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HIGH-FREQUENCY BEHAVIOUR OF CORNER SINGULARITIES IN HELMHOLTZ PROBLEMS[☆]

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Abstract. We analyze the singular behaviour of the Helmholtz equation set in a non-convex polygon. Classically, the solution of the problem is split into a regular part and one singular function for each re-entrant corner. The originality of our work is that the "amplitude" of the singular parts is bounded explicitly in terms of frequency. We show that for high frequency problems, the "dominant" part of the solution is the regular part. As an application, we derive sharp error estimates for finite element discretizations. These error estimates show that the "pollution effect" is not changed by the presence of singularities. Furthermore, a consequence of our theory is that locally refined meshes are not needed for high-frequency problems, unless a very accurate solution is required. These results are illustrated with numerical examples that are in accordance with the developed theory.

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1. Introduction

Time-harmonic waves are used in a wide range of applications including resource prospection, noise scattering, radar, cloaking and medical imaging [11, 13]. Consequently, many efforts have been made to develop efficient discretization techniques to approximate the solutions to such problems in an accurate and robust fashion. Popular approaches to carry out such a discretization include the finite difference method (FDM) [40], the finite element method (FEM) [12, 25] and the boundary element method (BEM) [14, 36].

For instance, wave propagation problems arising in scattering applications are set in the exterior of the scatterer. An inhomogeneous Dirichlet, Neumann or Robin-type boundary condition is prescribed on the boundary of the scatterer to model the illuminating wave, while the Sommerfeld condition is imposed "at infinity" to prevent non-physical ingoing waves (alternatively absorbing boundary conditions or perfectly matched layers are used to approximate the Sommerfeld condition on the exterior boundary of a surrounding domain Ω_0 [7, 16, 19]) and

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close the problem [13]. Scattering wave propagation problems thus naturally take place in non-convex domains. As a result, when the scatterer is not smooth, the solution is singular close to its corners.

Motivated by these examples, we carefully analyze the singular behaviour of the solution around the vertices of the boundary (for instance the vertices of a polygonal scatterer). This study is especially important, as the presence of singularities strongly impact the performance of numerical methods. Indeed, in the general case, the lack of regularity of the solution is numerically translated by a decreased convergence rate, unless especially refined (graded) meshes are used [5, 33].

Here we focus on acoustic 2D Helmholtz operator $-\omega^2/\mathscr{V}^2 - \Delta$, with varying wavespeed $\mathscr{V}$ (corresponding to heterogeneous media). Then, the singularities of the Helmholtz problem have the same form as the ones of the Laplace operator $-\Delta$. It allows us to use the vast literature on the subject (see for instance [22, 32]). However, though the theory of singularities is very well established for the Laplace operator, an essential feature is missing: the behaviour of the singularities with respect to the frequency ω.

Indeed, numerical methods are very sensitive to the frequency. In particular, solving high frequency problems accurately is a very computationally expensive task. This is linked to the fact that as the frequency increases, the Helmholtz operator has additional negative eigenvalues, so that it is challenging to ensure the stability of discrete numerical schemes. Intuitively, the solution is more oscillatory at higher frequencies, and these oscillations are difficult to capture numerically.

In this paper, we focus on FEM discretizations. In this context, the difficulty to solve at high frequencies is usually called the "pollution effect": for high frequency problems, unless the mesh is heavily refined, there is a gap between the interpolation error, and the error of the finite element solution. It means that the solution obtained by the finite-element procedure is much less accurate than the best representation of the continuous solution in the finite element space.

The "gap" between the interpolation and finite element errors is called the pollution error. On Cartesian grid based meshes, the pollution error can be computed thanks to a dispersion analysis [3, 28]. Furthermore, there exists an asymptotic range $h \leq h_0$ in which this gap disappears, and the finite element solution is almost as accurate as the interpolant. The behaviour of $h_0 = h_0(\omega)$ with respect to the frequency is thus a key to analyze the performance of the finite element method. When the domain is smooth, this analysis has been carried out for Lagrange finite elements of arbitrary order p [30, 31], and it is known that $h_0 \simeq \omega^{-1-1/p}$.

Of course, the regularity of the solution and the dependence of the Sobolev norms of the solution on ω play a central role in the above-mentioned analysis. As a result, it is not obvious how the singularities of the solution can affect the pollution effect, in a domain with corners. For instance, a scattering problem with re-entrant corners is discretized using a "plane wave" method in [4]. Therein, the authors propose a convergence analysis, and characterize the asymptotic range $h \leq h_0(\omega)$ in which the pollution effect vanishes. Because of the singularities and the use of uniform meshes, they obtain the condition that $h \lesssim \omega^{-5/2}$, which is more restrictive than the condition $h \lesssim \omega^{-2}$ known for $\mathcal{P}_1$ elements in smooth domains [30, 31].

Under minimal assumptions (namely the validity of a stability estimate and constant coefficients near the corner points) and following [22, 32], we here propose a splitting of the solution into a regular part in $H^2(\Omega)$ and one singular function for each re-entrant corner. Our main achievement is a precise description of the "amplitude" of the singularities depending on the frequency. We also show that the regular part of the solution behaves as the solution of a Helmholtz problem in a smooth domain. These results are derived using slight modifications of arguments used in an other context in [23]. Similar results have been obtained in [8, 9, 26] using slightly different techniques in the context of BEM. However, our analysis allows us to treat more general data and provide sharper estimates (we provide more details in the end of Sect. 3). In the context FEM, prior works taking into account the singularities of the solution include [4, 20]. To the best of our knowledge, our results are new, and we prove that our bounds are sharp.

Furthermore, we take advantage of the above-mentioned splitting to derive sharp error estimates for $\mathcal{P}_1$ finite element discretizations of the problem. In particular, we prove that the asymptotic range (and thus the pollution effect) is not affected by the presence of singularities. The newly introduced splitting is the key to improve the error estimates given in [4].

We illustrate these results with numerical experiments. For $\mathcal{P}_1$ elements, the numerical results are in agreement with the theory. Furthermore, we numerically investigate higher order discretizations. Our main observation is that, for high frequencies, the singularities only have a small impact on the numerical schemes. In particular, unless a very accurate solution is needed, the use of graded meshes is not required.

Finally in order to show the large applicability of our assumptions, we check that the stability estimate holds for various choices of domains and wave speeds $\mathscr{V}$.

Our work is outlined as follow. In Section 2, we precisely state the problem we consider and state our basic assumptions (in particular a stability estimate in the $H^1(\Omega)$ norm). Sections 3 and 4 are dedicated to the analysis of the singularities of the problem. The particular case of a disc sector featuring one singular corner at the origin is first analyzed in Section 3. This result is then applied, by localization, to analyze the case of a general polygonal domain in Section 4. We provide stability conditions and error estimates for finite element discretizations in Section 5, and Section 6 is devoted to numerical experiments. In the appendices, we collect some useful properties of Bessel functions and prove the stability estimate in some particular cases.

2. The setting

In this work, we consider wave propagation problems modelized by the Helmholtz equation in a domain Ω:

$$\left\{\begin{array}{rcll} -\dfrac{\omega^2}{\mathscr{V}^2}u - \Delta u & = & f & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} - \dfrac{i\omega}{\mathscr{V}}u & = & 0 & \text{on } \Gamma_{\mathrm{Diss}}, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{Dir}}, \end{array}\right. \tag{2.1}$$

where $f : \Omega \to \mathbb{C}$ is a given source term, and Γ_{Dir} and Γ_{Diss} are two disjoint open subsets of the boundary $\partial\Omega$ of Ω such that $\overline{\Gamma_{\mathrm{Dir}}} \cup \overline{\Gamma_{\mathrm{Diss}}} = \partial\Omega$. In addition, $\omega > 0$ is the angular frequency, and $\mathscr{V} : \Omega \to \mathbb{R}$ is the wavespeed. We assume that $\mathscr{V} \in C^1(\bar{\Omega})$ satisfies $0 < \mathscr{V}_{\min} \leq \mathscr{V} \leq \mathscr{V}_{\max} < +\infty$ for two constants $\mathscr{V}_{\min}, \mathscr{V}_{\max}$.

Classically, assuming that $f \in L^2(\Omega)$, we recast (2.1) into the variational problem that consists in looking for $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ solution to

$$B(u,v) = (f,v), \quad \forall v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega), \tag{2.2}$$

where

$$B(u,v) = -\omega^2(\mathscr{V}^{-2}u, v) - i\omega\langle \mathscr{V}^{-1}u, v\rangle_{\Gamma_{\mathrm{Diss}}} + (\nabla u, \nabla v),$$

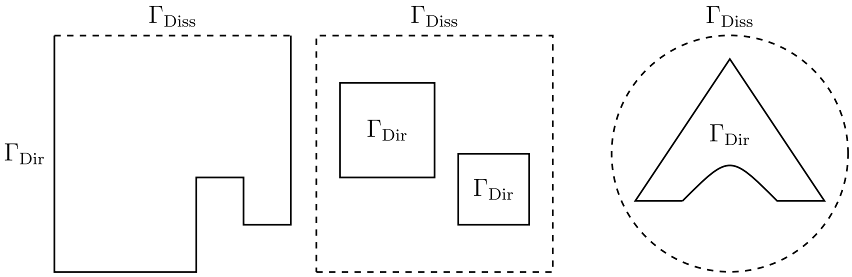
and

$$H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) = \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \Gamma_{\mathrm{Dir}}\},$$

γ_0 being the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$.

In most applications, the boundary Γ_{Dir} is imposed, and represents an obstacle or the basis of a cavity. On the other hand, the boundary Γ_{Diss} is artificially designed to approximate the Sommerfeld condition. The aim of our work is to analyze the singularities that can arise due to corners of the boundary Γ_{Dir}. We do not consider singularities due to Γ_{Diss}, since one can usually design this artificial boundary to avoid reentrant corners and thus, singularities.

The key technical assumptions required by our analysis are that close to each reentrant corner of Γ_{Dir}, the boundary is polygonal, and $\mathscr{V}$ is constant. A part from these technical assumptions, general domains and velocity parameters can be treated, as long as some stability estimate on $\|u\|_{1,\Omega}$ is available. In the following, we rigorously summarize our main assumptions.



(A) A cavity ($\sigma = 2$) (B) A trapping obstacle ($\sigma = 2$) (C) A non-trapping obstacle ($\sigma = 0$)

FIGURE 1. Example of domains satisfying Assumptions 2.1 and 2.2.

Assumption 2.1. We assume that there exist two non negative constants σ and ω_0 (independent of ω) such that for all $f \in L^2(\Omega)$ and $\omega \geq \omega_0$, problem (2.2) admits a unique solution $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. Furthermore, we assume that

$$\omega \|u\|_{0,\Omega} + |u|_{1,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}) \omega^\sigma \|f\|_{0,\Omega}, \tag{2.3}$$

for some positive constant $C(\Omega, \omega_0, \mathscr{V})$ that may depend on Ω, ω_0, $\mathscr{V}$, but neither on ω nor on f.

