Theorem 3.1.

4. NUMERICAL RESULTS

In this section, we test our scheme on four test problems to corroborate our theoretical results. The first two problems have the periodic boundary condition (2.1b) while the last two are furnished with Dirichlet boundary condition (2.9). The spatial discretization polynomial degree k is set to be between 0 and 3. For time discretization, we use the second-order midpoint rule described in Section 2.1.3.

4.1. Tests with periodic boundary condition

We consider the linear equation in Problem 1 and the nonlinear equation in Problem 2 with periodic boundary conditions. The characteristics for the two problems are given in Table 1.

In the first two tests, we solve Problems 1 and 2 using our scheme, and the components of the stabilization function are taken to be $\tau_{rq}^{\pm} = -1, \tau_{su}^{\pm} = 1$, $\tau_{\mathcal{F}} = |F'(\widehat{u}_h)|$ and all other τ are zero, which satisfy (2.8) and thus

TABLE 2. Errors and orders of convergence for Problem 1 in Test 1.

k	e_u	Order	e_q	Order	e_p	Order	e_r	Order	e_s	Order
0	3.79e-1	–	7.03e-1	–	9.92e-1	–	1.21e-0	–	1.41e-0	–
	2.53e-1	0.58	4.14e-1	0.76	6.19e-1	0.68	8.22e-1	0.56	1.03e-0	0.45
	1.38e-1	0.88	2.18e-1	0.92	3.45e-1	0.84	4.81e-1	0.78	6.20e-1	0.73
	7.04e-2	0.97	1.10e-1	0.99	1.780e-1	0.94	2.56e-1	0.91	3.34e-1	0.89
	3.51e-2	1.00	5.49e-2	1.01	9.12e-2	0.98	1.31e-1	0.97	1.72e-1	0.96
1	3.27e-2	–	5.01e-2	–	5.21e-2	–	5.70e-2	–	6.58e-2	–
	1.09e-2	1.58	1.30e-2	1.95	1.32e-2	1.99	1.35e-2	2.07	1.42e-2	2.21
	3.16e-3	1.78	3.28e-3	1.99	3.29e-3	2.00	3.31e-3	2.02	3.36e-3	2.08
	8.20e-4	1.94	8.21e-4	2.00	8.22e-4	2.00	8.24e-4	2.01	8.26e-4	2.02
	2.05e-4	2.00	2.05e-4	2.00	2.06e-4	2.00	2.06e-4	2.00	2.06e-4	2.01
2	2.60e-3	–	3.12e-3	–	3.20e-3	–	3.18e-3	–	3.21e-3	–
	3.76e-4	2.79	4.05e-4	2.95	4.00e-4	3.00	3.99e-4	2.99	4.30e-4	2.90
	4.98e-5	2.92	4.98e-5	3.02	5.00e-5	3.00	5.00e-5	3.00	5.02e-5	3.10
	6.25e-6	2.99	6.24e-6	3.00	6.25e-6	3.00	6.25e-6	3.00	6.23e-6	3.01
	7.81e-7	3.00	7.81e-7	3.00	7.81e-7	3.00	7.81e-7	3.00	7.78e-7	3.00
3	1.65e-4	–	1.76e-4	–	1.77e-4	–	1.76e-4	–	1.80e-4	–
	1.20e-5	3.79	1.21e-5	3.86	1.21e-5	3.87	1.21e-5	3.86	1.25e-5	3.85
	8.06e-7	3.89	8.05e-7	3.91	8.06e-7	3.91	8.06e-7	3.91	8.12e-7	3.94
	5.04e-8	4.00	5.03e-8	4.00	5.04e-8	4.00	5.04e-8	4.00	5.07e-8	4.00
	3.15e-9	4.00	3.15e-9	4.00	3.15e-9	4.00	3.15e-9	4.00	3.19e-9	3.99

TABLE 3. Errors and orders of convergence for Problem 2 in Test 2.

k	e_u	Order	e_q	Order	e_p	Order	e_r	Order	e_s	Order
0	3.88e-1	–	6.29e-1	–	9.64e-1	–	1.22e-0	–	1.47e-0	–
	2.60e-1	0.57	3.55e-1	0.83	5.74e-1	0.75	7.95e-1	0.62	1.03e-0	0.51
	1.40e-1	0.89	1.77e-1	1.00	3.01e-1	0.93	4.41e-1	0.85	5.90e-1	0.80
	7.10e-2	0.98	8.65e-2	1.03	1.51e-1	0.99	2.28e-1	0.95	3.11e-1	0.92
	3.55e-2	1.00	4.23e-2	1.03	7.53e-2	1.01	1.15e-1	0.98	1.59e-1	0.96
1	5.26e-2	–	5.56e-2	–	5.19e-2	–	5.70e-2	–	1.97e-1	–
	4.58e-3	2.22	1.35e-2	2.04	1.31e-2	1.98	1.31e-2	2.01	5.76e-2	1.78
	1.07e-3	2.10	3.31e-3	2.03	3.29e-3	2.00	3.28e-3	2.00	1.51e-2	1.94
	2.63e-4	2.03	8.23e-4	2.01	8.22e-4	2.00	8.22e-4	2.00	3.81e-3	1.98
	6.55e-5	2.01	2.06e-4	2.00	2.06e-4	2.00	2.06e-4	2.00	9.55e-4	2.00
2	1.44e-3	–	3.31e-3	–	3.28e-3	–	3.46e-3	–	9.11e-3	–
	1.79e-4	3.01	4.05e-4	2.96	4.03e-4	3.02	4.10e-4	3.08	1.22e-3	2.91
	2.23e-5	3.00	4.98e-5	3.02	5.01e-5	3.00	5.03e-5	3.03	1.50e-4	3.02
	2.79e-6	3.00	6.24e-6	3.00	6.25e-6	3.00	6.26e-6	3.01	1.88e-5	2.99
	3.49e-7	3.00	7.81e-7	3.00	7.81e-7	3.00	7.82e-7	3.00	2.35e-6	3.00
3	1.25e-4	–	2.21e-4	–	3.14e-4	–	5.53e-4	–	1.24e-4	–
	9.68e-6	3.69	1.62e-5	3.77	2.52e-5	3.64	4.85e-5	3.51	1.14e-5	3.44
	6.86e-7	3.82	1.11e-6	3.87	1.73e-6	3.86	3.22e-6	3.91	7.06e-6	4.01
	4.28e-8	4.00	6.90e-8	4.00	1.08e-7	4.00	2.00e-7	4.01	4.35e-7	4.02
	2.68e-9	4.00	4.31e-9	4.00	6.75e-9	4.00	1.26e-8	3.99	2.75e-8	3.98

the conditions in Theorem 2.1. In the third test, we solve Problem 1 with $\tau_{rq}^{\pm} = -1, \tau_{su}^{+} = 1$, $\tau_{su}^{-} = -0.5$ and all other τ zero, which violate the conditions in Theorem 2.1.

