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A C^0-NONCONFORMING QUADRILATERAL FINITE ELEMENT FOR THE FOURTH-ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM

Yuan Bao[1], Zhaoliang Meng[1,*] and Zhongxuan Luo[1,2]

Abstract. In this paper, a C^0 nonconforming quadrilateral element is proposed to solve the fourth-order elliptic singular perturbation problem. For each convex quadrilateral Q, the shape function space is the union of $S_2^1(Q^*)$ and a bubble space. The degrees of freedom are defined by the values at vertices and midpoints on the edges, and the mean values of integrals of normal derivatives over edges. The local basis functions of our element can be expressed explicitly by a new reference quadrilateral rather than by solving a linear system. It is shown that the method converges uniformly in the perturbation parameter. Lastly, numerical tests verify the convergence analysis.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Denote by $\partial\Omega$ the boundary of Ω. We discuss the following problem:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f \quad \text{in } \Omega, \\ u = \varepsilon \frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the standard Laplace operator, $\partial/\partial \boldsymbol{n}$ denotes the normal derivative along the boundary $\partial\Omega$, and $\varepsilon \in [0,1]$ is a real parameter. Notice that if $\varepsilon = 0$, (1.1) formally degenerates to Poisson's equation. If $\varepsilon \neq 0$, the problem is the fourth-order elliptic singular perturbation problem, which we investigate in this work.

Since problem (1.1) is fourth order, standard conforming finite element methods [2, 23] require globally C^1 continuity. For the biharmonic problems, conforming elements such as the Argyris element [3] and the 16-dof Bogner–Fox–Schmit (BFS) element [5] require high degree polynomials which are rather expensive. Cheaper but complicate conforming elements for fourth-order problems include the 12-dof Hsieh–Clough–Tocher element and the singular Zienkiewicz triangular element [4] and so on. In order to overcome the C^1 difficulty, nonconforming finite element methods are often used [6, 14, 27]. The Morley element method is appealing for fourth-order

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[1] School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P.R. China.
[2] School of Software, Dalian University of Technology, Dalian 116620, P.R. China.

[*] Corresponding author: mzhl@dlut.edu.cn

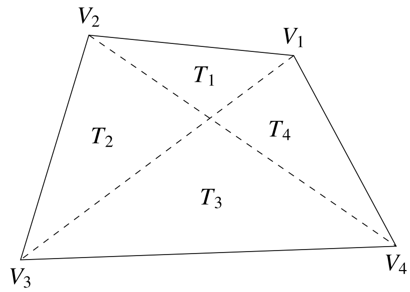

FIGURE 1. A quadrilateral's associated subdivision Q^*.

problems with the fewest number of degrees of freedom on each element. However, when the Morley method is applied to a second-order elliptic problem, as shown in [20], it will diverge. For more detailed properties of the Morley method we refer to [19, 26].

Concerning rectangular nonconforming elements for fourth-order problems, the incomplete biquadratic element [25] has been used as an analogue to the Morley element for rectangular meshes. The classical 12-dof Adini element [1], which contains all cubic polynomials on rectangles, has been well-known for rectangular meshes. Powell-Sabin types of macro elements on rectangles in two and three dimensions have been developed by Hu *et al.* in [15].

Even though the triangular or rectangular meshes are popular to use, in many cases where the geometry of the problem has a quadrilateral nature, one wishes to use quadrilateral meshes with proper elements. Fraeijs De Veubeke [10] presented a new scheme for plate bending element by decomposing the convex quadrilateral into the union of four triangles as shown in Figure 1. Park and Sheen [21] proposed a Morley-type finite element for quadrilateral meshes to solve biharmonic problems. For each quadrilateral Q, the finite element space was defined by the span of $P_2(Q)$ plus two functions in $P_3(Q)$. Many successful plate elements have been constructed, but not all of them can be used for (1.1) directly.

The fourth-order elliptic singular perturbation problem (1.1) has been studied in [7, 9, 13, 20, 22, 24, 28–30], and so on. In [20], a triangular element with nine degrees of freedom was presented. Nilssen *et al.* showed that the triangular element method converges uniformly in the perturbation parameter. Chen *et al.* [9] gave a convergence theorem for non C^0 nonconforming finite element for the perturbation problem (1.1). Besides, a nine parameter triangular element and a twelve parameter rectangular element were proposed with double set parameters [8]. The main trick in [9] is to impose the element to be C^0 one and the mean values of integrals of normal derivatives over edges to be continuous. In [28], Wang *et al.* presented a modified Morley element for problem (1.1). This method still uses triangle Morley element or rectangle Morley element, but the linear approximation of finite element functions is used in the part of the bilinear form corresponding to the second order differential term. Using double set parameter method [8], Xie *et al.* [30] gave a robust C^0 triangular element for problem (1.1). As a subsequent work, Chen *et al.* constructed an anisotropic nonconforming element for the fourth-order singular perturbation problem in [7]. Although many finite elements have been constructed for (1.1), most of them solve the problem on the triangular or rectangular meshes.

In this paper, we present a new quadrilateral element to solve the fourth-order elliptic singular perturbation problem. Our finite element space is locally $S_2^1(Q^*) \bigoplus \mathrm{Span}\{\varphi, l_{13}\varphi, l_{24}\varphi, l_{13}l_{24}\varphi\}$, where l_{13} and l_{24} are two linear polynomials vanishing at the vertices V_1, V_3 and V_2, V_4 and φ is defined in Section 2. We define the DOFs as the eight values at the vertices and midpoints on each edge, and the four mean values of integrals of normal derivatives over edges. Twelve local basis functions are defined on each quadrilateral element. By introducing a new reference domain $\tilde{Q}$ and an affine map as in [18], we can express the local basis functions explicitly without solving linear systems locally. Besides, all the integrations can be done over the reference domain, which is more efficient since the Jacobian determinant is constant. More precisely, we compute out the local stiffness matrix

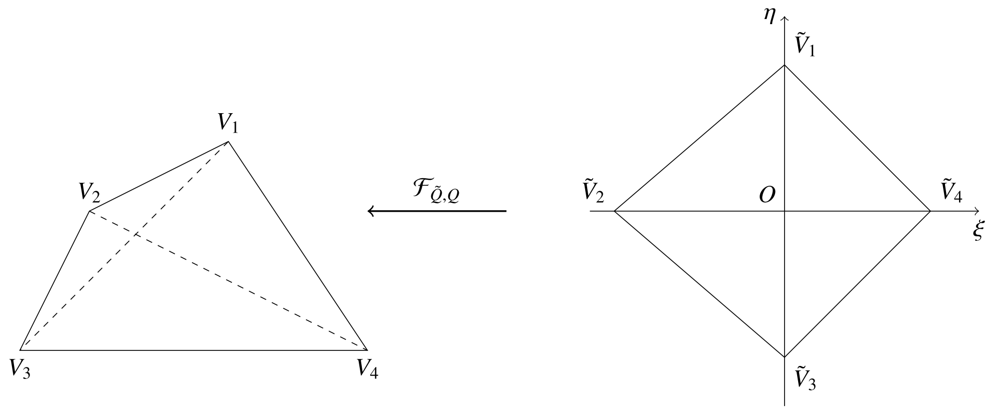

FIGURE 2. An affine map from a reference quadrilateral $\tilde{Q}$ to a quadrilateral Q.

for each quadrilateral element. Our C^0 finite element method is convergent uniformly with respect to ε for the fourth-order singular perturbation problem.

This paper is arranged as follows. In the next section, we present our quadrilateral element and prove the unisolvency through an affine quadrilateral. Then in Section 3, we define twelve local basis functions for our quadrilateral element and develop a set of formulae to compute the stiffness matrix. Section 4 is devoted to the convergence analysis. Numerical experiments are shown in Section 5. Lastly, we give the conclusion.

2. A new nonconforming quadrilateral element

In this section, we introduce the new quadrilateral nonconforming element and prove the unisolvency.

2.1. The reference element

Let Q be a convex quadrilateral shown as in Figure 1, where V_1, V_2, V_3, V_4 denote the vertices with counterclockwise indices, E_j designates the edge between V_j to V_{j+1} modulo 4, and M_j is the midpoint of E_j, $j = 1, \ldots, 4$. Denote by Q^* the subdivision of Q by connecting its diagonals such that Q^* is decomposed into four non-overlapping triangles $T_j, j = 1, \ldots, 4$, and designate by O the intersection point of two diagonals. Let l_{13} and l_{24} be the linear polynomials satisfying

$$l_{13}(V_1) = l_{13}(V_3) = l_{24}(V_2) = l_{24}(V_4) = 0, \quad l_{13}(V_4) = l_{24}(V_1) = 1.$$

Furthermore, let $l_{13}(V_2) = h_1, l_{24}(V_3) = h_2$. Note that Q is convex if and only if $h_1 < 0$ and $h_2 < 0$.