Assumption 2.2. The curves Γ_{Diss} and Γ_{Dir} are piecewise C^1. If $\mathbf{y}$ is a point at which Γ_{Diss} is not C^1, then the angle formed by Γ_{Diss} around $\mathbf{y}$ is less than or equal to π. Furthermore, there exists a set of points $\{\mathbf{x}_j\}_{j=1}^N$ and a real number $\ell > 0$ such that, for each point $\mathbf{y}$ at which Γ_{Dir} is not C^1, either (a) the angle formed by Γ_{Dir} around $\mathbf{y}$ is less than or equal to π, or (b) $\mathbf{y} = \mathbf{x}_j$ for some $j \in \{1, \ldots, N\}$, and Γ_{Dir} is polygonal and $\mathscr{V}$ is constant over $B(\mathbf{x}_j, \ell)$. Finally, we assume that either $\overline{\Gamma_{\mathrm{Dir}}} \cap \overline{\Gamma_{\mathrm{Diss}}} = \emptyset$, or that the angle formed by Γ_{Dir} and Γ_{Diss} is less than or equal to $\pi/2$ each time they meet.

Assumption 2.1 is satisfied in a number of applications, as illustrated by Figure 1. For the cavity problem, we have adapted the proof from [18] in Appendix B.1 to the case of parameters $\mathscr{V}$ such that $\partial_{\mathbf{x}_2} \mathscr{V} \leq 0$. The proof for the non-trapping obstacle is classical when $\mathscr{V}$ is constant. In Appendix B.2, we show that the result is still valid as long as $\nabla \mathscr{V} \cdot \mathbf{x} \leq (1 - \delta)\mathscr{V}$ for some $\delta \in (0, 1)$. Finally, we show that Assumption 2.1 holds for some trapping obstacles by adapting arguments from [10].

Assumption 2.2 ensures that the singularities only take place on the corners of Γ_{Dir}, which is reasonable in most applications, since the boundary Γ_{Diss} is artificially placed. On the other hand, the assumptions that Γ_{Dir} is polygonal and that $\mathscr{V}$ is constant in a neighborhood of such corners is only technical, but mandatory for our analysis. The domains presented in Figure 1 also satisfy Assumption 2.2.

Scattering of a plane wave by an obstacle immersed in a homogeneous medium is an important particular case. In this scenario, one considers an obstacle K and a domain $\Omega_0 \supset \overline{K}$. Then, the domain of interest is $\Omega = \Omega_0 \setminus \overline{K}$ with $\Gamma_{\mathrm{Dir}} = \partial K$ and $\Gamma_{\mathrm{Diss}} = \partial \Omega_0$. We look for v solution to

$$\begin{cases} -\omega^2 v - \Delta v = 0 & \text{in } \Omega, \\ \nabla v \cdot \mathbf{n} - i\omega v = 0 & \text{on } \Gamma_{\mathrm{Diss}}, \\ v = g & \text{on } \Gamma_{\mathrm{Dir}}, \end{cases} \tag{2.4}$$

where $g(\mathbf{x}) = e^{i\omega \mathbf{x}\cdot\mathbf{d}}$ for some unitary vector $\mathbf{d}$ that represents the direction of the plane wave. While our assumptions do not directly cover (2.4), we can introduce the function $\eta(\mathbf{x}) = e^{i\omega\mathbf{x}\cdot\mathbf{d}}\chi(\mathbf{x})$, where $\chi \in C^\infty(\overline{\Omega})$ is a cutoff function such that $\chi = 1$ in a neighborhood of Γ_{Dir} and $\chi = 0$ in a neighborhood of Γ_{Diss}. Then, the

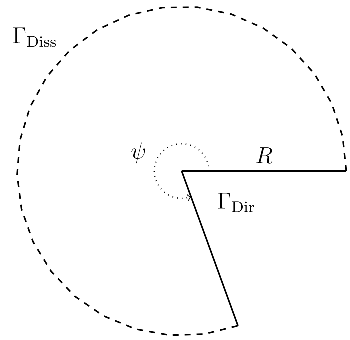


FIGURE 2. The disc sector $D_{\psi,R}$.

function $u = v - \eta$ is solution to (2.1) with the source term

$$f(\mathbf{x}) = e^{i\omega\mathbf{x}\cdot\mathbf{d}}\Delta\chi(\mathbf{x}) + 2i\omega e^{i\omega\mathbf{x}\cdot\mathbf{d}}\mathbf{d}\cdot\nabla\chi(\mathbf{x}).$$

We note that such a function f belongs to $L^2(\Omega)$ with

$$\|f\|_{0,\Omega} \leq C(\Omega)\omega. \tag{2.5}$$

Since $\eta \in C^\infty(\overline{\Omega})$, u and v have the same singularities. As a result, our analysis also holds for plane wave scattering problems satisfying Assumptions 2.1 and 2.2 if one uses (2.5) in the right-hand-side of the estimates.

When the domain Ω is convex, or smooth, one easily obtains a bound for the $H^2(\Omega)$ norm of the solution from (2.3) by applying a shift theorem for the Laplace operator. However, the domain we consider do not have such shifting properties. The analysis of higher order derivatives of the solution must therefore be carried out carefully, by explicitly analyzing the corner singularities.

3. The case of a disc sector

We consider a disc sector $D_{\psi,R} \subset \mathbb{R}^2$ of radius R and opening ψ. Its boundary is split into two parts $\Gamma_{\rm Diss}$ and $\Gamma_{\rm Dir}$ corresponding respectively to the circular and straight portions of $\partial D_{\psi,R}$, see Figure 2. Hence, the domain is defined by

$$D_{\psi,R} = \{(r\cos\theta, r\sin\theta) \quad | \quad 0 < r < R, \quad 0 < \theta < \psi\},$$

and the boundary is specified as $\Gamma_{\rm Diss} = S_{\psi,R}$ and $\Gamma_{\rm Dir} = \overline{I_{0,R}} \cup \overline{I_{\psi,R}}$, where

$$S_{\psi,R} = \{(R\cos\theta, R\sin\theta) \quad | \quad 0 < \theta < \psi\},$$

and

$$I_{0,R} = \{(r,0) \quad | \quad 0 < r < R\}, \quad I_{\psi,R} = \{(r\cos\psi, r\sin\psi) \quad | \quad 0 < r < R\}.$$

For the sake of simplicity, we assume that $\pi < \psi < 2\pi$, and $\mathscr{V} = 1$, so that a singularity occurs at the origin for the solution of problem (2.1) in $D_{\psi,R}$ with the previous choice of $\Gamma_{\rm Diss}$ and $\Gamma_{\rm Dir}$. We observe that because the two angles where $\Gamma_{\rm Diss}$ and $\Gamma_{\rm Dir}$ meet are $\pi/2$, the solution to the problem lies in $H^{1+\alpha-\epsilon}(\Omega)$ with $\alpha = \pi/\psi > 1/2$ and any $\epsilon > 0$ (see below). Furthermore, since $\mathbf{x}\cdot\mathbf{n} = 0$ on $\Gamma_{\rm Dir}$ and $\mathbf{x}\cdot\mathbf{n} = |\mathbf{x}|$ on $\Gamma_{\rm Diss}$, we can apply Theorem B.2, and we see that Assumption 2.1 is satisfied with $\sigma = 0$.

When the domain Ω is regular, it can further be shown that $u \in H^2(\Omega)$ and the semi-norm $|u|_{2,\Omega}$ is explicitly controlled in terms of ω. Here, we consider a domain $D_{\psi,R}$ with a re-entrant corner, so that in general, the solution presents a singularity at the origin. Indeed, as mentioned before the solution belongs to $H^{1+\alpha-\epsilon}(D_{\psi,R})$ for any $\epsilon > 0$, but not to $H^{1+\alpha}(D_{\psi,R})$. In particular this solution does not belong to $H^2(D_{\psi,R})$ since $\alpha \in (\frac{1}{2}, 1)$.

Hereafter, we propose a splitting of the solution u into a regular part belonging to $\tilde{u}_R \in H^2(D_{\psi,R})$ and a singular part $S \in H^{1+\alpha-\epsilon}(D_{\psi,R})$. We show that $|\tilde{u}_R|_{2,D_{\psi,R}}$ behaves as $|u|_{2,D_{\psi,R}}$ when the domain is regular. Furthermore, we provide a novel estimate for the semi-norm $|S|_{1+\alpha-\epsilon}$.

In this section, we require some properties linked to Bessel functions that are established in Appendix A and listed below.

Proposition 3.1. *For all $\alpha \in (1/2, 1)$, there exists $\omega_0 > 0$ large enough such that the following properties hold*

$$\int_0^R |J_\alpha(\omega r)|^2 r \mathrm{d}r \leq C(\alpha, R, \omega_0)\omega^{-1}, \tag{3.1}$$

$$\int_0^R |H_\alpha^{(2)}(\omega r)|^2 r \mathrm{d}r = C(\alpha, R, \omega_0)\omega^{-1} + \mathcal{O}\left(\omega^{-3}\right), \tag{3.2}$$

$$\frac{Y_\alpha'(\omega R) + iY_\alpha(\omega R)}{J_\alpha'(\omega R) + iJ_\alpha(\omega R)} = -i + \mathcal{O}\left((\omega R)^{-3/2}\right), \tag{3.3}$$

$$J_\alpha(\epsilon\omega)Y_\alpha'(\epsilon\omega) - J_\alpha'(\epsilon\omega)Y_\alpha(\epsilon\omega) = \frac{2}{\epsilon\pi\omega}, \tag{3.4}$$

for all $\omega \geq \omega_0$ and $\epsilon > 0$.

3.1. Splitting of the solution

We propose a splitting of the solution u into a regular part $\tilde{u}_R \in H^2(D_{\psi,R})$ and a singular part in $H^{1+\alpha-\epsilon}(D_{\psi,R})$. The singular properties of the Helmholtz operator are strongly linked to the ones of the Laplacian. Hence, our analysis heavily relies on the theory developed by Grisvard [22]. More precisely, we can state that the solution u of (2.1) in $D_{\psi,R}$ (with $\mathscr{V} = 1$ and the previous choice of Γ_{Dir} and Γ_{Diss}) can be decomposed as

$$u = \tilde{u}_R + \tilde{c}_\omega(f)\chi(r)r^\alpha \sin(\alpha\theta), \tag{3.5}$$

where $\tilde{u}_R \in H^2(D_{\psi,R})$ is the regular part, $r^\alpha \sin(\alpha\theta) \in H^{1+\alpha-\epsilon}(D_{\psi,R})$ represents the singularity of the solution, $\tilde{c}_\omega(f) \in \mathbb{C}$ is a constant depending on the data of the problem, and χ is a C^∞ cutoff function that equals 1 in a neighborhood of the origin and 0 close to Γ_{Diss}.

Decomposition (3.5) is especially useful when analyzing Laplace problems, as $r^\alpha \sin(\alpha\theta)$ is a harmonic function. Also, this decomposition will be useful when analyzing the approximation properties of finite element spaces. However, it is tricky to directly estimate the constant $\tilde{c}_\omega(f)$. As a result, we will use the decomposition

$$u = u_R + c_\omega(f)J_\alpha(\omega r)\sin(\alpha\theta), \tag{3.6}$$

with $u_R \in H^2(D_{\psi,R})$ and $J_\alpha(\omega r)\sin(\alpha\theta)$ represents the singularity.

As we detail later, $J_\alpha(\omega r)$ and r^α have the same behaviour close to the origin, so that both functions can be used to describe the singularity. The advantage of decomposition (3.6) over (3.5) is that the representation of the singularity satisfies

$$\left(-\omega^2 - \Delta\right)\left(J_\alpha(\omega r)\sin(\alpha\theta)\right) = 0.$$

As a result, decomposition (3.6) is easier to handle, and we shall use it to estimate $c_\omega(f)$. We easily recover an estimate for $\tilde{c}_\omega(f)$ in a "post-processing" fashion.