TABLE 4. Errors and orders of convergence for Problem 1 when conditions in Theorem 2.1 are violated.

k	e_u	Order	e_q	Order	e_p	Order	e_r	Order	e_s	Order
0	0.17e-0	–	1.60e-0	–	1.79e-0	–	2.06e-0	–	2.41e-0	–
	0.13e-0	0.45	1.39e-0	0.20	1.59e-0	0.17	1.81e-0	0.19	2.03e-0	0.24
	7.86e-2	0.69	0.98e-0	0.51	1.12e-0	0.51	1.26e-0	0.52	1.41e-0	0.53
	4.29e-2	0.87	0.56e-0	0.79	0.65e-0	0.79	0.73e-0	0.79	0.81e-0	0.80
	2.19e-2	0.97	0.30e-0	0.93	0.34e-0	0.92	0.38e-0	0.93	0.42e-0	0.93
1	7.80e-2	–	5.50e-2	–	5.68e-2	–	6.76e-2	–	7.54e-2	–
	3.30e-2	1.24	1.37e-2	2.01	1.39e-2	2.03	1.49e-2	2.18	2.33e-2	1.69
	1.29e-2	1.35	3.38e-3	2.02	3.34e-3	2.02	3.49e-3	2.09	9.00e-3	1.37
	4.59e-3	1.50	8.35e-4	2.02	8.35e-4	2.02	8.42e-4	2.05	3.24e-3	1.47
	1.38e-3	1.74	2.07e-4	2.01	2.07e-4	2.01	2.07e-4	2.02	9.81e-4	1.73
2	7.25e-3	–	3.15e-3	–	3.10e-3	–	3.29e-3	–	6.40e-3	–
	1.38e-3	2.39	4.07e-4	2.95	3.96e-4	2.97	4.05e-4	3.02	1.05e-3	2.61
	2.54e-4	2.44	5.00e-5	3.03	4.99e-5	2.99	5.03e-5	3.01	1.82e-4	2.53
	4.04e-5	2.66	6.25e-6	3.00	6.24e-6	3.00	6.27e-6	3.00	2.88e-5	2.66
	5.43e-6	2.89	7.82e-7	3.00	7.81e-7	3.00	7.82e-7	3.00	3.89e-6	2.89
3	4.49e-4	–	1.55e-4	–	1.53e-4	–	1.58e-4	–	3.55e-4	–
	4.08e-5	3.45	1.00e-5	3.95	9.90e-6	3.95	1.00e-5	3.98	2.92e-5	3.60
	3.48e-6	3.55	6.33e-7	3.99	6.31e-7	3.97	6.33e-7	3.98	2.48e-6	3.56
	2.52e-7	3.78	3.95e-8	4.00	3.95e-8	4.00	3.95e-8	4.00	1.81e-7	3.78
	1.62e-8	3.96	2.47e-9	4.00	2.47e-9	4.00	2.47e-9	4.00	1.16e-8	3.96

TABLE 5. Setup for the next two problems.

	Problem 3	Problem 4
Domain	$[0, \pi]$	$[0, \pi]$
(α, β)	$(0, -1)$	$(1, -1)$
F	0	$u + u^2 + u^3$
$u(x, t)$	$t \cdot \sin(x)$	$t \cdot \sin(x)$

We take $h = 2^{-n}$ for $n = 3, 4, 5, 6, 7$ in all three tests. The step size for time discretization is $\Delta t = 0.1 \times h$ for $k = 0, 1$, $\Delta t = 0.1 \times h^2$ for $k = 2, 3$. This is just to ensure that the temporal errors are negligible comparing with the space error. We compute the orders of convergence of p_h, q_h, u_h, r_h, s_h at the final time $T = 0.1$.

Results. In the first two tests, we compute the orders of convergence for p_h, q_h, u_h, r_h, s_h at the final time $T = 0.1$ for the linear equation in Problem 1 and the nonlinear equation in Problem 2. The results are listed in the Tables 2 and 3 which clearly show optimal convergence. In the third test, when the conditions in the stability Theorem 2.1 are violated, we compute the errors and orders for p_h, q_h, u_h, r_h, s_h at the final time $T = 0.1$ and still observe optimal convergence rates. The results are listed in the Table 4.

4.2. Tests with Dirichlet boundary condition

Although our error analysis is for KdV equations with periodic boundary conditions, we would like to investigate the convergence rates of our HDG method for Dirichlet boundary conditions. In test Problems 3 and 4, the stabilization functions are taken to be $\tau_{su}^{\pm} = \tau_{sq}^{\pm} = \tau_{ru}^{\pm} = 1, \tau_{rq}^{\pm} = -1$, $\tau_{\mathcal{F}} = |F'(\widehat{u}_h)|$ and others zero. The step size in time is again set to be $\Delta t = 0.1 \times h$ for $k = 0, 1$, and $\Delta t = 0.1 \times h^2$ for $k = 2, 3$.

Test problems. The characteristics for the two problems are given in Table 5.

TABLE 6. Error and convergence orders for problem 3.

k	e_u	Order	e_q	Order	e_p	Order	e_r	Order	e_s	Order
0	9.56e-3	–	6.99e-3	–	2.16e-2	–	5.48e-2	–	9.03e-2	–
	7.20e-3	0.41	4.38e-3	0.67	1.20e-2	0.85	3.17e-2	0.79	6.98e-2	0.37
	4.63e-3	0.64	2.67e-3	0.71	6.27e-3	0.94	1.89e-2	0.74	4.95e-2	0.50
	2.67e-3	0.80	1.51e-3	0.83	3.31e-3	0.92	1.11e-2	0.76	3.00e-2	0.72
	1.44e-3	0.89	8.02e-4	0.91	1.73e-3	0.93	6.18e-3	0.85	1.65e-2	0.86
1	4.49e-4	–	7.58e-4	–	9.75e-4	–	1.25e-3	–	1.92e-3	–
	1.36e-4	1.72	1.76e-4	2.11	2.39e-4	2.03	3.21e-4	1.97	3.72e-4	2.37
	3.75e-5	1.86	4.24e-5	2.05	5.90e-5	2.02	8.12e-5	1.98	8.49e-5	2.13
	9.82e-6	1.93	1.04e-5	2.02	1.46e-5	2.01	2.04e-5	1.99	2.07e-5	2.04
	2.51e-6	1.97	2.59e-6	2.02	3.65e-6	2.01	5.12e-6	2.00	5.14e-6	2.01
2	1.01e-4	–	2.00e-4	–	2.43e-4	–	2.84e-4	–	3.52e-4	–
	1.60e-5	2.66	2.20e-5	3.19	2.94e-5	3.04	3.81e-5	2.89	4.00e-5	3.14
	2.24e-6	2.83	2.61e-6	3.08	3.61e-6	3.03	4.90e-6	2.96	5.01e-6	3.00
	2.96e-7	2.92	3.19e-7	3.03	4.46e-7	3.01	6.19e-7	2.98	6.24e-7	3.00
	3.81e-8	2.96	3.94e-8	3.01	5.55e-8	3.01	7.78e-8	2.99	7.80e-8	3.00
3	5.44e-6	–	8.81e-6	–	1.13e-5	–	1.41e-5	–	1.58e-5	–
	4.06e-7	3.74	5.05e-7	4.12	6.89e-7	4.03	9.19e-7	3.94	9.43e-7	4.06
	2.75e-8	3.88	3.05e-8	4.05	4.25e-8	4.02	5.84e-8	3.97	5.92e-8	3.99
	1.79e-9	3.94	1.88e-9	4.02	2.64e-9	4.01	3.68e-9	3.99	3.70e-9	4.00
	1.14e-10	3.97	1.17e-10	4.01	1.64e-10	4.01	2.31e-10	4.00	2.31e-10	4.00

TABLE 7. Error and convergence orders for problem 4.