In this paper, we take $\tilde{Q}$ (see Fig. 2) as a reference quadrilateral with four vertices

$$\tilde{V}_1 = (0, 1),\ \tilde{V}_2 = (h_1, 0),\ \tilde{V}_3 = (0, h_2),\ \tilde{V}_4 = (1, 0),$$

which is firstly introduced in [18]. As shown there, there exists a unique affine transformation $\mathcal{F}_{\tilde{Q},Q} : \tilde{Q} \longrightarrow Q$ such that $\mathcal{F}_{\tilde{Q},Q}(\tilde{V}_i) = V_i, i = 1, \ldots, 4$. Note the inverse of $\mathcal{F}_{\tilde{Q},Q}$ can be written as: $(\xi, \eta) = \mathcal{F}^{-1}_{\tilde{Q},Q}(x, y) = (l_{13}(x, y), l_{24}(x, y))$ where $(x, y) \in Q$ and $(\xi, \eta) \in \tilde{Q}$. Furthermore let $\tilde{l}_{13} = l_{13} \circ \mathcal{F}_{\tilde{Q},Q}$ and $\tilde{l}_{24} = l_{24} \circ \mathcal{F}_{\tilde{Q},Q}$, then $\tilde{l}_{13}(\xi, \eta) = \xi, \tilde{l}_{24}(\xi, \eta) = \eta$. Similarly, we denote the four edges of $\tilde{Q}$ by $\tilde{E}_j, j = 1, \ldots, 4$. The following integrals on the four edges and the quadrilateral hold:

$$
\begin{aligned}
&\fint_{\tilde{E}_4} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!}, \quad \fint_{\tilde{E}_1} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!} h_1^i, \\
&\fint_{\tilde{E}_2} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!} h_1^i h_2^j, \quad \fint_{\tilde{E}_3} \xi^i \eta^j \mathrm{d}\tilde{s} = \frac{i!j!}{(i+j+1)!} h_2^j,
\end{aligned} \tag{2.1}
$$

$$
\int_{\tilde{Q}} \xi^i \eta^j \mathrm{d}\xi \mathrm{d}\eta = \frac{i!j!}{(2+i+j)!}(1-h_1^{i+1})(1-h_2^{j+1}). \tag{2.2}
$$

Here and in what follows, we denote $\fint_D = \frac{1}{|D|}\int_D$ by the integral mean on domain D.

2.2. A new finite element

Let $l_j(x,y)$ be the linear polynomial which vanishes on the edge E_j and $l_j(O) = 1$, $j = 1, \ldots, 4$. Then define piecewise polynomial

$$
\phi|_{T_j} = l_j, \quad j = 1, \ldots, 4.
$$

Thus ϕ is continuous at four vertices and the intersection point O, which implies that ϕ belongs to $C^0(Q)$. Further define $\varphi = l_{13} l_{24} \phi$. Obviously, $\varphi \in C^1(Q)$.

The space of multivariate spline functions $S_2^1(Q^*)$ is defined by a set of functions which are piecewise polynomials of degree 2 possessing 1st order continuous partial derivatives in Q, that is

$$
S_2^1(Q^*) := \{ v \in C^1(Q) : \ v|_{T_j} \in P_2(T_j), j = 1, \ldots, 4 \}.
$$

Here, and in what follows, $P_m(T)$ denotes the polynomial space of degree less than or equal to m on T. Furthermore, $S_2^1(Q^*)$ can be expressed explicitly as follows:

$$
S_2^1(Q^*) = \mathrm{Span}\{1, x, y, xy, x^2, y^2, [l_{13}^+(x,y)]^2, [l_{24}^+(x,y)]^2\}, \tag{2.3}
$$

where l_{13}^+ and l_{24}^+ are two ramp functions defined as follows:

$$
l_{13}^+(x,y) = \max(l_{13}(x,y), 0) \quad \text{and} \quad l_{24}^+(x,y) = \max(l_{24}(x,y), 0).
$$

About the space $S_2^1(Q^*)$, we have

Lemma 2.1 ([16]). *The dimension of $S_2^1(Q^*)$ is eight. Furthermore, for any given real numbers $a_j, a_j', j = 1, \ldots, 4$, there exists a unique $f \in S_2^1(Q^*)$ such that*

$$
f(V_j) = a_j, f(M_j) = a_j', \quad j = 1, \ldots, 4. \tag{2.4}
$$

Lemma 2.1 implies that a function in the space $S_2^1(Q^*)$ can be determined by values at four vertices and four midpoints. Furthermore, the corresponding nodal basis functions can be obtained by the B-net method [11]. However, the B-net method is not easy for those unfamiliar with the theory of multivariate spline. Now we present the explicit representations of the nodal basis functions by the reference element proposed in Section 2.1.

Suppose that $\mathcal{F}_{\tilde{Q},Q}$ is the affine transformation above from $\tilde{Q}$ to Q, then set $\tilde{Q}^* = \mathcal{F}_{\tilde{Q},Q}^{-1}(Q^*)$. Now let us consider the space $S_2^1(\tilde{Q}^*)$ first. It is obvious that

$$
S_2^1(\tilde{Q}^*) = \mathrm{Span}\{\tilde{\phi}_{bj}, j = 1, \ldots, 8\} = \mathrm{Span}\{1, \xi, \eta, \xi\eta, \xi^2, \eta^2, (\xi^+)^2, (\eta^+)^2\}.
$$

Define $\boldsymbol{\Phi}(\xi,\eta) = [1, \xi, \eta, \xi\eta, \xi^2, \eta^2, (\xi^+)^2, (\eta^+)^2]$ and then we have

$$\tilde{B} = \begin{bmatrix} \boldsymbol{\Phi}(\tilde{V}_1) \\ \boldsymbol{\Phi}(\tilde{V}_2) \\ \boldsymbol{\Phi}(\tilde{V}_3) \\ \boldsymbol{\Phi}(\tilde{V}_4) \\ \boldsymbol{\Phi}(\tilde{M}_1) \\ \boldsymbol{\Phi}(\tilde{M}_2) \\ \boldsymbol{\Phi}(\tilde{M}_3) \\ \boldsymbol{\Phi}(\tilde{M}_4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & h_1 & 0 & 0 & h_1^2 & 0 & 0 & 0 \\ 1 & 0 & h_2 & 0 & 0 & h_2^2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & \frac{h_1}{2} & \frac{1}{2} & \frac{h_1}{4} & \frac{h_1^2}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 1 & \frac{h_1}{2} & \frac{h_2}{2} & \frac{h_1 h_2}{4} & \frac{h_1^2}{4} & \frac{h_2^2}{4} & 0 & 0 \\ 1 & \frac{1}{2} & \frac{h_2}{2} & \frac{h_2}{4} & \frac{1}{4} & \frac{h_2^2}{4} & \frac{1}{4} & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

By solving the corresponding linear systems, we can obtain the interpolation basis functions for $S_2^1(\tilde{Q}^*)$ as follows:

$$\begin{aligned}
\tilde{f}_1(\xi,\eta) =& \frac{1}{2(h_2-1)}\left(-1 + 2\eta + \frac{h_2}{h_1^2}\xi^2 - \frac{1}{h_2}\eta^2\right) + \frac{h_2(h_1^2-1)}{2h_1^2(h_2-1)}(\xi^+)^2 + \frac{3h_2-1}{2h_2}(\eta^+)^2, \\
\tilde{f}_2(\xi,\eta) =& \frac{1}{2(h_1-1)}\left(1 - 2\xi + \frac{4h_1-3}{h_1^2}\xi^2 - \frac{1}{h_2^2}\eta^2\right) + \frac{h_1-3}{2h_1^2}(\xi^+)^2 - \frac{h_2^2-1}{2h_2^2(h_1-1)}(\eta^+)^2, \\
\tilde{f}_3(\xi,\eta) =& \frac{1}{2(h_2-1)}\left(1 - 2\eta - \frac{1}{h_1^2}\xi^2 + \frac{4h_2-3}{h_2^2}\eta^2\right) - \frac{h_1^2-1}{2h_1^2(h_2-1)}(\xi^+)^2 + \frac{h_2-3}{2h_2^2}(\eta^+)^2, \\
\tilde{f}_4(\xi,\eta) =& \frac{1}{2(h_1-1)}\left(-h_1 + 2\xi - \frac{1}{h_1}\xi^2 + \frac{h_1}{h_2^2}\eta^2\right) + \frac{3h_1-1}{2h_1}(\xi^+)^2 + \frac{h_1(h_2^2-1)}{2h_2^2(h_1-1)}(\eta^+)^2, \\
\tilde{f}_5(\xi,\eta) =& \frac{2}{(1-h_1)(1-h_2)}\left(-h_2 + 2h_2\xi + 2\eta - 2\xi\eta - \frac{h_2(2h_1-1)}{h_1^2}\xi^2 - \frac{1}{h_2}\eta^2\right) \\
&- \frac{2h_2(h_1-1)}{h_1^2(h_2-1)}(\xi^+)^2 + \frac{2(h_2-1)}{h_2(h_1-1)}(\eta^+)^2, \\
\tilde{f}_6(\xi,\eta) =& \frac{2}{(1-h_1)(1-h_2)}\left(1 - 2\xi - 2\eta + 2\xi\eta + \frac{(2h_1-1)}{h_1^2}\xi^2 + \frac{(2h_2-1)}{h_2^2}\eta^2\right) \\
&+ \frac{2(h_1-1)}{h_1^2(h_2-1)}(\xi^+)^2 + \frac{2(h_2-1)}{h_2^2(h_1-1)}(\eta^+)^2, \\
\tilde{f}_7(\xi,\eta) =& \frac{2}{(1-h_1)(1-h_2)}\left(-h_1 + 2\xi + 2h_1\eta - 2h_1\xi\eta - \frac{1}{h_1}\xi^2 - \frac{h_1(2h_2-1)}{h_2^2}\eta^2\right) \\
&+ \frac{2(h_1-1)}{h_1(h_2-1)}(\xi^+)^2 - \frac{2h_1(h_2-1)}{h_2^2(h_1-1)}(\eta^+)^2, \\
\tilde{f}_8(\xi,\eta) =& \frac{2}{(1-h_1)(1-h_2)}\left(h_1h_2 - 2h_2\xi - 2h_1\eta + 2\xi\eta + \frac{h_2}{h_1}\xi^2 + \frac{h_1}{h_2}\eta^2\right) \\
&- \frac{2h_2(h_1-1)}{h_1(h_2-1)}(\xi^+)^2 - \frac{2h_1(h_2-1)}{h_2(h_1-1)}(\eta^+)^2,
\end{aligned}$$

which satisfy the following interpolation property

$$\tilde{f}_i(\tilde{V}_j) = \delta_{ij},\ \tilde{f}_i(\tilde{M}_j) = 0,\ \tilde{f}_{i+4}(\tilde{V}_j) = 0,\ \tilde{f}_{i+4}(\tilde{M}_j) = \delta_{ij}, \quad i,j = 1,\dots,4.$$

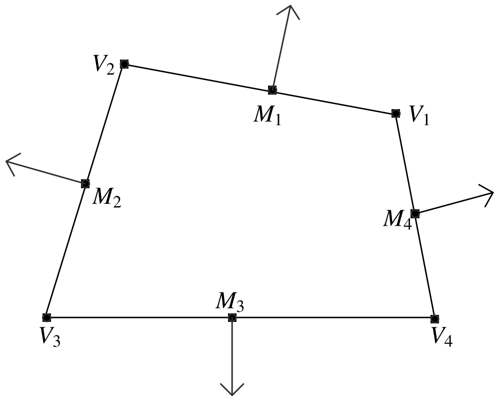

FIGURE 3. The degrees of freedom of the quadrilateral element.