In Theorem 3.2, we show that the solution u can be decomposed according to (3.5) or (3.6). Furthermore, we give a relation between the constants $c_\omega(f)$ and $\tilde{c}_\omega(f)$. Also, in order to simplify the notations, we introduce

$$s(\mathbf{x}) = J_\alpha(\omega r)\sin(\alpha\theta), \quad \tilde{s}(\mathbf{x}) = \chi(r) r^\alpha \sin(\alpha\theta). \tag{3.7}$$

Theorem 3.2. *For all $\omega \geq \omega_0$ and $f \in L^2(D_{\psi,R})$, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R})$ is solution to (2.2), there exist a function $\tilde{u}_R \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R}) \cap H^2(D_{\psi,R})$ and a constant $\tilde{c}_\omega(f) \in \mathbb{C}$ such that $u = \tilde{u}_R + \tilde{c}_\omega(f)\tilde{s}$.*

Furthermore, there exists a function $u_R \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R}) \cap H^2(D_{\psi,R})$ such that $u = u_R + c_\omega(f)s$, where

$$c_\omega(f) = 2^\alpha \Gamma(\alpha+1)\omega^{-\alpha}\tilde{c}_\omega(f). \tag{3.8}$$

Proof. The existence and uniqueness of $u \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R})$ being established, we can look at u as a solution to

$$\left\{ \begin{array}{rcll} -\Delta u &=& \tilde{f} & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma_{\mathrm{Dir}}, \\ \nabla u \cdot \mathbf{n} &=& g & \text{on } \Gamma_{\mathrm{Diss}}, \end{array} \right.$$

where $\tilde{f} = f + \omega^2 u \in L^2(D_{\psi,R})$, $g = i\omega u \in \tilde{H}^{1/2}(\Gamma_{\mathrm{Diss}})$. [1]

As g belongs to $\tilde{H}^{1/2}(\Gamma_{\mathrm{Diss}})$, by Theorem 1.5.2.8 of [22], there exists an element $\eta \in H^2(D_{\psi,R})$ such that

$$\gamma_0 \eta = 0, \quad \gamma_0 \left(\nabla\eta \cdot \mathbf{n}\right) = \tilde{g} = i\omega u \text{ on } \Gamma, \tag{3.9}$$

with the estimate

$$\|\eta\|_{2,\Omega} \leq C(\psi,R)\|\tilde{g}\|_{H^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})} = C(\psi,R)\omega\|u\|_{H^{\frac{1}{2}}(\Gamma_{\mathrm{Diss}})},$$

for some positive constant $C(\psi,R)$ that depends only on ψ and R. Hence by a trace theorem and estimate (2.3), we deduce that

$$\|\eta\|_{2,D_{\psi,R}} \leq C(\psi,R,\omega_0)\omega\|f\|_{0,D_{\psi,R}}. \tag{3.10}$$

Since $\gamma_0\eta = 0$, it is also clear that we have $\eta \in H^1_0(D_{\psi,R}) \subset H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R})$.

As a result, we see that $v = u - \eta \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R})$ is solution to

$$\left\{ \begin{array}{rcll} -\Delta v &=& h & \text{in } D_{\psi,R} \\ v &=& 0 & \text{on } \Gamma_{\mathrm{Dir}}, \\ \nabla v \cdot \mathbf{n} &=& 0 & \text{on } \Gamma_{\mathrm{Diss}}, \end{array} \right.$$

where $h = f + \omega^2 u + \Delta\eta \in L^2(D_{\psi,R})$. This allows us to apply Theorem 4.4.3.7 of [22], stating that $v \in \operatorname{span}\left\{H^2(D_{\psi,R}), \tilde{s}\right\}$. Hence, there exist a function $v_R \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R}) \cap H^2(D_{\psi,R})$ and a constant $\tilde{c}_\omega(f) \in \mathbb{C}$ such that $v = v_R + \tilde{c}_\omega(f)\tilde{s}$. Obviously, we obtain (3.5) by setting $u_R = v_R + \eta$.

Once (3.5) is established, (3.6) and (3.8) directly follow from a careful inspection of the definition of J_α. Indeed, if we isolate the first term in the development of J_α, we see that

$$J_\alpha(\omega r)\sin(\alpha\theta) = \frac{\omega^\alpha}{2^\alpha \Gamma(\alpha+1)} r^\alpha \sin(\alpha\theta) + \phi,$$

[1] As usual, for $s > 0$, a function g belongs to $\tilde{H}^s(\Gamma_{\mathrm{Diss}})$ if $\tilde{g}$, its extension by zero outside Γ_{Diss}, belongs to $H^s(\partial D_{\psi,R})$.

with $\phi \in H^1_{\Gamma_{\rm Dir}}(D_{\psi,R}) \cap H^2(D_{\psi,R})$. □

3.2. Estimation of $c_\omega(f)$

For each $\omega \geq \omega_0$, it is clear that the mapping $c_\omega : L^2(D_{\psi,R}) \to \mathbb{C}$ is continuous, and linear. Then, the Riesz representation theorem implies the existence of a unique $w_\omega \in L^2(D_{\psi,R})$ such that

$$c_\omega(f) = (f, w_\omega), \quad \forall f \in L^2(D_{\psi,R}). \tag{3.11}$$

Lemma 3.3. *For all $\omega \geq \omega_0$, w_ω can be characterized as the unique element of $L^2(D_{\psi,R})$ satisfying the following conditions:*

$$-\omega^2 w_\omega - \Delta w_\omega = 0 \ \textit{in} \ \mathcal{D}'(D_{\psi,R}), \tag{3.12}$$

$$\nabla w_\omega \cdot \mathbf{n} + i\omega w_\omega = 0 \ \textit{in} \ (\tilde{H}^{\frac{3}{2}}(\Gamma_{\rm Diss}))', \tag{3.13}$$

$$w_\omega = 0 \ \textit{in} \ (\tilde{H}^{\frac{1}{2}}(\Gamma_{\rm Dir} \setminus \{(0,0)\}))', \tag{3.14}$$

$$\big(-\omega^2 \eta s - \Delta(\eta s), w_\omega\big) = 1, \tag{3.15}$$

where $\eta \in C^\infty(D_{\psi,R})$ is any function such that $\eta = 1$ in a neighborhood of the origin and $\eta = 0$ in a neighborhood of $\Gamma_{\rm Diss}$. Further, here and below $\tilde{H}^{\frac{1}{2}}(\Gamma_{\rm Dir} \setminus \{(0,0)\})$ is the set of functions $g \in H^{\frac{1}{2}}(\Gamma_{\rm Dir})$ such that its restriction g_0 (resp. g_ψ) to $I_{0,R}$ (resp. $I_{\psi,,R}$) belongs to $\tilde{H}^{\frac{1}{2}}(I_{0,R})$ (resp. $\tilde{H}^{\frac{1}{2}}(I_{\psi,R})$).

Proof. First, let us prove that the function $w_\omega \in L^2(D_{\psi,R})$ defined in (3.11) satisfies conditions (3.12)–(3.15).

To prove (3.12), consider $\phi \in \mathcal{D}(D_{\psi,R})$, and define $f = -\omega^2\phi - \Delta\phi$. By definition of f, it is clear that $\phi = \phi_R + c_\omega(f)s$ with $\phi_R \in H^2(D_{\psi,R})$. But since $\phi \in \mathcal{D}(D_{\psi,R})$, we must have $c_\omega(f) = 0$. As a result,

$$c_\omega(-\omega^2\phi - \Delta\phi) = (-\omega^2\phi - \Delta\phi, w_\omega) = 0$$

for all $\phi \in \mathcal{D}(D_{\psi,R})$, which is precisely (3.12).

To analyse the boundary conditions satisfied by w_ω, we pick up an arbitrary function $\phi \in C^\infty(D_{\psi,R})$ such that $\phi = 0$ in a neighborhood of the origin, $\nabla\phi \cdot \mathbf{n} - i\omega\phi = 0$ on $\Gamma_{\rm Diss}$ and $\phi = 0$ on $\Gamma_{\rm Dir}$. Because ϕ is regular near the origin, for the same reason as above, we have

$$c_\omega(-\omega^2\phi - \Delta\phi) = (-\omega^2\phi - \Delta\phi, w_\omega) = 0. \tag{3.16}$$

The pair (w_ω, ϕ) satisfies the assumptions of Corollary 1.38 of [32]; this corollary yields

$$\int_{D_{\psi,R}} (w_\omega \Delta\phi - \Delta w_\omega \phi)\,\mathrm{d}x = \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle_{\Gamma_{\rm Diss}} + \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle_{\Gamma_{\rm Dir}} - \langle \phi, \nabla w_\omega \cdot \mathbf{n} \rangle_{\Gamma_{\rm Diss}} - \langle \phi, \nabla w_\omega \cdot \mathbf{n} \rangle_{\Gamma_{\rm Dir}}.$$

Then, by (3.12), we obtain

$$\begin{aligned}
0 &= (-\omega^2\phi - \Delta\phi, w_\omega) \\
&= \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle - \langle \phi, \nabla w_\omega \cdot \mathbf{n} \rangle \\
&= \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle_{\Gamma_{\rm Diss}} + \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle_{\Gamma_{\rm Dir}} - \langle \phi, \nabla w_\omega \cdot \mathbf{n} \rangle_{\Gamma_{\rm Diss}} - \langle \phi, \nabla w_\omega \cdot \mathbf{n} \rangle_{\Gamma_{\rm Dir}} \\
&= i\omega \langle \phi, w_\omega \rangle_{\Gamma_{\rm Diss}} + \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle_{\Gamma_{\rm Dir}} - \langle \phi, \nabla w_\omega \cdot \mathbf{n} \rangle_{\Gamma_{\rm Diss}} \\
&= \langle \nabla\phi \cdot \mathbf{n}, w_\omega \rangle_{\Gamma_{\rm Dir}} - \langle \phi, \nabla w_\omega \cdot \mathbf{n} + i\omega w_\omega \rangle_{\Gamma_{\rm Diss}},
\end{aligned}$$

for all $\phi \in C^\infty(D_{\psi,R})$ such that $\phi = 0$ in a neighborhood of the origin and satisfying the boundary conditions of Helmholtz problem (2.1). Since the traces of $v = \phi|_{\Gamma_{\text{Diss}}}$ and $z = \nabla\phi \cdot \mathbf{n}|_{\Gamma_{\text{Dir}}}$ runs in a dense subset of $\tilde{H}^{\frac{3}{2}}(\Gamma_{\text{Diss}})$ and $\tilde{H}^{\frac{1}{2}}(\Gamma_{\text{Dir}} \setminus \{(0,0)\})$, we deduce that

$$\langle z, w_\omega \rangle_{\Gamma_{\text{Dir}}} = 0, \quad \forall z \in \tilde{H}^{\frac{1}{2}}(\Gamma_{\text{Dir}} \setminus \{(0,0)\}),$$

and

$$\langle v, \nabla w_\omega \cdot \mathbf{n} + i\omega w_\omega \rangle_{\Gamma_{\text{Diss}}} = 0, \quad \forall v \in \tilde{H}^{\frac{3}{2}}(\Gamma_{\text{Diss}}).$$

By duality, we obtain (3.13) and (3.14).