k	e_u	Order	e_q	Order	e_p	Order	e_r	Order	e_s	Order
0	9.01e-3	–	7.66e-3	–	2.20e-2	–	5.53e-2	–	8.45e-2	–
	6.29e-3	0.52	4.22e-3	0.86	1.22e-2	0.85	3.11e-2	0.83	5.83e-2	0.54
	3.86e-3	0.70	2.47e-3	0.77	6.29e-3	0.96	1.67e-2	0.89	3.80e-2	0.62
	2.19e-3	0.82	1.40e-3	0.82	3.19e-3	0.98	8.94e-3	0.90	2.22e-2	0.77
	1.18e-3	0.89	7.53e-4	0.89	1.61e-3	0.98	4.70e-3	0.93	1.21e-2	0.88
1	3.50e-4	–	7.95e-4	–	9.73e-4	–	1.23e-3	–	2.50e-3	–
	9.46e-5	1.89	1.87e-4	2.09	2.39e-4	2.03	3.20e-4	1.94	5.72e-4	2.13
	2.48e-5	1.93	4.51e-5	2.05	5.90e-5	2.02	8.12e-5	1.98	1.40e-4	2.03
	6.37e-6	1.96	1.11e-5	2.03	1.46e-5	2.01	2.04e-5	1.99	3.50e-5	2.00
	1.61e-6	1.98	2.74e-6	2.01	3.65e-6	2.01	5.12e-6	2.00	8.75e-6	2.00
2	8.02e-5	–	2.09e-4	–	2.45e-4	–	2.93e-4	–	5.14e-4	–
	1.12e-5	2.84	2.37e-5	3.14	2.95e-5	3.05	3.86e-5	2.92	6.43e-5	3.00
	1.49e-6	2.91	2.82e-6	3.07	3.61e-6	3.03	4.92e-6	2.97	8.16e-6	2.98
	1.92e-7	2.95	3.44e-7	3.04	4.46e-7	3.01	6.21e-7	2.99	1.02e-6	3.00
	2.45e-8	2.98	4.25e-8	3.02	5.55e-8	3.01	7.79e-8	2.99	1.28e-7	3.00
3	4.17e-6	–	9.20e-6	–	1.13e-5	–	1.43e-5	–	2.51e-5	–
	2.79e-7	3.90	5.36e-7	4.10	6.90e-7	4.04	9.25e-7	3.95	1.59e-6	3.98
	1.81e-8	3.94	3.24e-8	4.05	4.25e-8	4.02	5.86e-8	3.98	1.01e-7	3.98
	1.16e-9	3.97	2.00e-9	4.03	2.64e-9	4.01	3.69e-9	3.99	6.30e-9	4.00
	7.30e-11	3.99	1.23e-10	4.01	1.64e-10	4.01	2.31e-9	4.00	3.94e-10	4.00

Results. We show the convergence orders in Tables 6 and 7 for the errors in all the five variables for the linear Problems 3 and the nonlinear Problem 4. Optimal convergence rates are achieved for both the linear and the nonlinear cases for all the variables.

5. Concluding remarks

In this paper, we introduce and analyze the first HDG method for fifth-order KdV type equations as well as fifth-order stationary equations. Our analysis show that the HDG approximate solutions have optimal convergence rates for linearized equations in the time-dependent case and superconvergence properties in the stationary case. Numerical results indicate that the method also has optimal convergence rates for nonlinear KdV equations. Our future work is to develop HDG methods for multidimensional KdV type equations and systems that involve third- or fifth-order derivatives.

Appendix A. Proof of Lemma 3.2

First, we derive a suitable error equation. Toward that end, we note that the exact solutions also satisfy (2.2). Subtracting (2.2) from its counterpart satisfied by the exact solution gives us

$$\begin{cases} (e_q, v) + (e_u, v_x) - \langle \widehat{e}_u, v \cdot n \rangle = 0, \\ (e_p, z) + (e_q, z_x) - \langle \widehat{e}_q, z \cdot n \rangle = 0, \\ (e_r, w) + (e_p, w_x) - \langle \widehat{e}_p, w \cdot n \rangle = 0, \\ (e_s, \phi) + (e_r, \phi_x) - \langle \widehat{e}_r, \phi \cdot n \rangle = 0, \\ (e_{u_t}, \psi) - (\alpha e_p + \beta e_s, \psi_x) + \langle \alpha \widehat{e}_p + \beta \widehat{e}_s, \psi \cdot n \rangle = 0. \end{cases}$$

We also have that

$$[\![\widehat{e}_p \cdot n]\!] = 0, \quad [\![\widehat{e}_r \cdot n]\!] = 0, \quad [\![\widehat{e}_s \cdot n]\!] = 0.$$

If we define the projections of the numerical flux errors as

$$\begin{aligned} &\widehat{\varepsilon}_u = \widehat{e}_u, \quad \widehat{\varepsilon}_q = \widehat{e}_q, \quad \widehat{\varepsilon}_p^- = \widehat{e}_p^-, \\ &\widehat{\varepsilon}_\omega^+ = \epsilon_\omega^+ + \tau_{\omega u}^+ (\widehat{\varepsilon}_u - \varepsilon_u^+) n^+ + \tau_{\omega q}^+ (\widehat{\varepsilon}_q - \varepsilon_q^+) n^+ \text{ for } \omega \in \{p, r, s\}, \\ &\widehat{\varepsilon}_\omega^- = \epsilon_\omega^- + \tau_{\omega u}^- (\widehat{\varepsilon}_u - \varepsilon_u^-) n^- + \tau_{\omega q}^- (\widehat{\varepsilon}_q - \varepsilon_q^-) n^- + \tau_{\omega p}^- (\widehat{\varepsilon}_p^- - \varepsilon_p^-) n^- \text{ for } \omega \in \{r, s\}, \end{aligned}$$

it is easy to show that

$$\widehat{\varepsilon}_\omega = \widehat{e}_\omega, \qquad \text{for } \omega \in \{u, q, p, r, s\}$$

by using the definition of the projection Π. Realizing that $e_\omega = \delta_\omega + \varepsilon_\omega$ and using the definition of the projection Π, we can rewrite the error equation above as

$$(\varepsilon_q, v) + (\delta_q, v) + (\varepsilon_u, v_x) - \langle \widehat{\varepsilon}_u, v \cdot n \rangle = 0, \tag{A.2a}$$
$$(\varepsilon_p, z) + (\delta_p, z) + (\varepsilon_q, z_x) - \langle \widehat{\varepsilon}_q, z \cdot n \rangle = 0, \tag{A.2b}$$
$$(\varepsilon_r, w) + (\delta_r, w) + (\varepsilon_p, w_x) - \langle \widehat{\varepsilon}_p, w \cdot n \rangle = 0, \tag{A.2c}$$
$$(\varepsilon_s, \phi) + (\delta_s, \phi) + (\varepsilon_r, \phi_x) - \langle \widehat{\varepsilon}_r, \phi \cdot n \rangle = 0, \tag{A.2d}$$
$$(\varepsilon_{u_t}, \psi) + (\delta_{u_t}, \psi) - (\alpha \varepsilon_p + \beta \varepsilon_s, \psi_x) + \langle \alpha \widehat{\varepsilon}_p + \beta \widehat{\varepsilon}_s, \psi \cdot n \rangle = 0, \tag{A.2e}$$

and that

$$[\![\widehat{\varepsilon}_p \cdot n]\!] = 0, \quad [\![\widehat{\varepsilon}_r \cdot n]\!] = 0, \quad [\![\widehat{\varepsilon}_s \cdot n]\!] = 0.$$