Thus

$$f_i(x,y) = \tilde{f}_i \circ \mathcal{F}^{-1}_{\tilde{Q},Q}, \quad i = 1,\ldots,8 \tag{2.5}$$

are the nodal basis functions for $S_2^1(Q^*)$.

Now we are ready to define our C^0 quadrilateral element.

Definition 2.2. The quadrilateral finite element (Q, P_Q, Φ_Q) is defined as follows:

- Q is a convex quadrilateral,
- $P_Q = S_2^1(Q^*) + \mathrm{Span}\{\varphi, l_{13}\varphi, l_{24}\varphi, l_{13}l_{24}\varphi\}$,
- The degrees of freedom are given by $\Phi_Q = \left\{u(V_j), u(M_j), \fint_{E_j} \frac{\partial u}{\partial \boldsymbol{n}_j} \mathrm{d}s : j = 1,\ldots,4\right\}$ (see Fig. 3), where $\boldsymbol{n}_j$ denotes the unit outward normal to E_j.

We first show our new quadrilateral element is well defined. For this, we need the following lemma.

Lemma 2.3. *The following matrix*

$$M = \begin{bmatrix} \fint_{E_1} l_{13}l_{24}\mathrm{d}s & \fint_{E_1} l_{13}^2 l_{24}\mathrm{d}s & \fint_{E_1} l_{13}l_{24}^2\mathrm{d}s & \fint_{E_1} l_{13}^2 l_{24}^2\mathrm{d}s \\ \fint_{E_2} l_{13}l_{24}\mathrm{d}s & \fint_{E_2} l_{13}^2 l_{24}\mathrm{d}s & \fint_{E_2} l_{13}l_{24}^2\mathrm{d}s & \fint_{E_2} l_{13}^2 l_{24}^2\mathrm{d}s \\ \fint_{E_3} l_{13}l_{24}\mathrm{d}s & \fint_{E_3} l_{13}^2 l_{24}\mathrm{d}s & \fint_{E_3} l_{13}l_{24}^2\mathrm{d}s & \fint_{E_3} l_{13}^2 l_{24}^2\mathrm{d}s \\ \fint_{E_4} l_{13}l_{24}\mathrm{d}s & \fint_{E_4} l_{13}^2 l_{24}\mathrm{d}s & \fint_{E_4} l_{13}l_{24}^2\mathrm{d}s & \fint_{E_4} l_{13}^2 l_{24}^2\mathrm{d}s \end{bmatrix}$$

is nonsingular.

Proof. Take $\tilde{Q}$ as the reference element as introduced in Section 2.1 and introduce a new matrix $\tilde{M}$ as

$$\tilde{M} = \begin{bmatrix} \fint_{\tilde{E}_1} \tilde{l}_{13}\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_1} \tilde{l}_{13}^2\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_1} \tilde{l}_{13}\tilde{l}_{24}^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_1} \tilde{l}_{13}^2\tilde{l}_{24}^2\mathrm{d}\tilde{s} \\ \fint_{\tilde{E}_2} \tilde{l}_{13}\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_2} \tilde{l}_{13}^2\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_2} \tilde{l}_{13}\tilde{l}_{24}^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_2} \tilde{l}_{13}^2\tilde{l}_{24}^2\mathrm{d}\tilde{s} \\ \fint_{\tilde{E}_3} \tilde{l}_{13}\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_3} \tilde{l}_{13}^2\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_3} \tilde{l}_{13}\tilde{l}_{24}^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_3} \tilde{l}_{13}^2\tilde{l}_{24}^2\mathrm{d}\tilde{s} \\ \fint_{\tilde{E}_4} \tilde{l}_{13}\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_4} \tilde{l}_{13}^2\tilde{l}_{24}\mathrm{d}\tilde{s} & \fint_{\tilde{E}_4} \tilde{l}_{13}\tilde{l}_{24}^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_4} \tilde{l}_{13}^2\tilde{l}_{24}^2\mathrm{d}\tilde{s} \end{bmatrix} = \begin{bmatrix} \fint_{\tilde{E}_1} \xi\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_1} \xi^2\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_1} \xi\eta^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_1} \xi^2\eta^2\mathrm{d}\tilde{s} \\ \fint_{\tilde{E}_2} \xi\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_2} \xi^2\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_2} \xi\eta^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_2} \xi^2\eta^2\mathrm{d}\tilde{s} \\ \fint_{\tilde{E}_3} \xi\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_3} \xi^2\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_3} \xi\eta^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_3} \xi^2\eta^2\mathrm{d}\tilde{s} \\ \fint_{\tilde{E}_4} \xi\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_4} \xi^2\eta\mathrm{d}\tilde{s} & \fint_{\tilde{E}_4} \xi\eta^2\mathrm{d}\tilde{s} & \fint_{\tilde{E}_4} \xi^2\eta^2\mathrm{d}\tilde{s} \end{bmatrix}. \tag{2.6}$$

Since $\mathcal{F}_{\tilde{Q},Q}$ is an affine map, we have $M = \tilde{M}$ through variable substitution. Thus, it only needs to prove that $\tilde{M}$ is nonsingular. According to formulas (2.1), we have

$$\begin{aligned}\tilde{M} &= \begin{bmatrix} \frac{1}{6}h_1 & \frac{1}{12}h_1^2 & \frac{1}{12}h_1 & \frac{1}{30}h_1^2 \\ \frac{1}{6}h_1h_2 & \frac{1}{12}h_1^2h_2 & \frac{1}{12}h_1h_2^2 & \frac{1}{30}h_1^2h_2^2 \\ \frac{1}{6}h_2 & \frac{1}{12}h_2 & \frac{1}{12}h_2^2 & \frac{1}{30}h_2^2 \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{12} & \frac{1}{30} \end{bmatrix} \\ &= \begin{bmatrix} h_1 & & & \\ & h_1h_2 & & \\ & & h_2 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & h_1 & 1 & h_1 \\ 1 & h_1 & h_2 & h_1h_2 \\ 1 & 1 & h_2 & h_2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & & & \\ & \frac{1}{12} & & \\ & & \frac{1}{12} & \\ & & & \frac{1}{30} \end{bmatrix},\end{aligned} \tag{2.7}$$

with $\det(\tilde{M}) = \frac{h_1^2h_2^2(h_1-1)^2(h_2-1)^2}{25920} \neq 0$, since $h_1 < 0$ and $h_2 < 0$. So $\tilde{M}$ is nonsingular. Thus we complete the proof. □

Next we proceed to show the following unisolvency result.

Theorem 2.4. *The set Φ_Q is P_Q-unisolvent.*

Proof. Suppose that $w \in P_Q$ satisfies

$$w(V_j) = w(M_j) = ⨍_{E_j} \frac{\partial w}{\partial \boldsymbol{n}_j} \mathrm{d}s = 0, \quad j = 1, \ldots, 4. \tag{2.8}$$

Due to the definition of P_Q we have

$$\begin{aligned} w(x,y) =& \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \beta_6 y^2 + \beta_7 [l_{13}^+(x,y)]^2 + \beta_8 [l_{24}^+(x,y)]^2 \\ &+ \beta_9\varphi + \beta_{10} l_{13}\varphi + \beta_{11} l_{24}\varphi + \beta_{12} l_{13} l_{24}\varphi, \end{aligned} \tag{2.9}$$

for some constants $\beta_i \in \mathbb{R}, i = 1, \ldots, 12$ to be determined. According to the definition of φ, one has

$$\varphi(V_j) = \varphi(M_j) = 0, \quad j = 1, \ldots, 4.$$

Hence if we set

$$s(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \beta_6 y^2 + \beta_7 [l_{13}^+(x,y)]^2 + \beta_8 [l_{24}^+(x,y)]^2,$$

then substituting (2.9) into (2.8) will lead to

$$s(V_j) = s(M_j) = 0, \quad j = 1, \ldots, 4. \tag{2.10}$$

Noticing that $s(x,y) \in S_2^1(Q^*)$ and using Lemma 2.1, we have

$$\beta_i = 0, \quad i = 1, \ldots, 8,$$

and

$$w(x,y) = \beta_9\varphi + \beta_{10} l_{13}\varphi + \beta_{11} l_{24}\varphi + \beta_{12} l_{13} l_{24}\varphi. \tag{2.11}$$