Let $\eta \in C^\infty(\overline{D_{\psi,R}})$ be a cutoff function like in (3.15). Clearly, because $\eta = 1$ near the origin, we have

$$\eta s = u_R + s,$$

with $u_R \in H^2(D_{\psi,R})$. Also, $\Delta(\eta s) \in L^2(D_{\psi,R})$ and ηs satisfies the boundary conditions of problem (2.1). As a result, it is clear that

$$c_\omega \left(-\omega^2(\eta s) - \Delta(\eta s)\right) = 1,$$

and (3.15) follows by definition (3.11) of w_ω.

We now prove the opposite statement that a function v_ω satisfying (3.12)–(3.15) is the function w_ω defined by (3.11). Indeed if $u \in H^1_{\Gamma_{\text{Dir}}}(D_{\psi,R})$ is the unique solution of (2.2), then the splitting (3.6) is equivalent to

$$u = u_R^* + c_\omega(f)\eta s,$$

with $u_R^* \in H^2(D_{\psi,R}) \cap H^1_{\Gamma_{\text{Dir}}}(D_{\psi,R})$. This splitting and condition (3.15) satisfied by v_ω directly yield

$$\begin{aligned} -(f, v_\omega) &= ((\Delta + \omega^2)u, v_\omega) \\ &= ((\Delta + \omega^2)u_R^*, v_\omega) - c_\omega(f). \end{aligned}$$

Then the conclusion follows if we can show that

$$((\Delta + \omega^2)u_R^*, v_\omega) = 0. \tag{3.17}$$

But it is not difficult to show that $H^3(D_{\psi,R}) \cap H^1_{\Gamma_{\text{Dir}}}(D_{\psi,R})$ is dense in $H^2(D_{\psi,R}) \cap H^1_{\Gamma_{\text{Dir}}}(D_{\psi,R})$, hence (3.17) holds if and only if

$$((\Delta + \omega^2)w, v_\omega) = \int_{\Gamma_{\text{Dir}}} (\nabla w \cdot \mathbf{n} v_\omega - w \nabla v_\omega \cdot \mathbf{n}), \forall w \in H^3(D_{\psi,R}) \cap H^1_{\Gamma_{\text{Dir}}}(D_{\psi,R}). \tag{3.18}$$

But for $w \in H^3(D_{\psi,R}) \cap H^1_{\Gamma_{\text{Dir}}}(D_{\psi,R})$, for a cut-off function η as before, we clearly have

$$\eta w \in W^{2,p}(D_{\psi,R}),$$

for any $p > 2$ and furthermore ηw and $\nabla(\eta w)$ is zero at the origin. Since $v_\omega \in L^q(D_{\psi,R})$ and $\Delta v_\omega \in L^q(D_{\psi,R})$ with $1 < q < 2$ such that $1/p + 1/q = 1$, we can apply Corollary 1.38 of [32] and find that

$$((\Delta + \omega^2)(\eta w), v_\omega) = 0.$$

On the other hand, since the angle between Γ_{Diss} and Γ_{Dir} is equal to $\pi/2$, v_ω belongs to H^2 far from the origin and consequently by standard Green's formula, we have

$$((\Delta+\omega^2)((1-\eta)w), v_\omega) = \int_{\Gamma_{\mathrm{Dir}}} (\nabla w\cdot \mathbf{n} v_\omega - w\nabla v_\omega\cdot\mathbf{n}).$$

The sum of these two last identities proves (3.18). □

As we consider a simple geometry, the analytical expression of w_ω is available, as shown in the following theorem.

Theorem 3.4. *We have*

$$w_\omega(\mathbf{x}) = -\alpha\left\{Y_\alpha(\omega r) - \frac{Y_\alpha'(\omega R)+iY_\alpha(\omega R)}{J_\alpha'(\omega R)+iJ_\alpha(\omega R)}J_\alpha(\omega r)\right\}\sin(\alpha\theta).$$

Proof. We are going to show that the function w_ω defined above satisfies conditions (3.12)–(3.15). For the sake of simplicity, let us write

$$s^\star(\mathbf{x}) = Y_\alpha(\omega r)\sin(\alpha\theta),$$

so that

$$w_\omega = -\alpha\left(s^\star - \frac{Y_\alpha'(\omega R)+iY_\alpha(\omega R)}{J_\alpha'(\omega R)+iJ_\alpha(\omega R)}s\right).$$

We observe that by construction, both s and $s^\star$ satisfy the Helmholtz PDE, as a result,

$$-\omega^2 w_\omega - \Delta w_\omega = 0,$$

and (3.12) is satisfied. As w_ω clearly belongs to $L^2(D_{\psi,R})$, we deduce that w_ω belongs to

$$D(\Delta, L^2(D_{\psi,R})) = \{v\in L^2(D_{\psi,R}) : \Delta v\in L^2(D_{\psi,R})\},$$

hence Theorem 1.37 of [32] gives a meaning of its trace on Γ_{Dir} as element of $(\tilde{H}^{\frac12}(\Gamma_{\mathrm{Dir}}\setminus\{(0,0)\}))'$. Furthermore, since $w_\omega\in C^\infty(\mathbb{R}^2\setminus(0,0))$, it is clear that $w_\omega=0$ on $\Gamma_{\mathrm{Dir}}\setminus(0,0)$, and (3.14) holds.

Boundary condition (3.13) is also satisfied by construction. Indeed, if $\mathbf{x}=(R\cos\theta, R\sin\theta)\in\Gamma_{\mathrm{Diss}}$, we have

$$(\nabla s\cdot\mathbf{n} + i\omega s)(\mathbf{x}) = \omega(J_\alpha'(\omega R)+iJ_\alpha(\omega R))\sin(\alpha\theta),$$

and

$$(\nabla s^\star\cdot\mathbf{n} + i\omega s^\star)(\mathbf{x}) = \omega(Y_\alpha'(\omega R)+iY_\alpha(\omega R))\sin(\alpha\theta),$$

so that $\nabla w_\omega\cdot\mathbf{n} + i\omega w_\omega = 0$ on Γ_{Diss}.

Hence it remains to show that (3.15) holds. We have

$$-\frac{1}{\alpha}\left(-\omega^2\eta s - \Delta(\eta s), w_\omega\right) = \left(-\omega^2\eta s - \Delta(\eta s), s^\star\right) + \frac{Y_\alpha'(\omega R)+iY_\alpha(\omega R)}{J_\alpha'(\omega R)+iJ_\alpha(\omega R)}\left(-\omega^2(\eta s) - \Delta(\eta s), s\right).$$

We are going to show that

$$\big(-\omega^2(\eta s) - \Delta(\eta s), s\big) = 0, \tag{3.19}$$

and

$$\big(-\omega^2(\eta s) - \Delta(\eta s), s^\star\big) = -\frac{1}{\alpha}, \tag{3.20}$$

which will conclude the proof. In spirit, the proof relies on simple integration by parts techniques. However, because we manipulate functions with low regularity close to the origin, these integrations by parts have to be done carefully.

The technique is then to subtract the ball $B(0,\epsilon)$ from $D_{\psi,R}$, so that all manipulated functions are C^∞ on $D_{\psi,R,\epsilon} = D_{\psi,R} \setminus B(0,\epsilon)$. Then, integration by parts is allowed on $D_{\psi,R,\epsilon}$ and the desired inner products are recovered by letting $\epsilon \to 0$.

The beginning of the proof of (3.19) and (3.20) is the same. Thus, let us set $\mu = s$ or $s^\star$. Because $s, \mu \in C^\infty(D_{\psi,R,\epsilon})$, and $-\omega^2\mu - \Delta\mu = 0$, double integration by parts yields

$$\int_{D_{\psi,R,\epsilon}} \big(-\omega^2 \eta s - \Delta(\eta s)\big)\,\mu = \int_{\partial D_{\psi,R,\epsilon}} \left(\nabla(\eta s)\cdot \mathbf{n}\mu - \eta s \nabla\mu \cdot \mathbf{n}\right).$$

Since $\eta s = \nabla(\eta s)\cdot\mathbf{n} = 0$ on $\varGamma_{\mathrm{Diss}}$, $s = \mu = 0$ on $\varGamma_{\mathrm{Dir}} \setminus B_\epsilon$, and $\eta = 1$ on $B_{1/2}$, we have

$$\begin{aligned}\int_{D_{\psi,R,\epsilon}} \big(-\omega^2 \eta s - \Delta(\eta s)\big)\,\mu &= \int_{\partial B_\epsilon} \left(\nabla(\eta s)\cdot \mathbf{n}\mu - \eta s \nabla\mu \cdot \mathbf{n}\right)\\ &= \int_{|\mathbf{x}|=\epsilon, 0<\theta<\psi} \left(\nabla s\cdot \mathbf{n}\mu - s \nabla\mu \cdot \mathbf{n}\right).\end{aligned} \tag{3.21}$$

Obviously, when $\mu = s$, the right-hand-side of (3.21) vanishes, so that

$$\int_{D_{\psi,R,\epsilon}} \big(-\omega^2 \eta s - \Delta(\eta s)\big)\, s = 0,$$

and (3.19) follows since

$$\begin{aligned}\big(-\omega^2(\eta s) - \Delta(\eta s), s\big) &= \int_{D_{\psi,R}} \big(-\omega^2 \eta s - \Delta(\eta s)\big)\,\overline{s}\\ &= \int_{D_{\psi,R}} \big(-\omega^2 \eta s - \Delta(\eta s)\big)\, s\\ &= \lim_{\epsilon\to 0}\int_{D_{\psi,R,\epsilon}} \big(-\omega^2 \eta s - \Delta(\eta s)\big)\, s.\end{aligned}$$

On the other hand, to prove (3.20), since

$$\nabla s\cdot\mathbf{n} = \omega J_\alpha'(\epsilon\omega)\sin(\alpha\theta), \quad \nabla s^\star\cdot\mathbf{n} = \omega Y_\alpha'(\epsilon\omega)\sin(\alpha\theta), \text{ on } \partial B_\epsilon,$$

by (3.21) with $\mu = s^\star$, we have

$$\int_{D_{\psi,R,\epsilon}} \left(-\omega^2(\eta s) - \Delta(\eta s)\right) s^\star = \epsilon\omega \left(J'_\alpha(\epsilon\omega)Y_\alpha(\epsilon\omega) - J_\alpha(\epsilon\omega)Y'_\alpha(\epsilon\omega)\right) \int_0^\psi \sin^2(\alpha\theta) \mathrm{d}\theta$$

Direct computations show that

$$\int_0^\psi \sin^2(\alpha\theta) = \frac{\pi}{2\alpha}.$$

Furthermore, recalling (3.4) from Proposition 3.1 we have

$$J'_\alpha(\epsilon\omega)Y_\alpha(\epsilon\omega) - J_\alpha(\epsilon\omega)Y'_\alpha(\epsilon\omega) = -\frac{2}{\epsilon\pi\omega},$$

and we obtain

$$\int_{D_{\psi,R,\epsilon}} \left(-\omega^2 s - \Delta s\right) s^\star = -\frac{1}{\alpha}.$$

Then, (3.20) holds by letting $\epsilon \to 0$. ☐

Corollary 3.5. *We have*

$$|w_\omega(\mathbf{x}) + i\alpha H^{(2)}_\alpha(\omega r)\sin(\alpha\theta)| \le C(\psi, R, \omega_0)\omega^{-3/2}|J_\alpha(\omega r)|\sin(\alpha\theta), \tag{3.22}$$

and

$$\|w_\omega\|_{0,D_{\psi,R}} = C(\psi, R, \omega_0)\left(\omega^{-1/2} + \mathcal{O}\left(\omega^{-3/2}\right)\right). \tag{3.23}$$

Proof. First, recalling (3.3) from Proposition 3.1, we have

$$\frac{Y'_\alpha(\omega R) + i\omega Y_\alpha(\omega R)}{J'_\alpha(\omega R) + i\omega J_\alpha(\omega R)} = i + \mathcal{O}\left((\omega R)^{-3/2}\right),$$

from which (3.22) follows.