To derive the energy identities, we will also need the time-derivatives of the equations (A.2a)–(A.2e).

$$(\varepsilon_{q_t}, v) + (\delta_{q_t}, v) + (\varepsilon_{u_t}, v_x) - \langle \widehat{\varepsilon}_{u_t}, v \cdot n \rangle = 0, \tag{A.3a}$$
$$(\varepsilon_{p_t}, z) + (\delta_{p_t}, z) + (\varepsilon_{q_t}, z_x) - \langle \widehat{\varepsilon}_{q_t}, z \cdot n \rangle = 0, \tag{A.3b}$$
$$(\varepsilon_{r_t}, w) + (\delta_{r_t}, w) + (\varepsilon_{p_t}, w_x) - \langle \widehat{\varepsilon}_{p_t}, w \cdot n \rangle = 0, \tag{A.3c}$$
$$(\varepsilon_{s_t}, \phi) + (\delta_{s_t}, \phi) + (\varepsilon_{r_t}, \phi_x) - \langle \widehat{\varepsilon}_{r_t}, \phi \cdot n \rangle = 0, \tag{A.3d}$$
$$(\varepsilon_{u_{tt}}, \psi) + (\delta_{u_{tt}}, \psi) - (\alpha\varepsilon_{p_t} + \beta\varepsilon_{s_t}, \psi_x) + \langle \alpha\widehat{\varepsilon}_{p_t} + \beta\widehat{\varepsilon}_{s_t}, \psi \cdot n \rangle = 0. \tag{A.3e}$$

We are now ready to prove the six energy identities in Lemma 3.2. The proofs are quite tedious, but very similar in nature. They also use similar techniques such as integration by parts, and error equations (A.2) and (A.3). For these reasons, we are simply listing the first steps of deriving these six energy identities. The rest for proving each energy identity are simply algebraic manipulations.

Indeed, we take

$$\psi = \varepsilon_u, \quad v = -(\alpha\varepsilon_p + \beta\varepsilon_s), \quad z = \alpha\varepsilon_q + \beta\varepsilon_r, \quad \phi = \beta\varepsilon_q, \quad w = -\beta\varepsilon_p$$

in (A.2a)–(A.2e), add them together to arrive at (3.3a), and

$$v = \varepsilon_q \text{ in (A.3a)}, \quad z = \varepsilon_{u_t} \text{ in (A.2b)}$$

for deriving (3.3b), and

$$z = \varepsilon_p \text{ in (A.3b)}, \quad w = \varepsilon_{q_t} \text{ in (A.2c)}, \quad v = -\varepsilon_r \text{ in (A.3a)}, \quad \phi = -\varepsilon_{u_t} \text{ in (A.2d)}$$

for getting (3.3c), and

$$w = \varepsilon_r \text{ in (A.3c)}, \quad \phi = \varepsilon_{p_t} \text{ in (A.2d)}, \quad z = -(\varepsilon_s + \frac{\alpha}{\beta}\varepsilon_p) \text{ in (A.3b)},$$

$$\psi = \frac{1}{\beta}\varepsilon_{q_t} \text{ in (A.2e)}, \quad v = -\frac{1}{\beta}\varepsilon_{u_t} \text{ in (A.3a)}$$

for concluding (3.3d), and

$$\phi = \varepsilon_s \text{ in (A.3d)}, \quad \psi = -\frac{1}{\beta}\varepsilon_{r_t} \text{ in (A.2e)}, \quad w = \frac{1}{\beta}\varepsilon_{u_t} \text{ in (A.3c)},$$

$$w = -\frac{\alpha}{\beta}\varepsilon_{r_t} \text{ in (A.2c)}, \quad v = \frac{1}{\beta}\varepsilon_{p_t} \text{ in (A.3a)}, \quad z = -\frac{1}{\beta}\varepsilon_{q_t} \text{ in (A.3b)}$$

to obtain (3.3e), and finally

$$\psi = \varepsilon_{u_t} \text{ in (A.3e)}, \quad v = -(\alpha\varepsilon_{p_t} + \beta\varepsilon_{s_t}) \text{ in (A.3a)}, \quad z = \alpha\varepsilon_{q_t} + \beta\varepsilon_{r_t} \text{ in (A.3b)},$$

$$\phi = \beta\varepsilon_{q_t} \text{ in (A.3d)}, \quad w = -\beta\varepsilon_{p_t} \text{ in (A.3c)}$$

for proving (3.3f) to conclude the proof of the six energy identities.

To finish proving this lemma, we only need to establish that $T \geq 0$ which is the purpose of the next lemma.

Lemma A.1. *Let T be defined as in Lemma 3.2, and the stabilization functions are chosen so that the only nonzero ones are*

$$\tau_{su}^- = \tau_{su}^+ = 1, \quad \tau_{rq}^- = \tau_{rq}^+ = -1,$$

then for $\beta = -1$ we have $T \geq 0$.

Proof. Let's recall from Lemma 3.2 that T contains all the boundary terms. For brevity, we define $\eta_\omega = \widehat{\varepsilon}_\omega - \varepsilon_\omega$ for $\omega \in \{u, q, p, r, s\}$, T can then be rewritten as

$$\begin{aligned} T =& \langle n, -\beta\eta_s\eta_u + \beta\eta_r\eta_q - \frac{\beta}{2}\eta_p^2\rangle + \langle n, \eta_q\eta_{ut}\rangle + \langle n, \eta_p\eta_{qt} - \eta_r\eta_{ut}\rangle \\ &+ \langle n, \eta_r\eta_{pt} - \eta_s\eta_{qt} - \frac{1}{2\beta}\eta_{ut}^2\rangle + \langle n, \eta_s\eta_{rt} + \frac{1}{\beta}\eta_{ut}\eta_{pt} - \frac{1}{2\beta}\eta_{qt}^2\rangle \\ &+ \langle n, -\beta\eta_{st}\eta_{ut} + \beta\eta_{rt}\eta_{qt} - \frac{\beta}{2}\eta_{pt}^2\rangle \\ :=& T^- + T^+, \end{aligned}$$

where T^- contains those terms on $\partial\mathfrak{T}_h^-$ with $n^- = 1$ and T^+ on $\partial\mathfrak{T}_h^+$ for $n^+ = -1$. It suffices to simplify each of them and show that they are nonnegative.