Next we will show that all constants in (2.11) are also zeros. Substituting $w(x,y)$ into

$$\fint_{E_j} \frac{\partial \omega}{\partial \boldsymbol{n}_j} \mathrm{d}s = 0, \quad j = 1, \dots, 4$$

will derive

$$N \begin{bmatrix} \beta_9 \\ \beta_{10} \\ \beta_{11} \\ \beta_{12} \end{bmatrix} = 0, \tag{2.12}$$

where

$$N = \begin{bmatrix} \fint_{E_1} \frac{\partial \varphi}{\partial \boldsymbol{n}_1} \mathrm{d}s & \fint_{E_1} \frac{\partial (l_{13}\varphi)}{\partial \boldsymbol{n}_1} \mathrm{d}s & \fint_{E_1} \frac{\partial (l_{24}\varphi)}{\partial \boldsymbol{n}_1} \mathrm{d}s & \fint_{E_1} \frac{\partial (l_{13}l_{24}\varphi)}{\partial \boldsymbol{n}_1} \mathrm{d}s \\ \fint_{E_2} \frac{\partial \varphi}{\partial \boldsymbol{n}_2} \mathrm{d}s & \fint_{E_2} \frac{\partial (l_{13}\varphi)}{\partial \boldsymbol{n}_2} \mathrm{d}s & \fint_{E_2} \frac{\partial (l_{24}\varphi)}{\partial \boldsymbol{n}_2} \mathrm{d}s & \fint_{E_2} \frac{\partial (l_{13}l_{24}\varphi)}{\partial \boldsymbol{n}_2} \mathrm{d}s \\ \fint_{E_3} \frac{\partial \varphi}{\partial \boldsymbol{n}_3} \mathrm{d}s & \fint_{E_3} \frac{\partial (l_{13}\varphi)}{\partial \boldsymbol{n}_3} \mathrm{d}s & \fint_{E_3} \frac{\partial (l_{24}\varphi)}{\partial \boldsymbol{n}_3} \mathrm{d}s & \fint_{E_3} \frac{\partial (l_{13}l_{24}\varphi)}{\partial \boldsymbol{n}_3} \mathrm{d}s \\ \fint_{E_4} \frac{\partial \varphi}{\partial \boldsymbol{n}_4} \mathrm{d}s & \fint_{E_4} \frac{\partial (l_{13}\varphi)}{\partial \boldsymbol{n}_4} \mathrm{d}s & \fint_{E_4} \frac{\partial (l_{24}\varphi)}{\partial \boldsymbol{n}_4} \mathrm{d}s & \fint_{E_4} \frac{\partial (l_{13}l_{24}\varphi)}{\partial \boldsymbol{n}_4} \mathrm{d}s \end{bmatrix}. \tag{2.13}$$

According to the definition of φ, we have

$$\int_{E_j} \frac{\partial (p\varphi)}{\partial \boldsymbol{n}_j} \mathrm{d}s = \int_{E_j} \frac{\partial (pl_{13}l_{24}l_j)}{\partial \boldsymbol{n}_j} \mathrm{d}s = \int_{E_j} \frac{\partial (pl_{13}l_{24})}{\partial \boldsymbol{n}_j} l_j \mathrm{d}s + \int_{E_j} \frac{\partial l_j}{\partial \boldsymbol{n}_j} (pl_{13}l_{24}) \mathrm{d}s = \frac{\partial l_j}{\partial \boldsymbol{n}_j} \int_{E_j} (pl_{13}l_{24}) \mathrm{d}s,$$

where $p = 1, l_{13}, l_{24}$ or $l_{13}l_{24}$. Thus N can be rewritten as

$$N = \mathrm{diag}\left(\frac{\partial l_1}{\partial \boldsymbol{n}_1}, \frac{\partial l_2}{\partial \boldsymbol{n}_2}, \frac{\partial l_3}{\partial \boldsymbol{n}_3}, \frac{\partial l_4}{\partial \boldsymbol{n}_4} \right) \cdot M. \tag{2.14}$$

According to Lemma 2.1, M is nonsingular which implies that N is nonsingular. This leads to

$$\beta_9 = \beta_{10} = \beta_{11} = \beta_{12} = 0,$$

and hence $w = 0$, which completes the proof. $\square$

3. Basis functions and stiffness matrix for the element Q

3.1. Basis functions for the shape function space P_Q

In this section, we will derive the explicit expression of the nodal basis functions. For convenience, define twelve linear functionals $\Psi_i, i = 1, \dots, 12$ on P_Q by

$$\Psi_i(f) = f(V_i), \quad \Psi_{i+4}(f) = f(M_i), \quad \Psi_{i+8}(f) = \fint_{E_i} \frac{\partial f}{\partial \boldsymbol{n}_i} \mathrm{d}s, \quad i = 1, \dots, 4.$$

Next define twelve local basis functions $P_j \in P_Q$ satisfying

$$\Psi_i(P_j) = \delta_{ij}, \quad i, j = 1, \dots, 12. \tag{3.1}$$

Firstly, let us give the explicit expressions of $P_j, j = 9, \ldots, 12$. Take P_9 as an example. It follows from the proof of Theorem 2.4 that $P_9 \in \text{Span}\{\varphi, l_{13}\varphi, l_{24}\varphi, l_{13}l_{24}\varphi\}$, which means that P_9 can be written as

$$P_9 = c_1\varphi + c_2 l_{13}\varphi + c_3 l_{24}\varphi + c_4 l_{13}l_{24}\varphi,$$

where $c_i, i = 1, \ldots, 4$ are coefficients to be determined. Thus according to the interpolation conditions (3.1), we can get linear equations

$$N \cdot \boldsymbol{c} = \boldsymbol{e}_1 \quad \text{with} \quad \boldsymbol{c} = [c_1, c_2, c_3, c_4]^T \quad \text{and} \quad \boldsymbol{e}_1 = [1, 0, 0, 0]^T.$$

Here N is given by (2.13) or (2.14). Notice in equation (2.14), $M = \tilde{M}$ where $\tilde{M}$ is given by equation (2.6) or in another form equation (2.7). Collecting above discussion, we have

$$\boldsymbol{c} = \frac{3}{h_1 \cdot |\tilde{Q}|} \cdot \left[\frac{\partial l_1}{\partial \boldsymbol{n}_1}\right]^{-1} \cdot [-h_2, 2h_2, 2, -5]^T,$$

and hence

$$P_9 = \frac{3}{h_1|\tilde{Q}|}\left[\frac{\partial l_1}{\partial \boldsymbol{n}_1}\right]^{-1} \cdot (-h_2 + 2h_2 l_{13} + 2l_{24} - 5l_{13}l_{24})l_{13}l_{24}\phi := k_1\bar{P}_9,$$

where

$$|\tilde{Q}| = \text{area}(\tilde{Q}) = (1-h_1)(1-h_2)/2, \quad k_1 = \frac{3}{h_1|\tilde{Q}|} \cdot \left[\frac{\partial l_1}{\partial \boldsymbol{n}_1}\right]^{-1},$$

and $\bar{P}_9$ is the remaining part. Similarly, we can get

$$\begin{aligned}
P_{10} &= \frac{3}{h_1h_2|\tilde{Q}|}\left[\frac{\partial l_2}{\partial \boldsymbol{n}_2}\right]^{-1} \cdot (1 - 2l_{13} - 2l_{24} + 5l_{13}l_{24})l_{13}l_{24}\phi := k_2\bar{P}_{10}\\
P_{11} &= \frac{3}{h_2|\tilde{Q}|}\left[\frac{\partial l_3}{\partial \boldsymbol{n}_3}\right]^{-1} \cdot (-h_1 + 2l_{13} + 2h_1 l_{24} - 5l_{13}l_{24})l_{13}l_{24}\phi := k_3\bar{P}_{11}\\
P_{12} &= \frac{3}{|\tilde{Q}|}\left[\frac{\partial l_4}{\partial \boldsymbol{n}_4}\right]^{-1} \cdot (h_1h_2 - 2h_2 l_{13} - 2h_1 l_{24} + 5l_{13}l_{24})l_{13}l_{24}\phi := k_4\bar{P}_{12},
\end{aligned}$$

where k_2, k_3, k_4 and $\bar{P}_{10}, \bar{P}_{11}, \bar{P}_{12}$ take the similar meaning with those k_1 and $\bar{P}_9$.

Let us turn to the computation of $k_i, i = 1, \ldots, 4$. Assume that D_i is the foot point of O on E_i. Since $l_i(x, y)$ is a linear polynomial and $l_i(O) = 1$, then

$$\frac{\partial l_i}{\partial \boldsymbol{n}_i} = \frac{l_i(D_i) - l_i(O)}{|OD_i|} = -\frac{1}{|OD_i|}.$$

Thus we have

$$3k_1^{-1} = h_1|\tilde{Q}| \cdot \frac{\partial l_1}{\partial \boldsymbol{n}_1} = -h_1|\tilde{Q}| \cdot \frac{1}{|OD_1|} = \frac{-\frac{1}{2}h_1|\tilde{Q}||E_1|}{\frac{1}{2}|OD_1||E_1|} = \frac{|\tilde{T}_1||\tilde{Q}||E_1|}{|T_1|} = \frac{|Q||E_1|}{\det(J)^2},$$

that is $k_1 = 3\det(J)^2/(|Q||E_1|)$. Other k_i's can be obtained by the same way. Explicitly, we have $k_i = 3\det(J)^2/(|Q||E_i|)$ for $i = 1, \ldots, 4$.