Then, because of (3.22), we have

$$\|w_\omega\|^2_{0,D_{\psi,R}} = C(\psi, R, \omega_0)\left(\int_0^R |H^{(2)}_\alpha(\omega r)|^2 r\mathrm{d}r + \mathcal{O}\left(\omega^{-3}\right)\int_0^R |J_\alpha(\omega r)|^2 r\mathrm{d}r\right).$$

Then, from (3.1) and (3.2), we have

$$\|w_\omega\|^2_{0,D_{\psi,R}} = C(\psi, R, \omega_0)\left(\omega^{-1} + \mathcal{O}\left(\omega^{-3}\right)\right),$$

and (3.23) follows. ☐

We are now ready to establish the main result of this section.

Theorem 3.6. *The estimate*

$$|c_\omega(f)| \leq C(\psi, R, \omega_0)\omega^{-1/2}\|f\|_{0,D_{\psi,R}}, \tag{3.24}$$

holds for all $\omega \geq \omega_0$ *and* $f \in L^2(D_{\psi,R})$.

Furthermore, estimate (3.24) *is optimal in the sense that for all* $\omega \geq \omega_0$, *there exists an* $f \in L^2(D_{\psi,R})$ *such that*

$$|c_\omega(f)| \geq C(\psi, R, \omega_0)\omega^{-1/2}\|f\|_{0,D_{\psi,R}}. \tag{3.25}$$

Proof. By definition, for all $\omega \geq \omega_0$ and $f \in L^2(D_{\psi,R})$, we have

$$|c_\omega(f)| = |(f, w_\omega)| \leq \|f\|_{0,D_{\psi,R}}\|w_\omega\|_{0,D_{\psi,R}},$$

and

$$|c_\omega(w_\omega)| = |(w_\omega, w_\omega)| = \|w_\omega\|^2_{0,D_{\psi,R}}.$$

But, Corollary 3.5 shows that

$$C_1(\psi, R, \omega_0)\omega^{-1/2} \leq \|w_\omega\|_{0,D_{\psi,R}} \leq C_2(\psi, R, \omega_0)\omega^{-1/2}, \quad \forall \omega \geq \omega_0,$$

assuming that ω_0 is large enough. As a result, we have (3.24), and (3.25) follows by taking $f = w_\omega$. $\square$

3.3. Behaviour of the regular part

So far, we have isolated the singular part of the solution u and described its behaviour with respect to the frequency. To complete the analysis, we now investigate the regular part $\tilde{u}_R \in H^2(D_{\psi,R})$.

Theorem 3.7. *For all* $\omega \geq \omega_0$ *and* $f \in L^2(D_{\psi,R})$, *if* $u \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R})$ *is solution to* (2.2), *there exist a function* $\tilde{u}_R \in H^2(D_{\psi,R})$ *and a constant* $\tilde{c}_\omega(f) \in \mathbb{C}$ *such that*

$$u = \tilde{u}_R + \tilde{c}_\omega(f)\tilde{s},$$

and it holds that

$$\|\tilde{u}_R\|_{2,D_{\psi,R}} \leq C(\psi, R, \omega_0)\omega\|f\|_{0,D_{\psi,R}}.$$

Proof. We proceed as in Theorem 3.2 and use the lifting $\eta \in H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi,R}) \cap H^2(D_{\psi,R})$ satisfying (3.9). Then, we let $v = u - \eta$ so that

$$\left\{\begin{array}{rcll} -\Delta v &=& h & \text{in } D_{\psi,R}, \\ \nabla v \cdot \mathbf{n} &=& 0 & \text{on } \Gamma_{\mathrm{Diss}}, \\ v &=& 0 & \text{on } \Gamma_{\mathrm{Dir}}, \end{array}\right.$$

with $h = \Delta u - \Delta\eta = f + \omega^2 u - \Delta\eta$.

But, $v = v_R + \tilde{c}_\omega(f)\tilde{s}$ with $v_R \in H^2(D_{\psi,R})$ satisfies

$$\left\{\begin{array}{rcll} -\Delta v_R &=& \tilde{h} & \text{in } D_{\psi,R}, \\ \nabla v_R \cdot \mathbf{n} &=& 0 & \text{on } \Gamma_{\mathrm{Diss}}, \\ v_R &=& 0 & \text{on } \Gamma_{\mathrm{Dir}}, \end{array}\right.$$

with $\tilde{h} = f + \omega^2 u - \Delta\eta - \tilde{c}_\omega(f)\Delta\tilde{s} \in L^2(D_{\psi,R})$. Then we see that v_R is solution to a Laplace problem with mixed boundary condition. Furthermore, since we have $v_R \in H^2(D_{\psi,R})$, we can apply the *a priori* bound derived by Grisvard in Theorem 4.3.1.4 of [22]:

$$\|v_R\|_{2,D_{\psi,R}} \leq C(\psi, R)\left(\|\tilde{h}\|_{0,D_{\psi,R}} + \|v_R\|_{0,D_{\psi,R}}\right).$$

By applying the Poincaré inequality, we further see that

$$\|v_R\|_{2,D_{\psi,R}} \leq C(\psi, R)\left(\|\tilde{h}\|_{0,D_{\psi,R}} + |v_R|_{1,D_{\psi,R}}\right) \leq C(\psi, R)\|\tilde{h}\|_{0,D_{\psi,R}},$$

and it only remains to estimate the $L^2(D_{\psi,R})$-norm of $\tilde{h}$. But, we have

$$\|\tilde{h}\|_{0,D_{\psi,R}} \leq \|f\|_{0,D_{\psi,R}} + \omega^2\|u\|_{0,D_{\psi,R}} + \|\Delta\eta\|_{0,D_{\psi,R}} + |\tilde{c}_\omega(f)|\|\Delta\tilde{s}\|_{0,D_{\psi,R}}.$$

Also, one trivially has

$$\|\Delta\tilde{s}\|_{0,D_{\psi,R}} \leq C(\psi, R),$$

further by (2.3) and (3.10), one deduces that

$$\|u\|_{0,D_{\psi,R}} \leq C(\psi, R, \omega_0)\omega^{-1}\|f\|_{0,D_{\psi,R}}, \quad \|\Delta\eta\|_{0,D_{\psi,R}} \leq C(\psi, R, \omega_0)\omega\|f\|_{0,D_{\psi,R}}.$$

Hence, by (3.24), we finally get

$$\|\tilde{h}\|_{0,D_{\psi,R}} \leq C(\psi, R, \omega_0)\left(1 + \omega^{\alpha-1/2} + \omega\right)\|f\|_{0,D_{\psi,R}},$$

and the result follows. Indeed, since $\alpha < 1$, we have $1 + \omega^{\alpha-1/2} + \omega \leq C(\omega_0)\omega$. □

As a conclusion, we summarize the key features of the presented splitting. First, the regular part $\tilde{u}_R \in H^2(D_{\psi,R})$ has the standard behaviour of the Helmholtz solution in the sense that $|\tilde{u}_R|_{2,D_{\psi,R}} \leq C\omega$. Second, we are able to isolate the singularity of u, that is represented by the function $S = c_\omega(f)\tilde{s}$ which only belongs to $H^{1+\alpha-\epsilon}(D_{\psi,R})$ (for all $\epsilon > 0$). Furthermore, the behaviour of S is controlled as $|S|_{1+\alpha-\epsilon} \leq C\omega^{\alpha-1/2}|\tilde{s}|_{1+\alpha-\epsilon} = C\omega^{\alpha-1/2}$.

The crucial observation is that the norm of the regular part grows faster for increasing frequencies than the one of the singular part. As a result, we can expect the regular part to be "dominant" in some sense for high frequency problems. In Section 5, we show that this observation has important consequences on numerical methods. Roughly speaking, the singular part (and the low convergence rate) is "invisible" until an asymptotic regime (for small mesh size) is reached. A particularly important consequence is that the "pollution effect", which is the main source of numerical error, is not affected by the singularity.

We finally mention that in the context of the integral formulation of problem (2.4), similar results have been obtained for the singular behaviour of the solution close to corners [8, 9, 26]. Specifically, in Theorem 2.3 of [9], a representation of the solution u as a series of Bessel functions is employed, whose first term corresponds to our singularity function s. In fact, it is shown that the singular coefficient behaves as

$$c_\omega \simeq \int_0^\psi u(R, \theta)\sin(\alpha\theta)\mathrm{d}\theta,$$

where u is the solution to the Helmholtz problem and R is a fixed radius, so that

$$|c_\omega| \leq C \sup_{\Omega} |u|,$$

where C is a constant independent of ω. It is further mentioned in [9] that the estimate

$$\sup_{\Omega} |u| \leq C,$$

is probably valid (as numerically observed), but that the sharpest available estimate is

$$\sup_{\Omega} |u| \leq C\omega^{1/2} \ln^{1/2} \omega,$$

so that the authors end up with the estimation $|c_\omega| \leq C\omega^{1/2} \ln^{1/2} \omega$ for the scattering Problem (2.4).

Though the singular decomposition we use is somehow similar to [9], our analysis exhibits the following improvements:

To the best of our understanding, the Bessel series employed to represent the solution in [9] is only valid if u satisfies the homogeneous Helmholtz equation inside Ω (with inhomogeneous boundary conditions), or at least if the load term vanishes in a neighborhood of the corner. While this assumption is satisfactory for scattering problems, our analysis handles arbitrary load terms, that are important in other applications.

The analysis of the authors from [9] is based on a bound of $\sup |u|$, which is not always available or sharp. On the other hand, we require a bound in the $L^2(\Omega)$-norm of the solution. As depicted in Appendix B, sharp $L^2(\Omega)$ estimates are available for a wide range of problems.

For the case of the scattering problem (2.4), we obtain the estimate $|c_\omega| \leq C\omega^{1/2}$, which is sharper than the one obtained in [9]. We note that, however, this result is not fully satisfactory, in the sense that it is expected that the optimal result for (2.4) is $|c_\omega| \leq C$. Nevertheless, our analysis applies to a wider range of load terms, and we provide example for which our bound is sharp. As a result, though our analysis is sharp for the general set of problems we consider, it is not clear whether or not our result is optimal for the scattering problem (2.4).

4. The general case

We now consider the general case of Helmholtz problems that satisfies Assumptions 2.1 and 2.2.

As previously mentioned, singularities can happen at the vertices of Γ_{Dir}, and the solution does not belong to $H^2(\Omega)$. However, we will prove that it admits a splitting

$$u = u_R + \sum_{j=1}^{N} S_j,$$

where $u_R \in H^2(\Omega)$, N is the number of non-convex vertices of Γ_{Dir} and S_j is a singular function associated with the corner $\mathbf{x}_j$ of Γ_{Dir}. Furthermore, we estimate the norms of u_R and S_j in the same fashion than in Section 3.

In contrast to the case of a disc sector which features one singular point, each vertex $\mathbf{x}_j$ of Γ_{Dir} for which the interior angle ψ_j is $> \pi$ (also called a non-convex vertex) is a singular point in the case consider here. From the analysis point of view, the main difference is that here, one must localize the functions representing the singularities (see the definition of $\tilde{s}_j$ below (4.1)). In addition, we now consider the general stability estimate in $\mathcal{O}(\omega^\sigma)$.