$$\begin{aligned} T^- =& \langle 1, \eta_u^2\rangle\left(-\beta\tau_{su}^-\right) + \langle 1, \eta_q^2\rangle\left(\beta\tau_{rq}^-\right) + \langle 1, \eta_p^2\rangle\left(-\frac{\beta}{2}\right) \\ &+ \langle 1, \eta_{ut}^2\rangle\left(-\frac{1}{2\beta} - \beta\tau_{su}^-\right) + \langle 1, \eta_{qt}^2\rangle\left(-\frac{1}{2\beta} + \beta\tau_{rq}^-\right) + \langle 1, \eta_{pt}^2\rangle\left(-\frac{\beta}{2}\right) \\ &+ \langle 1, \eta_u\eta_{ut}\rangle\left(-\tau_{ru}^- + \tau_{su}^-\tau_{ru}^-\right) + \langle 1, \eta_q\eta_{qt}\rangle\left(-\tau_{sq}^- + \tau_{sq}^-\tau_{rq}^-\right) + \langle 1, \eta_p\eta_{pt}\rangle\left(\tau_{rp}^- + \tau_{sp}^-\tau_{rp}^-\right) \\ &+ \langle 1, \eta_u\eta_q\rangle\left(-\beta\tau_{sq}^- + \beta\tau_{ru}^-\right) + \langle 1, \eta_u\eta_p\rangle\left(-\beta\tau_{sp}^-\right) + \langle 1, \eta_p\eta_q\rangle\left(\beta\tau_{rp}^-\right) \\ &+ \langle 1, \eta_q\eta_{ut}\rangle\left(1 - \tau_{rq}^- + \tau_{sq}^-\tau_{ru}^-\right) + \langle 1, \eta_u\eta_{qt}\rangle\left(-\tau_{su}^- + \tau_{su}^-\tau_{rq}^-\right) \\ &+ \langle 1, \eta_p\eta_{qt}\rangle\left(1 - \tau_{sp}^- + \tau_{sp}^-\tau_{rq}^-\right) + \langle 1, \eta_q\eta_{pt}\rangle\left(\tau_{rq}^- + \tau_{sq}^-\tau_{rp}^-\right) \\ &+ \langle 1, \eta_u\eta_{pt}\rangle\left(\tau_{ru}^- + \tau_{su}^-\tau_{rp}^-\right) + \langle 1, \eta_p\eta_{ut}\rangle\left(-\tau_{rp}^- + \tau_{sp}^-\tau_{ru}^-\right) \\ &+ \langle 1, \eta_{ut}\eta_{pt}\rangle\left(\frac{1}{\beta} - \beta\tau_{sp}^-\right) + \langle 1, \eta_{qt}\eta_{ut}\rangle\left(-\beta\tau_{sq}^- + \beta\tau_{ru}^-\right) + \langle 1, \eta_{qt}\eta_{pt}\rangle\left(\beta\tau_{rp}^-\right). \end{aligned}$$

Therefore, if we choose $\tau_{su}^- = 1, \quad \tau_{rq}^- = -1$, other $\tau_{\alpha\omega}^-$ be zero, and $\beta = -1$, we can complete the square for T^-

$$T^- = \langle 1, (\eta_u - \eta_{qt})^2 + \frac{1}{2}(\eta_p + \eta_{qt})^2 + \frac{1}{2}(\eta_{pt} - \eta_q - \eta_{ut})^2 + \frac{1}{2}(\eta_q + \eta_{ut})^2 + \frac{1}{2}\eta_{ut}^2\rangle.$$

We conclude that $T^- \geq 0$. For T^+, we have

$$\begin{aligned} T^+ =& \langle 1, \eta_u^2\rangle\left(-\beta\tau_{su}^+ + \frac{\beta}{2}{\tau_{pu}^+}^2\right) + \langle 1, \eta_q^2\rangle\left(\beta\tau_{rq}^+ + \frac{\beta}{2}{\tau_{pq}^+}^2\right) \\ &+ \langle 1, \eta_{ut}^2\rangle\left(\frac{1}{2\beta} + \frac{1}{\beta}\tau_{pu}^+ - \beta\tau_{su}^+ + \frac{\beta}{2}{\tau_{pu}^+}^2\right) + \langle 1, \eta_{qt}^2\rangle\left(\frac{1}{2\beta} + \beta\tau_{rq}^+ + \frac{\beta}{2}{\tau_{pq}^+}^2\right) \\ &+ \langle 1, \eta_u\eta_{ut}\rangle\left(-\tau_{ru}^+ - \tau_{ru}^+\tau_{pu}^+ - \tau_{su}^+\tau_{ru}^+\right) + \langle 1, \eta_q\eta_{qt}\rangle\left(\tau_{pq}^+ - \tau_{rq}^+\tau_{pq}^+ - \tau_{sq}^+ - \tau_{sq}^+\tau_{rq}^+\right) \\ &+ \langle 1, \eta_u\eta_q\rangle\left(-\beta\tau_{sq}^+ + \beta\tau_{ru}^+ + \beta\tau_{pu}^+\tau_{pq}^+\right) + \langle 1, \eta_{ut}\eta_q\rangle\left(-1 - \tau_{rq}^+ - \tau_{rq}^+\tau_{pu}^+ - \tau_{sq}^+\tau_{ru}^+\right) \\ &+ \langle 1, \eta_{qt}\eta_u\rangle\left(\tau_{pu}^+ - \tau_{ru}^+\tau_{pq}^+ - \tau_{su}^+ - \tau_{su}^+\tau_{rq}^+\right) \\ &+ \langle 1, \eta_{qt}\eta_{ut}\rangle\left(\frac{1}{\beta}\tau_{pq}^+ - \beta\tau_{sq}^+ + \beta\tau_{ru}^+ + \beta\tau_{pu}^+\tau_{pq}^+\right). \end{aligned}$$

Therefore, if we choose $\tau_{su}^+ = 1, \quad \tau_{rq}^+ = -1$, other $\tau_{\alpha\omega}^+$ be zero, then for $\beta = -1$, we see that

$$T^+ = \langle 1, \eta_u^2 + \eta_q^2 + \frac{1}{2}\eta_{ut}^2 + \frac{1}{2}\eta_{qt}^2\rangle.$$

□

Appendix B. Proof of Lemma 3.3

We prove it by arguing that the distance between Π and the L^2-projection is optimal. Indeed, for any $\omega \in L^2(I_i)$, we denote its L^2-projection onto $P^k(I_i)$ by $(\omega)_k$. Then we have

$$\delta_\omega = g_\omega + d_\omega, \quad \text{where } g_\omega := \omega - (\omega)_k,\ d_\omega := (\omega)_k - \Pi\omega,\ \text{for } \omega \in \{u, q, p, r, s\},$$

and we only need to estimate d_ω as g_ω is of order $k+1$.

Obviously, $d_\omega \in P^k(I_i)$. From the definition of the projection Π, we have that $(d_\omega, v) = 0$ for any $v \in P^{k-1}(I_i)$. This means that

$$d_\omega = c_\omega \cdot L_k \tag{B.1}$$

where L_k is the scaled Legendre polynomial of degree k. Next, we rewrite the definition of the projection as

$$\begin{cases} d_s - \tau_{su}^- d_u - \tau_{sq}^- d_q - \tau_{sp}^- d_p = -\left(g_s - \tau_{su}^- g_u - \tau_{sq}^- g_q - \tau_{sp}^- g_p\right) & \text{on } x_i^-, \\ d_r - \tau_{ru}^- d_u - \tau_{rq}^- d_q - \tau_{rp}^- d_p = -\left(g_r - \tau_{ru}^- g_u - \tau_{rq}^- g_q - \tau_{rp}^- g_p\right) & \text{on } x_i^-, \\ d_p + \tau_{pu}^+ d_u + \tau_{pq}^+ d_q = -\left(g_p + \tau_{pu}^+ g_u + \tau_{pq}^+ g_q\right) & \text{on } x_{i-1}^+, \\ d_s + \tau_{su}^+ d_u + \tau_{sq}^+ d_q = -\left(g_s + \tau_{su}^+ g_u + \tau_{sq}^+ g_q\right) & \text{on } x_{i-1}^+, \\ d_r + \tau_{ru}^+ d_u + \tau_{rq}^+ d_q = -\left(g_r + \tau_{ru}^+ g_u + \tau_{rq}^+ g_q\right) & \text{on } x_{i-1}^+. \end{cases}$$