It is left to give the expressions of $P_i, i = 1, \ldots, 8$. Recall that the basis functions of $S_2^1(Q^*)$ are given by equation (2.5). Immediately, P_i can be expressed by

$$P_i = f_i - \sum_{j=1}^{4} \left(\fint_{E_j} \frac{\partial f_i}{\partial \boldsymbol{n}_j} \mathrm{d}s \right) P_{j+8} = f_i - \sum_{j=1}^{4} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_j}(M_j) \right) P_{j+8}, \ i = 1, \ldots, 8.$$

Here we use the fact that f_i is piecewise quadratic and hence $\partial f_i/\partial \boldsymbol{n}_j$ is linear on E_j. Now it only needs to compute $\frac{\partial f_i}{\partial \boldsymbol{n}_j}(M_j)$. Set unit tangential vector $\boldsymbol{\tau}_j = \overrightarrow{V_j V_{j+1}}/|E_j|, j = 1, \ldots, 4$ modulo 4. Since f_i is a quadratic polynomial which can be determined by the values at two vertices and midpoint on the edge E_j, the derivative f_i along $\boldsymbol{\tau}_j$ can be obtained. By certain computation, one has

$$|E_j| \cdot \frac{\partial f_i}{\partial \boldsymbol{\tau}_j}(V_k) = \begin{cases} -3, & i = j = k, \\ 3, & i = k, j = k-1, \\ 1, & i = j = k-1, \\ -1, & i = j+1 = k+1, \\ 4, & i = j+4 = k+4, \\ -4, & i = j+4 = k-3, \\ 0, & \text{otherwise}. \end{cases}$$

According to the definition of directional derivative, one has

$$\begin{aligned} \frac{\partial f_i}{\partial \boldsymbol{n}_j}(V_j) &= \frac{1}{\langle \boldsymbol{\tau}_{j-1}, \boldsymbol{n}_j \rangle} \frac{\partial f_i}{\partial \boldsymbol{\tau}_{j-1}}(V_j) - \frac{\langle \boldsymbol{\tau}_{j-1}, \boldsymbol{\tau}_j \rangle}{\langle \boldsymbol{\tau}_{j-1}, \boldsymbol{n}_j \rangle} \frac{\partial f_i}{\partial \boldsymbol{\tau}_j}(V_j), \\ \frac{\partial f_i}{\partial \boldsymbol{n}_j}(V_{j+1}) &= \frac{1}{\langle \boldsymbol{\tau}_{j+1}, \boldsymbol{n}_j \rangle} \frac{\partial f_i}{\partial \boldsymbol{\tau}_{j+1}}(V_{j+1}) - \frac{\langle \boldsymbol{\tau}_j, \boldsymbol{\tau}_{j+1} \rangle}{\langle \boldsymbol{\tau}_{j+1}, \boldsymbol{n}_j \rangle} \frac{\partial f_i}{\partial \boldsymbol{\tau}_j}(V_{j+1}), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. Then

$$\frac{\partial f_i}{\partial \boldsymbol{n}_j}(M_j) = \frac{1}{2} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_j}(V_j) + \frac{\partial f_i}{\partial \boldsymbol{n}_j}(V_{j+1}) \right) := \frac{1}{2} \langle B_j, F_{ij} \rangle,$$

where

$$B_j = \left(\frac{1}{|E_{j-1}| \langle \boldsymbol{\tau}_{j-1}, \boldsymbol{n}_j \rangle}, -\frac{\langle \boldsymbol{\tau}_{j-1}, \boldsymbol{\tau}_j \rangle}{|E_j| \langle \boldsymbol{\tau}_{j-1}, \boldsymbol{n}_j \rangle}, \frac{1}{|E_{j+1}| \langle \boldsymbol{\tau}_{j+1}, \boldsymbol{n}_j \rangle}, -\frac{\langle \boldsymbol{\tau}_j, \boldsymbol{\tau}_{j+1} \rangle}{|E_{j-1}| \langle \boldsymbol{\tau}_{j+1}, \boldsymbol{n}_j \rangle} \right),$$

and

$$F_{ij} = \left(|E_{j-1}| \frac{\partial f_i}{\partial \boldsymbol{\tau}_{j-1}}(V_j), |E_j| \frac{\partial f_i}{\partial \boldsymbol{\tau}_j}(V_j), |E_{j+1}| \frac{\partial f_i}{\partial \boldsymbol{\tau}_{j+1}}(V_{j+1}), |E_j| \frac{\partial f_i}{\partial \boldsymbol{\tau}_j}(V_{j+1}) \right).$$

More precisely, let F_k be a matrix whose ith row is F_{ik}, then we can write

$$[F_1|F_2|F_3|F_4] = \left[\begin{array}{cccc|cccc|cccc|cccc} 3 & -3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -3 & 3 \\ 0 & -1 & -3 & 3 & 3 & -3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -3 & 3 & 3 & -3 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -3 & 3 & 3 & -3 & 0 & 1 \\ 0 & 4 & 0 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & -4 & -4 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & -4 \end{array}\right].$$

3.2. Stiffness matrix for the element Q

Now we turn to consider the computational aspect. Set $A = [a_{ij}]_{12\times 12}$ and $B = [b_{ij}]_{12\times 12}$ are the local stiffness matrices corresponding to the bilinear form $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, respectively, that is

$$a_{ij} = \int_Q D^2 P_i : D^2 P_j, \quad b_{ij} = \int_Q DP_i \cdot DP_j,$$

where $D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j}$, we may also use $(x, y) = (x_1, x_2)$ for brevity.

Generally, the computation of quadrilateral element can be realized on the reference domain $[-1, 1]^2$ and the bilinear transformation is required. Next, we will show that all the computation can be finished efficiently on our reference quadrilateral $\tilde{Q}$.

Let $l_{13}(x, y) = \alpha_1 x + \beta_1 y + \gamma_1$ and $l_{24}(x, y) = \alpha_2 x + \beta_2 y + \gamma_2$. $\mathcal{F}_{\tilde{Q},Q}$ is the linear transformation given in Section 2.1. Thus the derivatives of the function f defined on Q can be derived as follows:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial \tilde{f}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{f}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \alpha_1 \frac{\partial \tilde{f}}{\partial \xi} + \alpha_2 \frac{\partial \tilde{f}}{\partial \eta} = \boldsymbol{\alpha} D_{\xi,\eta} \tilde{f}, \\ \frac{\partial f}{\partial y} &= \frac{\partial \tilde{f}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{f}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \beta_1 \frac{\partial \tilde{f}}{\partial \xi} + \beta_2 \frac{\partial \tilde{f}}{\partial \eta} = \boldsymbol{\beta} D_{\xi,\eta} \tilde{f}, \end{aligned} \tag{3.2}$$

where $\tilde{f} = f \circ \mathcal{F}_{\tilde{Q},Q}, \boldsymbol{\alpha} = [\alpha_1, \alpha_2], \boldsymbol{\beta} = [\beta_1, \beta_2]$ and $D_{\xi,\eta}\tilde{f}$ denotes the gradient of $\tilde{f}$ with respect to ξ and η. The Jacobian matrix will be calculated as

$$J = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}^{-1}, \tag{3.3}$$

with its determinant

$$\det(J) = 1/(\alpha_1 \beta_2 - \alpha_2 \beta_1).$$

In what follows, for convenience, let $R_i = f_i$ for $i = 1, \ldots, 8$ and $R_i = P_i$ for $i = 9, \ldots, 12$. Taking $\tilde{R}_i = R_i \circ \mathcal{F}_{\tilde{Q},Q}$ and using formulas (2.2), the matrix $\tilde{G} = [\tilde{G}_{ij}]_{12\times 12}$ can be exactly calculated, where $\tilde{G}_{ij}$'s are sub-matrices of 2×2 and given by

$$\tilde{G}_{ij} = \int_{\tilde{Q}} \left[D_{\xi,\eta} \tilde{R}_i\right] \left[D_{\xi,\eta} \tilde{R}_j\right]^T \mathrm{d}\xi \mathrm{d}\eta, \quad i, j = 1, \ldots, 12.$$

Thus for $i, j = 1, \ldots, 12$, we have

$$\begin{aligned}(DR_i, DR_j)_Q &= \int_Q \left(\frac{\partial R_i}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_i}{\partial y}\frac{\partial R_j}{\partial y} \right) \mathrm{d}x\mathrm{d}y \\ &= \int_{\tilde{Q}} \left((\boldsymbol{\alpha} D_{\xi,\eta}\tilde{R}_i)(\boldsymbol{\alpha} D_{\xi,\eta}\tilde{R}_j) + (\boldsymbol{\beta} D_{\xi,\eta}\tilde{R}_i)(\boldsymbol{\beta} D_{\xi,\eta}\tilde{R}_j) \right) \det(J)\mathrm{d}\xi\mathrm{d}\eta \\ &= \left(\boldsymbol{\alpha}\tilde{G}_{ij}\boldsymbol{\alpha}^T + \boldsymbol{\beta}\tilde{G}_{ij}\boldsymbol{\beta}^T \right) \det(J) := \bar{b}_{ij}.\end{aligned}$$

Then we have $b_{ij} = \bar{b}_{ij}$ for $i, j = 9, \ldots, 12$, and

$$\begin{aligned}b_{ij} = (DP_i, DP_j)_Q &= \left(D\left(f_i - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) P_t \right), DP_j \right)_Q \\ &= (Df_i, DP_j)_Q - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) (DP_t, DP_j)_Q = \bar{b}_{ij} - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) b_{tj},\end{aligned}$$

and $b_{ji} = b_{ij}$ for $i = 1, \ldots, 8, j = 9, \ldots, 12$, and

$$\begin{aligned}b_{ij} = (DP_i, DP_j)_Q &= \left(D\left(f_i - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) P_t \right), DP_j \right)_Q \\ &= (Df_i, DP_j)_Q - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) (DP_t, DP_j)_Q \\ &= \left(Df_i, D\left(f_j - \sum_{s=9}^{12} \left(\frac{\partial f_j}{\partial \boldsymbol{n}_{s-8}}(M_{s-8}) \right) P_s \right) \right)_Q - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) b_{tj} \\ &= (Df_i, Df_j)_Q - \sum_{s=9}^{12} \left(\frac{\partial f_j}{\partial \boldsymbol{n}_{s-8}}(M_{s-8}) \right) (Df_i, DP_s)_Q - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) b_{tj} \\ &= \bar{b}_{ij} - \sum_{s=9}^{12} \left(\frac{\partial f_j}{\partial \boldsymbol{n}_{s-8}}(M_{s-8}) \right) \bar{b}_{is} - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) b_{tj} \\ &= \bar{b}_{ij} - \sum_{t=9}^{12} \left\{ \left(\frac{\partial f_j}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) \bar{b}_{it} + \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) b_{tj} \right\},\end{aligned}$$

for $i, j = 1, \ldots, 8$.