4.1. Notations

In the following, we denote by $\chi \in C^\infty(\mathbb{R})$ a cutoff function so that $\chi(\rho) = 1$ if $\rho < \ell/3$ and $\chi(\rho) = 0$ if $\rho > 2\ell/3$.

To each corner $\mathbf{x}_j$, we associate the singular function $\tilde{s}_j$ defined by

$$\tilde{s}_j(\mathbf{x}) = \chi_j(\mathbf{x}) r_j^{\alpha_j} \sin(\alpha_j \theta_j), \tag{4.1}$$

where $\chi_j(\mathbf{x}) = \chi(r_j(\mathbf{x}))$, $r_j(\mathbf{x})$ is the distance from $\mathbf{x}$ to $\mathbf{x}_j$ and $\alpha_j = \pi/\psi_j$.

We further write $D_j = \Omega \cap B(\mathbf{x}_j, \ell)$. One sees that D_j is a disc sector centered at $\mathbf{x}_j$, of opening ψ_j and radius ℓ. As a result, D_j is obtained from $D(\psi_j, \ell)$ by rotation and translation. Hence, we are able to apply results from Section 2 when localizing the analysis in D_j.

4.2. Splitting of the solution

By a localization argument, we give a splitting of the solution into a regular part in $H^2(\Omega)$ and N singular functions, associated with each non-convex corner of Γ_{Dir}.

Theorem 4.1. *For all $\omega \geq \omega_0$ and $f \in L^2(\Omega)$, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ is solution to (2.2), there exist a function $\tilde{u}_R \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \cap H^2(\Omega)$ and constants $c^j_\omega(f) \in \mathbb{C}$ such that*

$$u = \tilde{u}_R + \sum_{j=1}^N \tilde{c}^j_\omega(f) \tilde{s}_j,$$

and it holds that

$$|\tilde{c}^j_\omega(f)| \leq C(\Omega, \omega_0, \mathscr{V}) \omega^{\sigma + \alpha_j - 1/2} \|f\|_{0,\Omega}, \tag{4.2}$$

for all $j = 1, \ldots, N$, while

$$|\tilde{u}_R|_{2,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}) \omega^{\sigma+1} \|f\|_{0,\Omega}. \tag{4.3}$$

Proof. The proof heavily relies on a localization argument and the results of the previous section. Indeed for all $j = 1, \ldots, N$, we set

$$u_j = \chi_j u.$$

Up to an isometric change of coordinates, we see that u_j belongs to $H^1_{\Gamma_{\mathrm{Dir}}}(D_{\psi_j,L})$, and is the variational solution of the problem (2.2) in $D_{\psi_j,L}$ with data $f_j = \chi_j f - 2\nabla\chi_j \cdot \nabla u - u\Delta\chi_j$, namely

$$\left\{ \begin{array}{rcll} -\omega_j^2 u_j - \Delta u_j & = & f_j & \text{in } D_{\psi_j,\ell}, \\ \nabla u_j \cdot \mathbf{n} - i\omega_j u_j & = & 0 & \text{on } S_{\psi_j,\ell}, \\ u_j & = & 0 & \text{on } I_{0,\ell} \cup I_{\psi_j,\ell}, \end{array} \right. \tag{4.4}$$

with $\omega_j = \omega/\mathscr{V}_j$, where $\mathscr{V}_j$ is the constant value of $\mathscr{V}$ over $B(\mathbf{x}_j, \ell)$. First let us notice that the estimate (2.3) yields

$$\|f_j\|_{0,D_j} \leq C(\Omega, \omega_0) \omega^\sigma \|f\|_{0,\Omega}. \tag{4.5}$$

Hence applying Theorem 3.2 to problem (4.4) one gets the splitting

$$u_j = \tilde{u}_{R,j} + \tilde{c}^j_{\omega_j}(f_j)\tilde{s}_j, \tag{4.6}$$

with $\tilde{u}_{R,j} \in H^2(\Omega_j)$ and $\tilde{c}^j_\omega(f_j) \in \mathbb{C}$. Furthermore with the help of Theorems 3.6 and 3.7 (and the estimate (4.5)), one has

$$|\tilde{c}^j_{\omega_j}(f_j)| \le C(\psi_j, \ell, \omega_0)\omega_j^{\alpha_j - 1/2}\|f_j\|_{0,\Omega_j} \le C(\Omega, \omega_0, \mathscr{V})\omega^{\sigma+\alpha_j-1/2}\|f\|_{0,\Omega}, \tag{4.7}$$

and

$$\|\tilde{u}_{R,j}\|_{2,\Omega_j} \le C(\psi_j, \ell, \omega_0)\omega\|f_j\|_{0,\Omega_j} \le C(\Omega, \omega_0)\omega^{\sigma+1}\|f\|_{0,\Omega}. \tag{4.8}$$

Finally setting $\chi = 1 - \sum_{j=1}^N \chi_j$, we define $U = \chi u$. It is clear that we have

$$\|U\|_{0,\Omega} \le \|u\|_{0,\Omega} \le \omega^{\sigma-1}\|f\|_{0,\Omega}, \tag{4.9}$$

and

$$|U|_{1,\Omega} \le |\chi|_{1,\infty}\|u\|_{0,\Omega} + |u|_{1,\Omega} \le C(\Omega, \omega_0, \mathscr{V})\omega^\sigma\|f\|_{0,\Omega}. \tag{4.10}$$

We can also look at U as the solution of (2.2) in $\mathcal{O}$ with data $F = \chi f - 2\nabla\chi \cdot \nabla u - u\Delta\chi$, where $\mathcal{O}$ is a smooth domain corresponding to Ω where we have rounded the non-convex corners $\mathbf{x}_j$ of Ω (without loss of generality, we can assume that the boundary of $\mathcal{O}$ has two connected components Γ^s_{Dir} and Γ_{Diss} and that $\mathcal{O} \subset \Omega$), namely

$$\left\{ \begin{array}{rcll} -\dfrac{\omega^2}{\mathscr{V}^2}U - \Delta U &=& F & \text{in } \mathcal{O}, \\ \nabla U \cdot \mathbf{n} - \dfrac{i\omega}{\mathscr{V}}U &=& 0 & \text{on } \Gamma_{\mathrm{Diss}} \\ U &=& 0 & \text{on } \Gamma^s_{\mathrm{Dir}}. \end{array} \right. \tag{4.11}$$

We note that since $U = 0$ in a neighborhood of each $\mathbf{x}_j$, we have

$$\|U\|_{0,\mathcal{O}} = \|U\|_{0,\Omega}, \quad \|U\|_{1,\mathcal{O}} = \|U\|_{1,\Omega},$$

and thus

$$\|F\|_{0,\mathcal{O}} \le C(\Omega, \omega_0)\omega^\sigma\|f\|_{0,\Omega}$$

from estimates (4.9) and (4.10).

By standard regularity results in $\mathcal{O}$, one has $U \in H^2(\mathcal{O})$ with

$$\|U\|_{2,\mathcal{O}} \le C(\mathcal{O})\left(\|U\|_{1,\mathcal{O}} + \|\Delta U\|_{0,\mathcal{O}} + \omega\|U\|_{\frac{1}{2},\Gamma_{\mathrm{Diss}}}\right).$$

Hence using a trace theorem, we find that

$$\|U\|_{2,\mathcal{O}} \le C(\mathcal{O}, \mathscr{V})\left((1+\omega)\|U\|_{1,\Omega} + \omega^2\|U\|_{0,\mathcal{O}} + \|F\|_{0,\mathcal{O}}\right).$$

By the estimates (4.9) and (4.10), we conclude that

$$\|U\|_{2,\mathcal{O}} \leq C(\Omega, \mathscr{V})\omega^{\sigma+1}\|f\|_{0,\Omega}. \tag{4.12}$$

Since $u = U + \sum_{j=1}^{N} u_j$, the conclusion follows from splitting (4.6), the regularity $U \in H^2(\mathcal{O})$ and estimates (4.7)–(4.12). $\square$

For the sake of completeness, we provide an additional result for the case of velocity parameters $\mathscr{V}$ that are not constant close to the re-entrant corners of Γ_{Dir}. We state this result separately, since we do not believe it is optimal: a factor $\omega^{1/2}$ is added in the estimate of the singular coefficient.

Theorem 4.2. *We assume that Γ_{Dir} and Γ_{Diss} satisfy Assumption 2.2 but we allow $\mathscr{V}$ to be a general $C^1(\overline{\Omega})$ function such that $0 < \mathscr{V}_{\min} \leq \mathscr{V} \leq \mathscr{V}_{\max} < +\infty$ for two fixed values $\mathscr{V}_{\min}$ and $\mathscr{V}_{\max}$. Then, for all $\omega \geq \omega_0 > 1/\ell^2$ and $f \in L^2(\Omega)$, if $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ is solution to (2.2), there exist a function $\tilde{u}_R \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \cap H^2(\Omega)$ and constants $c^j_\omega(f) \in \mathbb{C}$ such that*

$$u = \tilde{u}_R + \sum_{j=1}^{N} \tilde{c}^j_\omega(f)\tilde{s}_j,$$

and it holds that

$$|\tilde{c}^j_\omega(f)| \leq C(\Omega, \omega_0, \mathscr{V})\omega^{\sigma+\alpha_j}\|f\|_{0,\Omega}, \tag{4.13}$$

for all $j = 1, \ldots, N$, while

$$|\tilde{u}_R|_{2,\Omega} \leq C(\Omega, \omega_0, \mathscr{V})\omega^{\sigma+1}\|f\|_{0,\Omega}. \tag{4.14}$$

Proof. Since the proof is similar to the one of the previous theorem, we only highlight the differences.