Applying (B.1) and the fact that $L_k(x_i^-) = 1$ and $L_k(x_{i-1}^+) = (-1)^k$, this linear system can be written as

$$\mathbb{A}\vec{c} = \vec{b}$$

where

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & -\tau_{sp}^- & -\tau_{sq}^- & -\tau_{su}^- \\ 0 & 1 & -\tau_{rp}^- & -\tau_{rq}^- & -\tau_{ru}^- \\ 0 & 0 & (-1)^k & (-1)^k\tau_{pq}^+ & (-1)^k\tau_{pu}^+ \\ (-1)^k & 0 & 0 & (-1)^k\tau_{sq}^+ & (-1)^k\tau_{su}^+ \\ 0 & (-1)^k & 0 & (-1)^k\tau_{rq}^+ & (-1)^k\tau_{ru}^+ \end{pmatrix}$$

and

$$\vec{c} = \begin{pmatrix} c_s \\ c_r \\ c_p \\ c_q \\ c_u \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = -\begin{pmatrix} g_s - \tau_{su}^- g_u - \tau_{sq}^- g_q + \tau_{sp}^- g_p \\ g_r - \tau_{ru}^- g_u - \tau_{rq}^- g_q + \tau_{rp}^- g_p \\ g_p + \tau_{pu}^+ g_u + \tau_{pq}^- g_q \\ g_s + \tau_{su}^+ g_u + \tau_{sq}^- g_q \\ g_r + \tau_{ru}^+ g_u + \tau_{rq}^- g_q \end{pmatrix}.$$

Basic row operation of $\left(\mathbb{A}|\vec{b}\right)$ leads to

$$\left(\begin{array}{ccccc|c} 1 & 0 & -\tau_{sp}^- & -\tau_{sq}^- & -\tau_{su}^- & b_1 \\ 0 & 1 & -\tau_{rp}^- & -\tau_{rq}^- & -\tau_{ru}^- & b_2 \\ 0 & 0 & 1 & \tau_{pq}^+ & \tau_{pu}^+ & (-1)^k b_3 \\ 0 & 0 & 0 & \theta_q^s & \theta_u^s & \tilde{b}_4 \\ 0 & 0 & 0 & \theta_q^r & \theta_u^r & \tilde{b}_5 \end{array}\right),$$

where $\theta_\omega^\alpha = \tau_{\alpha\omega}^- + \tau_{\alpha\omega}^+ - \tau_{\alpha p}^- \tau_{p\omega}^+$, $\tilde{b}_4 = -b_1 + (-1)^k b_4 - (-1)^k \tau_{sp}^- b_3$, and $\tilde{b}_5 = -b_2 + (-1)^k b_5 - (-1)^k \tau_{rp}^- b_3$. We can then invoke the Cramer's rule to solve for $\vec{c}$ when the linear system is uniquely solvable, that is when

$$\Delta := \theta_q^s \theta_u^r - \theta_u^s \theta_q^r \neq 0.$$

Under this condition, we have

$$
\begin{aligned}
c_q &= \frac{\tilde{b}_4\theta_u^r - \tilde{b}_5\theta_u^s}{\Delta},\\
c_u &= \frac{\tilde{b}_5\theta_q^s - \tilde{b}_4\theta_q^r}{\Delta},\\
c_p &= (-1)^k b_3 - \tau_{pq}^+ c_q - \tau_{pu}^+ c_u,\\
c_r &= b_3 + \tau_{rp}^- c_p + \tau_{rq}^- c_q + \tau_{ru}^- c_u,\\
c_s &= b_1 + \tau_{sp}^- c_p + \tau_{sq}^- c_q + \tau_{su}^- c_u.
\end{aligned}
$$

We conclude that $\|c_\omega\| \le Ch^{k+1}$ thanks to the fact that $\|g_\omega\| \le Ch^{k+1}$ for any $\omega \in H^{k+1}(I_i)$. This completes the proof of Lemma 3.3.

Remark B.1. The choice of τ in the error analysis is that the only nonzero stabilization functions are $\tau_{su}^- = \tau_{su}^+ = 1$, $\tau_{sq}^- = \tau_{sq}^+ = -1$. For this case, we have $\theta_q^s = 0$, $\theta_u^s = 2$, $\theta_q^r = -2$. This means that $\Delta = 4 \neq 0$.

Appendix C. Proof of Lemma 3.4

The structure and techniques for proving the accuracy of the scheme (2.7) are similar to [6]. For completeness and brevity of this appendix, we are listing here all the results but only provide a sketch of each proof. The main results are superconvergence of the projection of the error and that of the numerical fluxes, as stated in the following two theorems.

Theorem C.1 (Error estimate for the linear steady-state fifth-order equation)**.** *Let ω be the solution to the steady-state equation (2.6) with $F = 0$, $\alpha = 0$ and $\beta = -1$, Π be the projection defined by (3.1), and ω_h be the approximate solution given by (2.7), the projection of the error ε_ω satisfy*

$$\|\varepsilon_\omega\| \le Ch^{k+1+\min\{k,1\}} \qquad \text{for } \omega \in \{u, q, p, r, s\}$$

Theorem C.2 (Superconvergence of the numerical traces)**.** *Under the assumption of Theorem C.1, the error of the numerical traces $\widehat{e}_\omega(x_i)$ satisfy*

$$|\widehat{e}_\omega(x_i)| \le Ch^{2k+1} \qquad \text{for } \omega \in \{u, q, p, r, s\}.$$

The proof of C.2 is very similar to that for the third-order equation [6]. We thus omit it and prove Theorem C.1 before moving to fulfill the main charge of this Appendix – establishing the main Lemma 3.4.

C.1. Proof for Theorem C.1.

The proof adopts a duality argument, see also [6]. For that purpose, we first introduce the dual problem

$$
\begin{aligned}
&\Psi_u' - \Psi_q = \eta_s, \quad \Psi_q' - \Psi_p = -\eta_r, \\
&\Psi_p' - \Psi_r = \eta_p, \quad \Psi_r' - \Psi_s = -\eta_q, \quad \beta\Psi_s' + \gamma\Psi_u = \eta_u \qquad \text{in } \Omega,\\
&\Psi_\omega(0,t) = \Psi_\omega(L,t) \quad \text{for } \omega \in \{u, q, p, r, s\},
\end{aligned}
\tag{C.1}
$$

where Ψ' denotes Ψ_x for any Ψ. Then we define a projection $\Pi^* : \left[H^1(\mathcal{T}_h)\right]^5 \longrightarrow \left[W_h^k\right]^5$ that is dual of projection Π defined by (3.1): For $(u, q, p, r, s) \in \left[H^1(\mathcal{T}_h)\right]^5$, we find in $\left[W_h^k\right]^5$ the element

$$(\Pi^* u, \Pi^* q, \Pi^* p, \Pi^* r, \Pi^* s)$$

whose restriction on element K satisfies the following condition

$$\begin{cases} (\delta^*\Psi_u, v)_{I_i} = 0,\ (\delta^*\Psi_q, z)_{I_i} = 0,\ (\delta^*\Psi_p, w)_{I_i} = 0, \\ (\delta^*\Psi_r, \phi)_{I_i} = 0,\ (\delta^*\Psi_s, \psi)_{I_i} = 0, \\ \delta^*\Psi_p^- - \tau_{rp}^-\delta\Psi_q n + \tau_{sp}^-\delta\Psi_u n = 0 & \text{at } x_i^- \\ \delta^*\Psi_r^- + \tau_{rq}^-\delta\Psi_q n - \tau_{sq}^-\delta\Psi_u n = 0 & \text{at } x_i^- \\ \delta^*\Psi_s^- - \tau_{ru}^-\delta\Psi_q n + \tau_{su}^-\delta\Psi_u n = 0 & \text{at } x_i^- \\ \delta^*\Psi_r^+ + \tau_{rq}^+\delta\Psi_q n - \tau_{sq}^+\delta\Psi_u n - \tau_{pq}^+\delta\Psi_p n = 0 & \text{at } x_{i-1}^+ \\ \delta^*\Psi_s^+ - \tau_{ru}^+\delta\Psi_q n + \tau_{su}^+\delta\Psi_u n + \tau_{pu}^+\delta\Psi_p n = 0 & \text{at } x_{i-1}^+ \end{cases}$$

for any $(v, z, w, \phi, \psi) \in P_{k-1}(I_i)$.