Let us turn to the computation of the first part of the stiffness matrix. The second order partial derivatives of the function f can be obtained as follows:

$$\frac{\partial^2 f}{\partial x^2} = \alpha_1^2 \frac{\partial^2 \tilde{f}}{\partial \xi^2} + \alpha_1\alpha_2 \frac{\partial^2 \tilde{f}}{\partial \xi \partial \eta} + \alpha_2\alpha_1 \frac{\partial^2 \tilde{f}}{\partial \eta \partial \xi} + \alpha_2^2 \frac{\partial^2 \tilde{f}}{\partial \eta^2} := \boldsymbol{\theta}_1 H^2_{\xi,\eta}\tilde{f}, \tag{3.4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \alpha_1\beta_1 \frac{\partial^2 \tilde{f}}{\partial \xi^2} + \alpha_1\beta_2 \frac{\partial^2 \tilde{f}}{\partial \xi \partial \eta} + \alpha_2\beta_1 \frac{\partial^2 \tilde{f}}{\partial \eta \partial \xi} + \alpha_2\beta_2 \frac{\partial^2 \tilde{f}}{\partial \eta^2} := \boldsymbol{\theta}_2 H^2_{\xi,\eta}\tilde{f}, \tag{3.5}$$

$$\frac{\partial^2 f}{\partial y^2} = \beta_1^2 \frac{\partial^2 \tilde{f}}{\partial \xi^2} + \beta_1\beta_2 \frac{\partial^2 \tilde{f}}{\partial \xi \partial \eta} + \beta_2\beta_1 \frac{\partial^2 \tilde{f}}{\partial \eta \partial \xi} + \beta_2^2 \frac{\partial^2 \tilde{f}}{\partial \eta^2} := \boldsymbol{\theta}_3 H^2_{\xi,\eta}\tilde{f}, \tag{3.6}$$

where $\boldsymbol{\theta}_1 = [\alpha_1^2, 2\alpha_1\alpha_2, \alpha_2^2], \boldsymbol{\theta}_2 = [\alpha_1\beta_1, \alpha_1\beta_2 + \alpha_2\beta_1, \alpha_2\beta_2], \boldsymbol{\theta}_3 = [\beta_1^2, 2\beta_1\beta_2, \beta_2^2]$ and $H^2_{\xi,\eta}\tilde{f} = [\frac{\partial^2 \tilde{f}}{\partial \xi^2}, \frac{\partial^2 \tilde{f}}{\partial \xi \partial \eta}, \frac{\partial^2 \tilde{f}}{\partial \eta^2}]^T$.

Similarly, the matrix $\tilde{H} = [\tilde{H}_{ij}]_{12\times 12}$ can be calculated exactly, where $\tilde{H}_{ij}$'s are sub-matrices of 3×3 and given by

$$\tilde{H}_{ij} = \int_{\tilde{Q}} [H^2_{\xi,\eta}\tilde{R}_i][H^2_{\xi,\eta}\tilde{R}_j]^T \mathrm{d}\xi \mathrm{d}\eta, \quad i,j = 1,\dots,12.$$

Hence for $i,j = 1,\dots,12$, we have

$$\begin{aligned}
&\int_Q D^2 R_i : D^2 R_j \mathrm{d}x\mathrm{d}y \\
&\quad = \int_Q \left(\frac{\partial^2 R_i}{\partial x^2}\frac{\partial^2 R_j}{\partial x^2} + 2\frac{\partial^2 R_i}{\partial x \partial y}\frac{\partial^2 R_j}{\partial x \partial y} + \frac{\partial^2 R_i}{\partial y^2}\frac{\partial^2 R_j}{\partial y^2} \right) \mathrm{d}x\mathrm{d}y \\
&\quad = \int_{\tilde{Q}} \Big[(\boldsymbol{\theta}_1 H^2_{\xi,\eta}\tilde{R}_i)(\boldsymbol{\theta}_1 H^2_{\xi,\eta}\tilde{R}_j) + 2(\boldsymbol{\theta}_2 H^2_{\xi,\eta}\tilde{R}_i)(\boldsymbol{\theta}_2 H^2_{\xi,\eta}\tilde{R}_j) + (\boldsymbol{\theta}_3 H^2_{\xi,\eta}\tilde{R}_i)(\boldsymbol{\theta}_3 H^2_{\xi,\eta}\tilde{R}_j) \Big] \det(J) \mathrm{d}\xi\mathrm{d}\eta \\
&\quad = \Big(\boldsymbol{\theta}_1 \tilde{H}_{ij} \boldsymbol{\theta}_1^T + 2\boldsymbol{\theta}_2 \tilde{H}_{ij} \boldsymbol{\theta}_2^T + \boldsymbol{\theta}_3 \tilde{H}_{ij} \boldsymbol{\theta}_3^T \Big) \det(J) := \bar{a}_{ij}.
\end{aligned}$$

Following similar argument above, we have

$$a_{ij} = \begin{cases} \bar{a}_{ij}, & i,j = 9,\dots,12, \\ \bar{a}_{ij} - \sum_{t=9}^{12} \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) a_{tj}, \quad a_{ji} = a_{ij}, & i = 1,\dots,8, j = 9,\dots,12, \\ \bar{a}_{ij} - \sum_{t=9}^{12} \left\{ \left(\frac{\partial f_j}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) \bar{a}_{it} + \left(\frac{\partial f_i}{\partial \boldsymbol{n}_{t-8}}(M_{t-8}) \right) a_{tj} \right\}, & i,j = 1,\dots,8. \end{cases}$$

Collecting above discussion, the components of the stiffness matrix $\varepsilon^2 a_{ij} + b_{ij}$ can be calculated exactly.

4. Convergence analysis

In this section, the convergence analysis of the element proposed in the previous section will be considered.

We introduce some notations first. The inner product on $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)$. For $m \geq 0$, we use $H^m = H^m(\Omega)$ to denote the Sobolev space, denote the corresponding norm and seminorm by $\|\cdot\|_m$ and $|\cdot|_m$, respectively. The space H_0^m is the closure in H^m of $C_0^\infty(\Omega)$.

The weak form of (1.1) is to find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{4.1}$$

where

$$a(u,v) = \int_\Omega D^2 u : D^2 v, \qquad b(u,v) = \int_\Omega Du \cdot Dv. \tag{4.2}$$

Owing to the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ [26], one sees that if the solution $u \in H_0^2(\Omega)$ of (4.1) has an additional regularity $H^3(\Omega)$, it fulfills

$$\int_\Omega (-\varepsilon^2 D(\Delta u) + Du) \cdot Dv = (f,v) \quad \forall v \in H_0^1(\Omega). \tag{4.3}$$

Let u^0 be the solution of the reduced problem, namely, taking $\varepsilon = 0$ in (1.1):

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Then Lemma 5.1 in [20] also holds when u is the weak solution of (1.1) due to a density argument.

Lemma 4.1 ([20])**.** *Assume Ω is convex. Let $u \in H_0^2(\Omega) \cap H^3(\Omega)$ be the solution of problem (4.1), while $u^0 \in H_0^1(\Omega) \cap H^2(\Omega)$ denotes the solution for the reduced problem (4.4). Then there exists a constant c, independent of ε and f, such that*

$$\varepsilon^{\frac{1}{2}}|u|_2 + \varepsilon^{\frac{3}{2}}|u|_3 \le c\,\|f\|_0 \quad \text{and} \quad |u - u^0|_1 \le c\varepsilon^{\frac{1}{2}}\,\|f\|_0$$

for all $f \in L^2(\Omega)$.

Let $\{\mathcal{T}_h\}$ be a regular family of decompositions of Ω into convex quadrilaterals. Denote by $\mathcal{E}_h$ the set of edges. For each edge $E \in \mathcal{E}_h$, set a unit vector $\boldsymbol{n}_E$ perpendicular to E. Let

$$\begin{aligned} V_{\varepsilon,h} = \{ & v : \Omega \to \mathbb{R} \mid v|_Q \in P_{\varepsilon,Q} \text{ for all } Q \in \mathcal{T}_h, \\ & v \text{ is continuous at all interior vertices and midpoints in } \mathcal{T}_h, \\ & \varepsilon \fint_E \frac{\partial v}{\partial \boldsymbol{n}_E} \mathrm{d}s \text{ is continuous over all interior edges } E \text{ in } \mathcal{T}_h\}, \\ V_{\varepsilon,h0} = \{ & v \in V_{\varepsilon,h} \mid v \text{ vanishes at all boundary vertices and midpoints} \\ & \text{and } \varepsilon \fint_E \frac{\partial v}{\partial \boldsymbol{n}_E} \mathrm{d}s \text{ vanishes over all boundary edges}\}, \end{aligned}$$

where $P_{\varepsilon,Q} = P_Q$ if $\varepsilon \neq 0$, and $P_{\varepsilon,Q} = S_2^1(Q^*)$ otherwise. Notice that when $\varepsilon = 0$ we use the 8-node quadrilateral spline finite element [17]. The local basis functions have been expressed explicitly in equation (2.5). Since we focus on the case $\varepsilon \neq 0$, we rewrite $V_h := V_{\varepsilon,h}, V_{h0} := V_{\varepsilon,h0}$.