For each corner $\mathbf{x}_j$, we define a cutoff function ρ_j such that $\rho_j(\mathbf{x}) = 1$ in a neighborhood of $\mathbf{x}_j$ and $\rho_j(\mathbf{x}) = 0$ if $|\mathbf{x} - \mathbf{x}_j| \geq \omega^{-1/2}$ (remark that $\omega^{-1/2} \leq \omega_0^{-1/2} \leq \ell$). We note that we can construct ρ_j in such a way that

$$|\rho_j|_{0,\infty,\Omega} \leq 1, \quad |\rho_j|_{1,\infty,\Omega} \leq C(\psi_j)\omega^{1/2}, \quad |\rho_j|_{2,\infty,\Omega} \leq C(\psi_j)\omega.$$

Then for $j = 1, \ldots, N$, we set $u_j = \rho_j u$, $\mathscr{V}_j = \mathscr{V}(\mathbf{x}_j)$, and $\omega_j = \omega/\mathscr{V}_j$. As in the previous theorem, we see that (up to an isometry) u_j is solution to

$$\left\{ \begin{array}{rcll} -\omega_j^2 u_j - \Delta u_j &=& f_j & \text{in } D_{\psi_j,\ell}, \\ \nabla u_j \cdot \mathbf{n} - i\omega_j u_j &=& 0 & \text{on } S_{\psi_j,\ell}, \\ u_j &=& 0 & \text{on } I_{0,\ell} \cup I_{\psi_j,\ell}, \end{array} \right.$$

where

$$f_j = \rho_j f - 2\nabla\rho_j \cdot \nabla u - u\Delta\rho_j - \omega^2\left(\mathscr{V}(\mathbf{x}_j)^{-2} - \mathscr{V}^{-2}\right)\rho_j u.$$

Since $\mathscr{V}^{-2}$ belongs to $C^1(\overline{\Omega})$, we can apply Taylor expansion, and find

$$\left|\mathscr{V}(\mathbf{x}_j)^{-2} - \mathscr{V}^{-2}(\mathbf{x})\right| \leq |\mathbf{x} - \mathbf{x}_j||\mathscr{V}^{-2}|_{1,\infty,\Omega} \leq 2\mathscr{V}_{\min}^{-3}|\mathbf{x} - \mathbf{x}_j||\mathscr{V}|_{1,\infty,\Omega} = C(\mathscr{V})|\mathbf{x} - \mathbf{x}_j|.$$

Since in addition $\rho_j(\mathbf{x}) = 0$ if $|\mathbf{x} - \mathbf{x}_j| \geq \omega^{-1/2}$, and $|\rho_j|_{0,\infty,\Omega} \leq 1$, we see that

$$\left| \left(\mathscr{V}(\mathbf{x}_j)^{-2} - \mathscr{V}^{-2} \right) \rho_j \right| \leq C(\mathscr{V}) \omega^{-1/2}.$$

Then, we recall that by Assumption 2.1, we have

$$\|u\|_{0,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}) \omega^{\sigma-1} \|f\|_{0,\Omega}, \quad |u|_{1,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}) \omega^{\sigma} \|f\|_{0,\Omega},$$

so that

$$\|f_j\|_{0,\Omega} \leq \omega^{\sigma+1/2} \|f\|_{0,\Omega},$$

and (4.13) follows from Theorem 3.2.

We use the same arguments than in the previous theorem to show (4.14). $\square$

5. Frequency explicit stability of finite element discretizations

5.1. The finite element space

In this section, we investigate the discretization of problem (2.2) by linear finite elements. For the sake of simplicity, we assume that $\Gamma_{\rm Dir}$ and $\Gamma_{\rm Diss}$ are polygonal. We consider meshes $\mathcal{T}_h$ of Ω made of triangles K satisfying

$$\operatorname{diam}(K) \leq h, \quad \operatorname{diam}(K) \leq \gamma \rho(K),$$

where γ is a constant independent of h, and

$$\operatorname{diam}(K) = \sup_{x,y \in K} |x - y|, \quad \rho(K) = \sup \{ r > 0 \mid \exists x \in K;\ B(x,r) \subset K \}.$$

The solution $u \in H^1_{\Gamma_{\rm Dir}}(\Omega)$ to problem (2.2) is then approximated by a function $u_h \in V_h$ satisfying

$$B(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{5.1}$$

where

$$V_h = \left\{ v_h \in H^1_{\Gamma_{\rm Dir}}(\Omega) \mid v_h|_K \in \mathcal{P}_1(K);\ \forall K \in \mathcal{T}_h \right\}$$

is the space of Lagrange linear elements build on $\mathcal{T}_h$. For more detail on the construction of V_h and its properties, we refer the reader to [12].

In this section, we will consider meshes such that

$$\omega h \leq 1. \tag{5.2}$$

This assumption is natural and means that the number of elements per wavelength is bounded from below. As we shall see, more restrictive conditions on h must be imposed to ensure that the finite element error remains bounded independently of ω, so that we can assume (5.2) without loss of generality.

In order to simplify notations, we introduce the $\omega-$dependent norm

$$|||v|||^2 = \omega^2 \|v\|^2_{0,\Omega} + |v|^2_{1,\Omega}, \quad \forall v \in H^1_{\Gamma_{\rm Dir}}(\Omega).$$

The $|||\cdot|||$ norm is equivalent to the standard $H^1(\Omega)$ norm (with a constant obviously depending on ω). This norm turns out to be the "natural" one to analyze the problem, since in view of stability estimate (2.3), the L^2 and H^1 terms of $|||u|||$ are "balanced" when u is a solution of the Helmholtz problem.

In the following, we denote by $\mathcal{I}_h$ the "quasi-interpolation" operator of Scott & Zhang [38]. We have $\mathcal{I}_h : H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \to V_h$, and it holds that (see Thm. 4.1 of [38])

$$|v - \mathcal{I}_h v|_{l,\Omega} \leq C(\Omega,\gamma) h^{1-l} |v|_{1,\Omega} \quad (l = 0, 1). \tag{5.3}$$

Furthermore, if $v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega) \cap H^2(\Omega)$, it holds that

$$|v - \mathcal{I}_h v|_{l,\Omega} \leq C(\Omega,\gamma) h^{2-l} |v|_{2,\Omega} \quad (l = 0, 1). \tag{5.4}$$

5.2. Preliminary discussion

It is well known that the main source of error in numerical discretizations of high frequency problems is numerical dispersion. This is known as the "pollution effect": unless the mesh is heavily refined, the finite-element solution is not quasi-optimal, it is "polluted".

To simplify the discussion, let us denote by

$$\eta = \frac{|||u - \mathcal{I}_h u|||}{\|f\|_{0,\Omega}},$$

the best approximation error. In a smooth domain, it is known [30, 31] that the parameter η is bounded as

$$\eta \lesssim \omega^{\sigma+1} h,$$

for linear Lagrange elements. Our main achievement is to establish that

$$\eta \lesssim \omega^{\sigma - 1/2} (\omega h)^{\alpha} + \omega^{\sigma}(\omega h), \tag{5.5}$$

in a domain presenting re-entrant corners.

For 1D problems, when $\sigma = 0$, the behaviour of the finite element solution and the pollution effect have been precisely analysed, see for instance [27, 28]. It is shown that if there are sufficiently many discretization points per wavelength (*i.e.* $\eta = \omega h$ is small enough), then

$$|||u - u_h||| \lesssim \left(\eta + \omega\eta^2\right) \|f\|_{0,\Omega} \simeq \left(\omega h + \omega^3 h^2\right) \|f\|_{0,\Omega}. \tag{5.6}$$

The pollution effect is clearly visible in (5.6), where the pollution term $\omega\eta^2$ is added to the best approximation error η. For large ω, the pollution term $\omega^3 h^2$ is dominant unless h is sufficiently small. This is called the "pre-asymptotic range", where the pollution error is the largest. On the other hand, the "asymptotic range" is achieved when $\omega\eta \leq C_0$ is small enough. Then, the finite element solution is quasi-optimal since $\omega\eta^2 \leq C_0\eta$.

If we insert bound (5.5) for non-convex domains into (5.6), we obtain

$$\begin{aligned} |||u - u_h||| &\lesssim \left(\eta + \omega\eta^2\right) \|f\|_{0,\Omega} \\ &\lesssim \left(\omega^{-1/2}\omega^{\alpha}h^{\alpha} + \omega h + \omega^{2\alpha}h^{2\alpha} + \omega^3 h^2\right) \|f\|_{0,\Omega}. \\ &\lesssim \left(\omega^{-1/2}\omega^{\alpha}h^{\alpha} + \omega h + \omega^3 h^2\right) \|f\|_{0,\Omega}. \end{aligned}$$

Hence, we see that the presence of singularities is reflected by the term $\omega^{-1/2}\omega^{\alpha}h^{\alpha}$. It is crucial to observe that this term is only significant in an asymptotic range where h is small. More precisely, the term $\omega^{-1/2}\omega^{\alpha}h^{\alpha}$ is dominant only when $\omega^{-1/2}\omega^\alpha h^\alpha \leq \omega h$, which corresponds to $h \leq \omega^{\frac{\alpha-3/2}{1-\alpha}} \leq \omega^{-2}$. Also, we see that the pollution term $\omega^3 h^2$ is not affected by the presence of singularities. We thus conclude that the dispersive behaviour of the finite element scheme remains unchanged in the presence of singularities. Furthermore, unless a highly accurate solution is required, the problem can be solved without using special techniques to "resolve" the singularities.

Unfortunately, the authors are not aware of a proof of (5.6) for general meshes in 2D. Nevertheless, in the following, we are able to give two interesting results.

First, we give asymptotic error estimates that are based on the so-called Schatz argument [37]. The methodology can be found, for instance, in [17] or [29]. Classically, if $\omega\eta$ is small enough, the finite element solution is optimal and it holds that $|||u - u_h||| \leq \eta \|f\|_{0,\Omega}$. We show that the condition that $\omega\eta$ is small is satisfied as soon as $\omega^{\sigma+2}h$ is small. Hence, the presence of singularities does not change the asymptotic range.

Second, though we are not able to prove a general pre-asymptotic error estimate like in [27], we can derive a weaker version thanks to a method recently introduced in [21]. The method relies on the introduction of a special elliptic projection. When applied in a smooth domain, it implies that if $\omega h \eta$ is small enough, then (5.6) holds. Hence, it provides the same error estimate, but the condition $\omega h\eta \leq C$ is imposed on the mesh step. When linear elements are consider, this condition is equivalent to $\omega^3 h^2 \leq C$, so that we obtain an optimal bound on the error. For higher p however, this condition is not optimal. Hereafter, we adapt this method to the case of domains with singular points. Unfortunately, the elliptic projection used in [21] is affected by the singularities. As a result, we can show that (5.6) holds, but only if $\omega h^\alpha \eta$ is small enough (which is more restrictive than the original condition $\omega h\eta \leq C$ for regular domains).

5.3. Interpolation of singularities

Before deriving our main results, we present an interpolation result for the singularity functions. The analysis is subtle as $s_j \in H^{1+s}(\Omega)$ holds for $s < \alpha$, but not in the limiting case $s = \alpha$. As a result, direct approximation results in Sobolev spaces do not provide interpolation error estimates in $\mathcal{O}(h^\alpha)$. We will use a regularity result from [6] involving Besov spaces giving the desired estimates.

Lemma 5.1. *For $l = 0$ or 1, we have*

$$|\tilde{s}_j - \mathcal{I}_h \tilde{s}_j|_{l,\Omega} \leq C(\Omega,\gamma) h^{1-l+\alpha_j}. \tag{5.7}$$

Proof. If we write $s_j(\mathbf{x}) = r_j^{\alpha_j}\sin(\alpha_j\theta_j)$ and $\tilde{s}_j = \chi_j s_j$, we see that $\Delta s_j = 0$. Since $\operatorname{supp} s_j \subset D_j$, we observe that $\tilde{s}_j$ is solution to

$$\begin{cases} -\Delta \tilde{s}_j &= g_j \quad \text{in } \Omega \\ \quad \tilde{s}_j &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.8}$$

where

$$g_j = -\nabla\chi \cdot \nabla s_j - \Delta\chi s_j.$$

We observe that $\nabla\chi$ and $\Delta\chi$ are supported in $B(\mathbf{x}_j, 2L/3) \setminus B(\mathbf{x}_j, L/3)$. Since s_j is smooth on that set, we clearly have $g_j \in L^2(\Omega)$ and $\|g_j\|_{0,D_j} \leq C(\Omega)$.