We have a lemma on the regularity of the dual problem, and a lemma on the approximation properties of the dual projection Π^*.

Lemma C.3 (Regularity of the dual problem)**.** *For the dual problem* (C.1)*, we have that*

$$\|\Psi_u\|_1 + \|\Psi_q\|_1 + \|\Psi_p\|_1 + \|\Psi_r\|_1 + \|\Psi_s\|_1 \le C_{\text{reg}}\|\eta\|,$$

where $\|\eta\| = \|\eta_u\| + \|\eta_q\| + \|\eta_p\| + \|\eta_r\| + \|\eta_s\|$.

Lemma C.4. *For the dual projection* Π^**, we have that*

$$\|\delta^*\Psi_\omega\| \le Ch\|\Psi_\omega\|_1, \quad \text{for } \omega \in \{u, q, p, r, s\}.$$

The proof of the second lemma is very similar to that of Π, therefore we only prove the first lemma here. We integrate the last equation of the dual problem (C.1) over the spatial domain Ω. Thanks to periodicity, we have

$$\gamma\bar{\Psi}_u = \bar{\eta}_u, \quad \text{where } \bar{\omega} := \int_\Omega \omega \mathrm{d}x \quad \text{for any } \omega \in L^2(\Omega),$$

which implies that $\|\bar{\Psi}_u\| = \|\gamma^{-1}\bar{\eta}_u\| \le \frac{C}{\gamma}\|\eta_u\|$. By Poincaré inequality and the first equation of (C.1), we have $\|\Psi_u - \bar{\Psi}_u\| \le C\|\Psi_u'\| \le \|\Psi_q\| + \|\eta_s\|$ which means

$$\|\Psi_u\| \le \|\Psi_q\| + \|\eta_s\| + \frac{C}{\gamma}\|\eta_u\|.$$

Similarly, we have

$$\begin{aligned} \|\Psi_q\| &\le \|\Psi_p\| + \|\eta_r\| + C\|\eta_s\| \\ \|\Psi_p\| &\le \|\Psi_r\| + \|\eta_p\| + C\|\eta_r\| \\ \|\Psi_r\| &\le \|\Psi_s\| + \|\eta_q\| + C\|\eta_p\| \\ \|\Psi_s\| &\le \frac{1}{|\beta|}\left(\gamma\|\Psi_u\| + \|\eta_u\|\right) + C\|\eta_q\|. \end{aligned}$$

We conclude that

$$\|\Psi_\omega\| \le \frac{C\gamma}{|\beta|}\|\Psi_u\| + C\|\eta\| \quad \text{for } \omega \in \{u, q, p, r, s\}. \tag{C.2}$$

Next, we multiply the five equations of (C.1) by $(\beta\Psi_s, -\beta\Psi_r, \beta\Psi_p, -\beta\Psi_q, \Psi_u)$ respectively, integrate them on Ω, and add all five together to obtain

$$\begin{aligned} \gamma\|\Psi_u\|^2 &= \beta\int_\Omega \left(\Psi_s\eta_s + \Psi_r\eta_r + \Psi_p\eta_p + \Psi_q\eta_q + \Psi_u\eta_u\right)\mathrm{d}x \\ &\le \frac{|\beta|\epsilon}{2}\left(\|\Psi_s\|^2 + \|\Psi_r\|^2 + \|\Psi_p\|^2 + \|\Psi_q\|^2 + \|\Psi_u\|^2\right) + \frac{|\beta|\epsilon^{-1}}{2}\|\eta\|^2. \end{aligned}$$

We then invoke (C.2) and let ϵ be small enough to arrive at $\|\Psi_u\| \le C\|\eta\|$ which implies, due to (C.2), that

$$\|\Psi_\omega\| \le C\|\eta\| \quad \text{for } \omega \in \{u, q, p, r, s\}.$$

Finally to conclude the proof of this lemma, we note from the first equation of (C.1) that

$$|\Psi_u|_1 \le \|\Psi_q\| + \|\eta_s\| \le C\|\eta\|.$$

Similarly, $|\Psi_\omega|_1 \le C\|\eta\|$ for $\omega \in \{q, p, r, s\}$.

With these two lemmas proven, we now go back to prove Theorem C.1. The first ingredient we need is the error equation that is similar to those for the time-dependent case (A.2a)–(A.2e). For simplicity,we assume that $\beta = -1$.

$$\begin{aligned}
&(\varepsilon_q, v) + (\delta_q, v) + (\varepsilon_u, v') - \langle \widehat{\varepsilon}_u, v \cdot n\rangle = 0,\\
&(\varepsilon_p, z) + (\delta_p, z) + (\varepsilon_q, z') - \langle \widehat{\varepsilon}_q, z \cdot n\rangle = 0,\\
&(\varepsilon_r, w) + (\delta_r, w) + (\varepsilon_p, w') - \langle \widehat{\varepsilon}_p, w \cdot n\rangle = 0,\\
&(\varepsilon_s, \phi) + (\delta_s, \phi) + (\varepsilon_r, \phi') - \langle \widehat{\varepsilon}_r, \phi \cdot n\rangle = 0,\\
&(\gamma\varepsilon_u, \psi) + (\gamma\delta_u, \psi) + (\varepsilon_s, \psi') - \langle \widehat{\varepsilon}_s, \psi \cdot n\rangle = 0.
\end{aligned} \tag{C.3}$$

The proof of theorem C.1 is through the classical duality argument. Indeed, we define

$$\|\varepsilon\|^2 := (\varepsilon_u, \eta_u) + (\varepsilon_q, \eta_q) + (\varepsilon_p, \eta_p) + (\varepsilon_r, \eta_r) + (\varepsilon_s, \eta_s)$$

Using the dual problem (C.1), we can rewrite it as $\|\varepsilon\|^2 = \Theta_I + \Theta_{II}$ where

$$\begin{aligned}
\Theta_I &= (\varepsilon_u, \Psi_s') - (\varepsilon_q, \Psi_r') + (\varepsilon_p, \Psi_p') - (\varepsilon_r, \Psi_q') + (\varepsilon_s, \Psi_u')\\
\Theta_{II} &= (\varepsilon_u, \gamma\Psi_u) + (\varepsilon_q, \Psi_s) - (\varepsilon_p, \Psi_r) + (\varepsilon_r, \Psi_p) - (\varepsilon_s, \Psi_q)
\end{aligned}$$