Then the nonconforming finite element approximation of (4.1) is: find $u_h \in V_{h0}$ such that

$$\varepsilon^2 a_h(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}, \tag{4.5}$$

where

$$a_h(u_h, v_h) = \sum_{Q \in \mathcal{T}_h} \int_Q D^2 u_h : D^2 v_h,\ b_h(u_h, v_h) = \sum_{Q \in \mathcal{T}_h} \int_Q D u_h \cdot D v_h. \tag{4.6}$$

Define a seminorm $|||\cdot|||_{\varepsilon,h}$ by

$$|||w|||_{\varepsilon,h}^2 = \varepsilon^2 a_h(w, w) + b_h(w, w) = \varepsilon^2 |w|_{2,h}^2 + |w|_{1,h}^2,$$

where $|\cdot|_{i,h}^2 = \sum_{Q \in \mathcal{T}_h} |\cdot|_{i,Q}^2, i = 1, 2$. It is easy to verify that $|||\cdot|||_{\varepsilon,h}$ is a norm on V_{h0}. Therefore the problem (4.5) has a unique solution by the Lax-Milgram lemma.

Define the global interpolation operator $\mathit{\Pi}_h : H_0^2 \to V_{h0}$, where $\mathit{\Pi}_Q = \mathit{\Pi}_h|_Q$ for $Q \in \mathcal{T}_h$. For each $E \in \mathcal{E}_h$, let $[\cdot]_E$ present the jump of a function on E, $\boldsymbol{n}$ and $\boldsymbol{\tau}$ denote the unit outward normal and tangential vectors on E, respectively.

Since Π_Q preserves functions in $P_2(Q)$, using the Bramble-Hilbert lemma, there exists a constant c independent of h such that

$$\sum_{Q\in\mathcal{T}_h} |v-\Pi_h v|_{j,Q} \le ch^{k-j}|v|_k \quad \text{for} \quad v\in H^k, \tag{4.7}$$

where $j=0,1,2$ and $k=2,3$. From [20] and a Bramble-Hilbert argument, one has

$$|v-\Pi_h v|_1 \le ch^{\frac{1}{2}}|v|_1^{\frac{1}{2}}|v|_2^{\frac{1}{2}} \quad \text{for} \quad v\in H_0^2. \tag{4.8}$$

The continuity requirement on V_{h0} implies that

$$\int_E \left[\frac{\partial w_h}{\partial \boldsymbol{n}}\right]_E \mathrm{d}s = 0 \quad \text{for any} \quad w_h\in V_{h0}. \tag{4.9}$$

Using (4.9) and a series of estimates as in [20], we derive

$$\sum_{E\in\mathcal{E}_h}\int_E \left(\Delta\psi - \partial^2\psi/\partial\boldsymbol{\tau}^2\right)\left[\frac{\partial w_h}{\partial \boldsymbol{n}}\right]_E \mathrm{d}s \le c\frac{h}{\varepsilon}|\psi|_3\,|||w_h|||_{\varepsilon,h}\,, \tag{4.10}$$

for all $\psi\in H^3, w_h\in V_{h0}$. As an alternative, we have the bound

$$\sum_{E\in\mathcal{E}_h}\int_E \left(\Delta\psi - \partial^2\psi/\partial\boldsymbol{\tau}^2\right)\left[\frac{\partial w_h}{\partial \boldsymbol{n}}\right]_E \mathrm{d}s \le c\frac{h^{1/2}}{\varepsilon}|\psi|_2^{1/2}|\psi|_3^{1/2}\,|||w_h|||_{\varepsilon,h}\,, \tag{4.11}$$

for all $\psi\in H^3, w_h\in V_{h0}$.

The following theorem presents that for any fixed $\varepsilon\in(0,1]$, the finite element method we proposed converges linearly with respect to h.

Theorem 4.2. *Let u and $u_h\in V_{h0}$ be solutions of (4.1) and (4.5), respectively. Assume that u is in $H_0^2(\Omega)\cap H^3(\Omega)$ for a given $f\in L^2(\Omega)$. Then there exists a constant c independent of ε and h such that*

$$|||u-u_h|||_{\varepsilon,h} \le c\begin{cases} (h^2+\varepsilon h)|u|_3, \\ h(\varepsilon|u|_3+|u|_2). \end{cases} \tag{4.12}$$

Proof. From the second Strang lemma,

$$|||u-u_h|||_{\varepsilon,h} \le c\left(\inf_{v_h\in V_{h0}}|||u-v_h|||_{\varepsilon,h} + \sup_{w_h\in V_{h0}}\frac{|E_{\varepsilon,h}(u,w_h)|}{|||w_h|||_{\varepsilon,h}}\right), \tag{4.13}$$

where

$$E_{\varepsilon,h}(u,w_h) = \varepsilon^2 a_h(u,w_h) + b_h(u,w_h) - (f,w_h). \tag{4.14}$$

Owing to $P_2(Q) \subset P_Q$, the interpolation theory leads to

$$\inf_{v_h\in V_{h0}} |||u-v_h|||_{\varepsilon,h} \le |||u-\Pi_h u|||_{\varepsilon,h} = (\varepsilon^2|u-\Pi_h u|_{2,h}^2 + |u-\Pi_h u|_{1,h}^2)^{\frac{1}{2}} \le c\begin{cases} (h^2+\varepsilon h)|u|_3, \\ h(\varepsilon|u|_3+|u|_2). \end{cases} \tag{4.15}$$

It remains to estimate $E_{\varepsilon,h}(u,w_h)$.

Invoking the definition of $a_h(u,w_h)$, one has

$$a_h(u,w_h) = -(D\Delta u, Dw_h) + \sum_{Q\in\mathcal{T}_h}\int_{\partial Q}\left(\Delta u - \frac{\partial^2 u}{\partial \boldsymbol{\tau}^2}\right)\frac{\partial w_h}{\partial \boldsymbol{n}}\mathrm{d}s. \tag{4.16}$$

Since $u\in H^3(\Omega)$ it follows from (4.16) and the identify (4.3) that

$$E_{\varepsilon,h}(u,w_h) = \varepsilon^2\sum_{E\in\mathcal{E}_h}\int_E\left(\Delta u - \frac{\partial^2 u}{\partial \boldsymbol{\tau}^2}\right)\left[\frac{\partial w_h}{\partial \boldsymbol{n}}\right]_E\mathrm{d}s. \tag{4.17}$$

It therefore follows from (4.10) that

$$E_{\varepsilon,h}(u,w_h) \le c\varepsilon h|u|_3\, |||w_h|||_{\varepsilon,h}\,. \tag{4.18}$$

Collecting the estimates (4.15) and (4.18), we complete the proof. $\square$

Remark 4.3. If $u\in H^3(\Omega)$, note the fact that in the limit when ε tends to 0, the first bound in Theorem 4.1 gives the estimate

$$|u-u_h|_1 \le ch^2|u|_3.$$

Lemma 4.1 leads to the subsequent uniform convergence property for our nonconforming finite element method.

Theorem 4.4. *Assume that Ω is convex, $f\in L^2(\Omega)$ and $u\in H_0^2(\Omega)\cap H^3(\Omega)$ be the solution of (4.1) and $u_h\in V_{h0}$ be the solution of (4.5), respectively. Then there exists a constant c, independent of ε, h and f, such that*

$$|||u-u_h|||_{\varepsilon,h} \le ch^{\frac{1}{2}}\,\|f\|_0.$$

Proof. The second Strang lemma (4.13) is still valid. From (4.7) and Lemma 4.1, we have

$$\varepsilon|u-\Pi_h u|_2 = \varepsilon|u-\Pi_h u|_2^{\frac{1}{2}}|u-\Pi_h u|_2^{\frac{1}{2}} \le c\varepsilon h^{\frac{1}{2}}|u|_2^{\frac{1}{2}}|u|_3^{\frac{1}{2}} \le ch^{\frac{1}{2}}\,\|f\|_0.$$

We proceed to estimate the H^1-part of $|||u-\Pi_h u|||_{\varepsilon,h}$. Let u^0 denote the solution for the reduced problem (4.4) as Lemma 4.1. Using a triangle inequality we obtain

$$|u-\Pi_h u|_1 \le |u-u^0-\Pi_h(u-u^0)|_1 + |u^0-\Pi_h u^0|_1.$$

From [12] we get the fact that

$$\left\|u^0\right\|_2 \le c\,\|f\|_0.$$

From (4.8) and Lemma 4.1, one has

$$\begin{aligned}|u-u^0-\Pi_h(u-u^0)|_1 &\le ch^{\frac{1}{2}}|u-u^0|_1^{\frac{1}{2}}|u-u^0|_2^{\frac{1}{2}}\\ &\le ch^{\frac{1}{2}}\|f\|_0,\end{aligned}$$

while

$$|u^0-\Pi_h u^0|_1 \le ch|u^0|_2 \le ch\|f\|_0.$$

Hence, we have showed that

$$\inf_{v_h\in V_{h0}} |||u-v_h|||_{\varepsilon,h} \le ch^{\frac{1}{2}}\|f\|_0. \tag{4.19}$$

Furthermore, the consistency error $E_{\varepsilon,h}(u,w_h)$ have been expressed as (4.17). From (4.11) we obtain

$$E_{\varepsilon,h}(u,w_h) \le c\varepsilon h^{\frac{1}{2}}|u|_2^{\frac{1}{2}}|u|_3^{\frac{1}{2}}\,|||w_h|||_{\varepsilon,h}\,.$$

It therefore follows from Lemma 4.1 that

$$E_{\varepsilon,h}(u,w_h) \le ch^{\frac{1}{2}}\|f\|_0\,|||w_h|||_{\varepsilon,h}\,. \tag{4.20}$$

Collecting the estimates (4.19) and (4.20), we obtain

$$|||u-u_h|||_{\varepsilon,h} \le ch^{\frac{1}{2}}\|f\|_0.$$

We complete the proof. $\square$

5. Numerical experiments

In this section, we provide numerical results with the proposed element. The domain Ω is first divided into n^2 squares of size $h\times h$ with $h=1/n$. Then three types of quadrilateral meshes are employed: uniform meshes shown in Figure 4, the randomly perturbed quadrilateral meshes depicted in Figure 5 and the trapezoid meshes as shown in Figure 6. Numerical solutions are computed both in these meshes.