Invoking

$$(\varepsilon_\omega, \Psi_v') = (\varepsilon_\omega, \Pi^*\Psi_v') + (\varepsilon_\omega, \delta^*\Psi_v') = (\varepsilon_\omega, \Pi\Psi_v') + \langle \varepsilon_\omega, \delta^*\Psi_v \cdot n\rangle$$

by using integration by parts and the definition of Π^*, we can further rewrite Θ_I as $\Theta_I = I_1 + I_2$ with

$$\begin{aligned}
I_1 &= (\varepsilon_u, \Pi^*\Psi_s') - (\varepsilon_q, \Pi^*\Psi_r') + (\varepsilon_p, \Pi^*\Psi_p') - (\varepsilon_r, \Pi^*\Psi_q') + (\varepsilon_s, \Pi^*\Psi_u')\\
I_2 &= \langle \varepsilon_u, \delta^*\Psi_s \cdot n\rangle - \langle \varepsilon_q, \delta^*\Psi_r \cdot n\rangle + \langle \varepsilon_p, \delta^*\Psi_p \cdot n\rangle - \langle \varepsilon_r, \delta^*\Psi_q \cdot n\rangle + \langle \varepsilon_s, \delta^*\Psi_u \cdot n\rangle.
\end{aligned}$$

We now apply the error equation (C.3) with $v = \Pi^*\Psi_s, z = \Pi^*\Psi_r, w = \Pi^*\Psi_p, \phi = \Pi^*\Psi_q, \psi = \Pi^*\Psi_u$ to rewrite I_1 as

$$\begin{aligned}
I_1 = &-(\varepsilon_q, \Pi^*\Psi_s) + (\varepsilon_p, \Pi^*\Psi_r) - (\varepsilon_r, \Pi^*\Psi_p) + (\varepsilon_s, \Pi^*\Psi_q) - (\gamma\varepsilon_u, \Pi^*\Psi_u)\\
&-(\delta_q, \Pi^*\Psi_s) + (\delta_p, \Pi^*\Psi_r) - (\delta_r, \Pi^*\Psi_p) + (\delta_s, \Pi^*\Psi_q) - (\gamma\delta_u, \Pi^*\Psi_u)\\
&+\langle \widehat{\varepsilon}_u, \Pi^*\Psi_s \cdot n\rangle - \langle \widehat{\varepsilon}_q, \Pi^*\Psi_r \cdot n\rangle + \langle \widehat{\varepsilon}_p, \Pi^*\Psi_p \cdot n\rangle - \langle \widehat{\varepsilon}_r, \Pi^*\Psi_q \cdot n\rangle + \langle \widehat{\varepsilon}_s, \Pi^*\Psi_u \cdot n\rangle.
\end{aligned}$$

Adding I_1, I_2, and Θ_{II}, keeping in mind that $\widehat{\varepsilon}_\omega$ and Ψ_ω are single-valued and periodic for $\omega \in \{u, q, p, r, s\}$, and performing simple manipulations, we have

$$\|\varepsilon\|^2 = \theta_1 + \theta_2 + \theta_3,$$

where

$$\begin{aligned}
\theta_1 &= (\gamma\varepsilon_u, \delta^*\Psi_u) + (\varepsilon_q, \delta^*\Psi_s) - (\varepsilon_p, \delta^*\Psi_r) + (\varepsilon_r, \delta^*\Psi_p) - (\varepsilon_s, \delta^*\Psi_q),\\
\theta_2 &= -(\delta_q, \Pi^*\Psi_s) + (\delta_p, \Pi^*\Psi_r) - (\delta_r, \Pi^*\Psi_p) + (\delta_s, \Pi^*\Psi_q) - (\gamma\delta_u, \Pi^*\Psi_u),\\
\theta_3 &= -\langle \delta^*\Psi_s, (\widehat{\varepsilon}_u - \varepsilon_u) \cdot n\rangle + \langle \delta^*\Psi_r, (\widehat{\varepsilon}_q - \varepsilon_q) \cdot n\rangle - \langle \delta^*\Psi_p, (\widehat{\varepsilon}_p - \varepsilon_p) \cdot n\rangle\\
&\quad + \langle \delta^*\Psi_q, (\widehat{\varepsilon}_r - \varepsilon_r) \cdot n\rangle - \langle \delta^*\Psi_u, (\widehat{\varepsilon}_s - \varepsilon_s) \cdot n\rangle.
\end{aligned}$$

We next estimate θ_i's one by one. We bound θ_1 using the approximation property of Π^*.

$$|\theta_1| \le Ch\left(\|\varepsilon_u\| + \|\varepsilon_q\| + \|\varepsilon_p\| + \|\varepsilon_r\| + \|\varepsilon_s\|\right)\|\eta\|.$$

We estimate θ_2 by realizing that $(\delta_v, \Pi^*\Psi_w) = (\delta_v, -\delta^*\Psi + \delta^{k-1}\Psi_w)$. Here $\delta^{k-1}\Psi_w := \Psi_w - (\Psi_w)_{k-1}$ with $\|\delta^{k-1}\Psi_\omega\| \le Ch^{\min\{1,k\}}\|\eta\|$ for $\omega \in \{u,q,p,r,s\}$. Due to this bound and the properties of the projections Π and Π^*, we have

$$|\theta_2| \le Ch^{k+1+\min\{1,k\}}\|\eta\|.$$

Lastly, we realize that $\theta_3 = 0$ thanks to the definition of $\widehat{\varepsilon}_s$, $\widehat{\varepsilon}_r$, $\widehat{\varepsilon_p^{\sim}}$ and Π^*.

To conclude, we take $\eta_\omega = \varepsilon_\omega$ for $\omega \in \{u,q,p,r,s\}$ to obtain

$$\|\varepsilon\|^2 \le |\theta_1| + |\theta_2| \le Ch\|\varepsilon\|^2 + Ch^{k+1+\min\{k,1\}}\|\varepsilon\|.$$

When h is small enough, we have that $\|\varepsilon\| \le Ch^{k+1+\min\{k,1\}}$.

C.2. Proof of the Lemma 3.4.

We take $t = 0$ and $\psi = \varepsilon_{ut}(0)$ in (A.2e) to obtain

$$(\varepsilon_{ut}(0), \varepsilon_{ut}(0)) + (\delta_{ut}(0), \varepsilon_{ut}(0)) - (\beta\varepsilon_s(0), \varepsilon_{ut}(0)_x) + \langle \beta\widehat{\varepsilon}_s(0), \varepsilon_{ut}(0)\cdot n\rangle = 0$$

By Cauchy inequality, inverse inequality, and the trace inequality, we have

$$\|\varepsilon_{ut}(0)\|^2 \le C\|\delta_{ut}(0)\|\|\varepsilon_{ut}(0)\| + Ch^{-1}\|\varepsilon_s(0)\|\|\varepsilon_{ut}(0)\| + Ch^{-1/2}\|\widehat{\varepsilon}_s(0)\|_{\partial\Omega_h}\|\varepsilon_{ut}(0)\|$$

which means

$$\|\varepsilon_{ut}(0)\|^2 \le C\|\delta_{ut}(0)\|^2 + Ch^{-2}\|\varepsilon_s(0)\|^2 + Ch^{-1}\|\widehat{\varepsilon}_s(0)\|_{\partial\Omega_h}. \tag{C.4}$$

The estimates in Theorems C.1 and C.2 imply that

$$\|\varepsilon_s(0)\| \le Ch^{k+2}, \quad \|\widehat{\varepsilon}_s(0)\| \le Ch^{2k+1}.$$

Moreover, the properties of the projection Π states that $\|\delta_{ut}(0)\| \le Ch^{k+1}$. Plugging these three estimates into (C.4) gives

$$\|\varepsilon_{ut}(0)\| \le Ch^{k+1},$$

which concludes the proof of the lemma.

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