Example 5.1. Consider problem (1.1) with $\Omega=[0,1]^2\subset\mathbb{R}^2$ and $f=\varepsilon^2\Delta^2u-\Delta u$, where $u=(\sin(\pi x_1)\sin(\pi x_2))^2$.

The errors measured by the seminorm $|||\cdot|||_{\varepsilon,h}$ for different values of ε and meshes are shown in Tables 1–3, respectively. For a comparison we also consider the biharmonic problem $\Delta^2u=f$ by using the element space P_Q. From these tables below, we see that these numerical results are consistent with our theoretical analysis. More precisely, the new quadrilateral element method converges for all $\varepsilon\in(0,1]$ and behaves very well for the biharmonic problem.

Example 5.2. Assume $\Omega=[0,1]^2\subset\mathbb{R}^2$ and $u(x_1,x_2)=\varepsilon(e^{-x_1/\varepsilon}+e^{-x_2/\varepsilon})-x_1^2x_2$, a direct computation shows $f=\varepsilon^2\Delta^2u-\Delta u=2x_2$ whenever $\varepsilon\neq 0$.

Clearly, u does not satisfy the homogeneous boundary condition in (1.1). Nevertheless, the nonhomogeneous counterpart can be naturally applied in our programming. Indeed, the seminorms $|u|_2$ and $|u|_3$ will explode

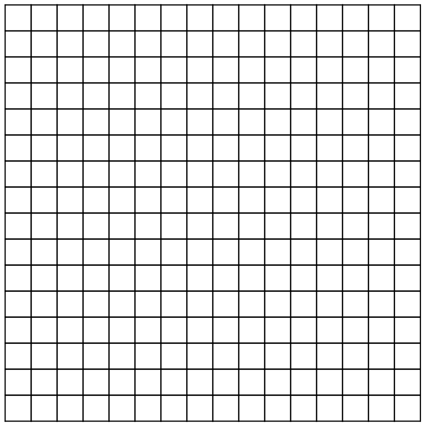

FIGURE 4. A uniform quadrilateral mesh.

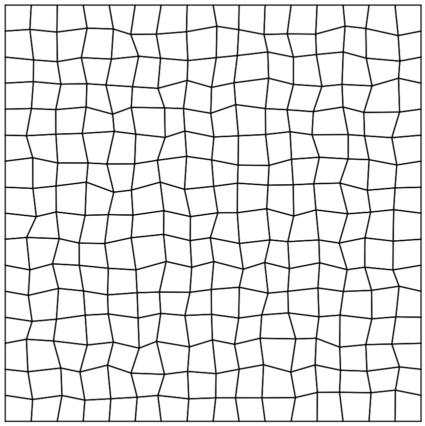

FIGURE 5. A randomly perturbed quadrilateral mesh.

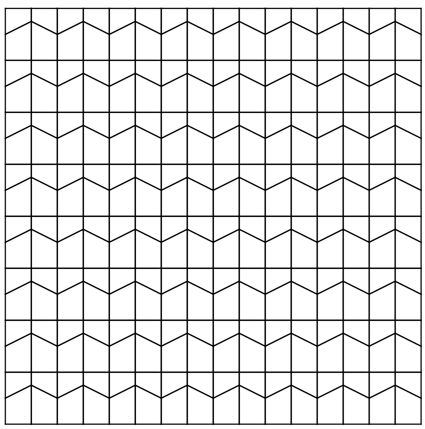

FIGURE 6. A trapezoid mesh.

TABLE 1. The error measured by the energy norm for uniform meshes.

$\varepsilon\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	Rate
2^0	5.6010	2.7883	1.3913	0.6954	0.3476	1.0005
2^{-2}	1.4360	0.7014	0.3484	0.1739	0.0869	1.0025
2^{-4}	0.4246	0.1825	0.0879	0.0436	0.0217	1.0115
2^{-6}	0.2145	0.0620	0.0242	0.0112	0.0055	1.1115
2^{-8}	0.1888	0.0438	0.0117	0.0037	0.0015	1.3026
2^{-10}	0.1868	0.0422	0.0103	0.0026	7.2316e−04	1.8461
Biharmonic	5.5899	2.7869	1.3911	0.6953	0.3476	1.0002

TABLE 2. The error measured by the energy norm for randomly perturbed meshes.

$\varepsilon\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	Rate
2^0	6.0561	3.0974	1.5011	0.7621	0.3801	1.0036
2^{-2}	1.4848	0.7617	0.3797	0.1914	0.0925	1.0491
2^{-4}	0.4884	0.2027	0.0971	0.0474	0.0238	1.0618
2^{-6}	0.2309	0.0709	0.0270	0.0123	0.0060	1.1343
2^{-8}	0.2123	0.0537	0.0138	0.0044	0.0017	1.3720
2^{-10}	0.2033	0.0508	0.0124	0.0032	8.6860e−04	1.8813
Biharmonic	5.9454	2.9859	1.5104	0.7599	0.3803	0.9987

TABLE 3. The error measured by the energy norm for trapezoid meshes.

$\varepsilon\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	Rate
2^0	6.6177	3.4086	1.6871	0.8407	0.4198	1.0019
2^{-2}	1.7100	0.8604	0.4228	0.2103	0.1050	1.0075
2^{-4}	0.5218	0.2299	0.1075	0.0528	0.0263	1.0257
2^{-6}	0.2710	0.0869	0.0316	0.0139	0.0067	1.0529
2^{-8}	0.2399	0.0645	0.0172	0.0053	0.0020	1.4060
2^{-10}	0.2375	0.0623	0.0154	0.0039	0.0011	1.8260
Biharmonic	6.6003	3.4060	1.6867	0.8407	0.4198	1.0019

TABLE 4. The error measured by the energy norm for uniform meshes.

$\varepsilon\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	Rate
2^0	0.1845	0.0918	0.0459	0.0229	0.0115	1.0010
2^{-2}	0.1329	0.0639	0.0313	0.0156	0.0078	1.0227
2^{-4}	0.2032	0.1198	0.0619	0.0309	0.0154	0.9305
2^{-6}	0.1491	0.1265	0.0957	0.0575	0.0304	0.5735
2^{-8}	0.1521	0.1013	0.0732	0.0618	0.0470	0.4236
2^{-10}	0.1706	0.1144	0.0758	0.0505	0.0364	0.5571

when ε tends to zero. The errors $|||u - u_h|||_{\varepsilon,h}$ for different values of ε and meshes are depicted in Tables 4–6. These numerical results are conformable to the theoretical analysis of Theorem 4.2.

Table 5. The error measured by the energy norm for randomly perturbed meshes.

$\varepsilon\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	Rate
2^{0}	0.1982	0.1001	0.0499	0.0254	0.0127	0.9910
2^{-2}	0.1320	0.0675	0.0336	0.0167	0.0084	0.9935
2^{-4}	0.2013	0.1216	0.0632	0.0323	0.0164	0.9044
2^{-6}	0.1550	0.1282	0.0974	0.0592	0.0314	0.5758
2^{-8}	0.1496	0.1037	0.0733	0.0627	0.0474	0.4145
2^{-10}	0.1730	0.1126	0.0754	0.0498	0.0371	0.5553

Table 6. The error measured by the energy norm for trapezoid meshes.

$\varepsilon\backslash h$	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	Rate
2^{0}	0.2222	0.1091	0.0542	0.0270	0.0135	1.0102
2^{-2}	0.1746	0.0802	0.0377	0.0183	0.0090	1.0695
2^{-4}	0.2445	0.1480	0.0763	0.0371	0.0181	0.9389
2^{-6}	0.1787	0.1479	0.1122	0.0690	0.0366	0.5719
2^{-8}	0.1773	0.1196	0.0870	0.0719	0.0546	0.4249
2^{-10}	0.1938	0.1311	0.0876	0.0593	0.0431	0.5422

6. Conclusion

In this paper, we propose a new nonconforming finite element method to solve the elliptic fourth-order singular perturbation equation. Our method is based on quadrilateral mesh, and the constructed finite element is a C^0 element. Namely, the element is H^1-conforming. The local basis functions of our element can be expressed explicitly, which is an advantage. Besides, all the integrations can be done through the reference domain. By using the finite element method constructed in this paper, the fourth-order singular perturbation problem is convergent uniformly with respect to ε.

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