The main ingredient of the proof is then Theorem 4.1 of [6], which gives a regularity result for the solution of Laplace problem (5.8) in the Besov space

$$B^{1+\alpha} = [H^2(\Omega) \cap H_0^1(\Omega), H_0^1(\Omega)]_{1-\alpha,\infty}$$

obtained by real interpolation. In particular, we can state that

$$\|s_j\|_{B^{1+\alpha}} \leq C(\Omega)\|g_j\|_{0,D_j}.$$

Since $H_0^1(\Omega) \subset H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, (5.3) and (5.4) hold for all $v \in H_0^1(\Omega)$. Hence the linear operator $T = (\mathrm{Id} - \mathcal{I}_h)$ is linear and bounded from $H_0^1(\Omega) \to H^l(\Omega)$ with norm $M_0 = C(\Omega,\gamma)h^{1-l}$ and from $H^2(\Omega) \cap H_0^1(\Omega) \to H^l(\Omega)$ with norm $M_1 = C(\Omega,\gamma)h^{2-l}$. Since $B^{1+\alpha}$ is an exact interpolation space (see, for instance [2]), we have that T is bounded from $B^{1+\alpha}$ to $H^l(\Omega)$ with norm $M_\alpha = M_0^\alpha M_1^{1-\alpha} = C(\Omega,\gamma)h^{1-l+\alpha_j}$. It follows that

$$\|v - \mathcal{I}_h v\|_{l,\Omega} \leq C(\Omega,\gamma)h^{1-l+\alpha_j}\|v\|_{B^{1+\alpha}},$$

for all $v \in B^{1+\alpha}$, and the result follows by taking $v = \tilde{s}_j$. □

5.4. Asymptotic error estimate

We start by deriving an asymptotic error estimate. The first step consists in estimating the best approximation error. The right-hand side of estimate (5.9) contains the quantity η introduced above and we conclude that here

$$\eta \leq C(\Omega,\omega_0,\mathscr{V},\gamma)\left(\omega^{\sigma-1/2}(\omega h)^\alpha + \omega^\sigma(\omega h)\right).$$

Lemma 5.2. *For $\phi \in L^2(\Omega)$, define $u_\phi \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ as the solution to*

$$B(u_\phi, v) = (\phi, v), \quad \forall v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega).$$

Then we have

$$|||u_\phi - \mathcal{I}_h u_\phi||| \leq C(\Omega,\omega_0,\mathscr{V},\gamma)\left(\omega^{\sigma-1/2}(\omega h)^\alpha + \omega^\sigma(\omega h)\right)\|\phi\|_{0,\Omega}. \tag{5.9}$$

Furthermore, estimate (5.9) also holds for the function $u_\phi^\star$ defined as the unique element of $H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ solution to

$$B(v, u_\phi^\star) = (v, \phi), \quad \forall v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega).$$

Proof. We recall that we have the decomposition

$$u_\phi = \tilde{u}_R + \sum_{j=1}^N \tilde{c}_\omega^j(\phi)\tilde{s}_j,$$

where $\tilde{s}_j = \chi(r_j)r^{\alpha_j}\sin(\alpha_j\theta_j)$, $\tilde{u}_R \in H^2(\Omega) \cap H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. Hence, since

$$u_\phi - \mathcal{I}_h u_\phi = (\tilde{u}_R - \mathcal{I}_h\tilde{u}_R) + \sum_{j=1}^N \tilde{c}_\omega^j(\phi)\left(\tilde{s}_j - \mathcal{I}_h\tilde{s}_j\right),$$

we have

$$\omega^{1-l}|u_\phi - \mathcal{I}_h u_\phi|_{l,\Omega} \leq \omega^{1-l}\left(|\tilde{u}_R - \mathcal{I}_h\tilde{u}_R|_{l,\Omega} + \sum_{j=1}^N |\tilde{c}_\omega^j(\phi)||\tilde{s}_j - \mathcal{I}_h\tilde{s}_j|_{l,\Omega}\right),$$

for $l = 0, 1$. Recalling that $\omega h \leq 1$, we obtain from (5.4) and (4.3) that

$$\omega^{1-l}|\tilde{u}_R - \mathcal{I}_h\tilde{u}_R|_{l,\Omega} \leq C(\Omega,\gamma)\omega^{\sigma+1-l}h^{2-l}|\tilde{u}_R|_{2,\Omega}$$

$$\begin{aligned}&\leq C(\varOmega,\omega_0,\gamma)\omega^{\sigma+2-l}h^{2-l}\|\phi\|_{0,\varOmega}\\&\leq C(\varOmega,\omega_0,\gamma)\omega^{\sigma+1}h\|\phi\|_{0,\varOmega}.\end{aligned}\tag{5.10}$$

On the other hand, recalling (4.2) and (5.7), we have

$$\begin{aligned}\omega^{1-l}\sum_{j=1}^{N}|\tilde{c}^j_\omega(\phi)||\tilde{s}_j-\mathcal{I}_h\tilde{s}_j|_{l,\varOmega}&\leq C(\varOmega,\omega_0,\gamma)\omega^{1-l}\sum_{j=1}^{N}\omega^{\sigma+\alpha_j-1/2}h^{1-l+\alpha_j}\|\phi\|_{0,\varOmega}\\&\leq C(\varOmega,\omega_0,\gamma)\omega^{\sigma-1/2}\omega^{1-l}h^{1-l}\sum_{j=1}^{N}\omega^{\alpha_j}h^{\alpha_j}\|\phi\|_{0,\varOmega}.\end{aligned}$$

Since $\omega h\leq 1$, we have $\omega^{1-l}h^{1-l}\leq 1$ and $\omega^{\alpha_j}h^{\alpha_j}\leq\omega^\alpha h^\alpha$ for $j=1,\ldots,N$.

$$\begin{aligned}\omega^{1-l}\sum_{j=1}^{N}|\tilde{c}^j_\omega(\phi)||\tilde{s}_j-\mathcal{I}_h\tilde{s}_j|_{l,\varOmega}&\leq C(\varOmega,\omega_0,\gamma)\omega^{\sigma-1/2}\sum_{j=1}^{N}\omega^{\alpha}h^{\alpha}\|\phi\|_{0,\varOmega}\\&\leq C(\varOmega,\omega_0,\gamma)N\omega^{\sigma-1/2}\omega^{\alpha}h^{\alpha}\|\phi\|_{0,\varOmega}\\&\leq C(\varOmega,\omega_0,\gamma)\omega^{\sigma-1/2}\omega^{\alpha}h^{\alpha}\|\phi\|_{0,\varOmega}.\end{aligned}\tag{5.11}$$

Then (5.9) follows from (5.10) and (5.11). □

Thanks to the estimates of the best approximation error derived in Lemma 5.2, we obtain an asymptotic error estimate by applying the Schatz argument [17, 29, 37]. A crucial observation is that the asymptotic range is defined by the condition that $\omega^{\sigma+2}h$ is small enough. This condition is the same than in the case of a smooth domain. Then, in error estimate (5.13), the term $\omega^{\sigma-1/2}\omega^\alpha h^\alpha$ is added in comparison to the case of a smooth domain. As we discussed above, for high frequencies, this term is less important than the usual term ωh unless h is very small.

We also compare our results to the literature. In [4], a "plane wave" numerical method is analyzed, for domains satisfying Assumption 2.1 with $\sigma=0$. The authors consider uniform meshes made of squares, so that re-entrant corners of angle $3\pi/2$ ($\alpha=2/3$) are allowed. The $H^{5/3}$ norm of the continuous solution is estimated without considering the singularities explicitly. As a result, the obtained asymptotic error estimate only holds under the condition that $\omega^{5/2}h$ is small enough. In contrast, our asymptotic error estimate holds under the less restrictive condition that ω^2h is small. We also mention [20], where the authors obtain an asymptotic error estimate under the condition that ω^2h is small, and that the mesh is geometrically refined close to singular corners.

Theorem 5.3. *Assume that $\omega^{\sigma+2}h$ is small enough, then problem (5.1) admits a unique solution $u_h\in V_h$. Furthermore, the finite-element solution u_h is quasi-optimal:*

$$|||u-u_h|||\leq C(\varOmega,\omega_0,\mathscr{V},\gamma)|||u-\mathcal{I}_hu|||,\tag{5.12}$$

and the error estimate

$$|||u-u_h|||\leq C(\varOmega,\omega_0,\mathscr{V},\gamma)\omega^\sigma\left(\omega^{-1/2}\omega^\alpha h^\alpha+\omega h\right)\|f\|_{0,\varOmega}\tag{5.13}$$

holds.

Proof. The proof uses the standard Schatz argument. Let $u_h \in V_h$ be any solution to (5.1). We introduce $\xi \in H^1_{\Gamma_{\rm Dir}}(\Omega)$ solution to $B(v,\xi) = (v, u - u_h)$, for all $v \in H^1_{\Gamma_{\rm Dir}}(\Omega)$, so that

$$\|u - u_h\|_{0,\Omega}^2 = B(u - u_h, \xi) = B(u - u_h, \xi - \mathcal{I}_h \xi).$$

By definition of ξ, recalling (5.9), we have

$$|||\xi - \mathcal{I}_h\xi||| \leq C(\Omega, \omega_0, \mathscr{V}, \gamma) \left(\omega^{\sigma+1} h + \omega^{\sigma-1/2}\omega^\alpha h^\alpha\right) \|u - u_h\|_{0,\Omega},$$

and therefore

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= B(u - u_h, \xi - \mathcal{I}_h\xi) \\ &\leq C(\Omega)|||u - u_h||| \cdot |||\xi - \mathcal{I}_h\xi||| \\ &\leq C(\Omega, \omega_0, \mathscr{V}, \gamma) \left(\omega^{\sigma+1} h + \omega^{\sigma-1/2}\omega^\alpha h^\alpha\right) |||u - u_h||| \cdot \|u - u_h\|_{0,\Omega}. \end{aligned}$$

It follows that

$$\|u - u_h\|_{0,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}, \gamma) \left(\omega^{\sigma+1} h + \omega^{\sigma-1/2}\omega^\alpha h^\alpha\right) |||u - u_h|||. \tag{5.14}$$

Now, we write that

$$\begin{aligned} |||u - u_h|||^2 &\leq \operatorname{Re} B(u - u_h, u - u_h) + \frac{2\omega^2}{\mathscr{V}_{\min}^2} \|u - u_h\|_{0,\Omega}^2 \\ &= \operatorname{Re} B(u - u_h, u - \mathcal{I}_h u_h) + \frac{2\omega^2}{\mathscr{V}_{\min}^2} \|u - u_h\|_{0,\Omega}^2 \\ &\leq C(\Omega, \omega_0, \mathscr{V}, \gamma) \Big\{ |||u - u_h||| \cdot |||u - \mathcal{I}_h u||| \\ &+ \omega^2 \left(\omega^{\sigma+1} h + \omega^{\sigma-1/2}\omega^\alpha h^\alpha\right)^2 |||u - u_h|||^2 \Big\}, \end{aligned}$$

and simplifying by $|||u - u_h|||$, we obtain

$$\left\{1 - C(\Omega, \omega_0, \mathscr{V}, \gamma) \left(\omega^{\sigma+2} h + \omega^{\sigma+1/2}\omega^\alpha h^\alpha\right)^2\right\} |||u - u_h||| \leq C(\Omega, \omega_0, \mathscr{V}, \gamma)|||u - \mathcal{I}_h u|||. \tag{5.15}$$

Recalling that $\omega h \leq 1$, since $\alpha \geq 1/2$, we have $\omega^{\sigma+1/2}\omega^\alpha h^\alpha \leq \omega^{\sigma+2} h$. Hence, assuming that $\omega^{\sigma+2} h$ is small enough, we have

$$C(\Omega, \omega_0, \mathscr{V}, \gamma) \left(\omega^{\sigma+2} h + \omega^{\sigma+1/2}\omega^\alpha h^\alpha\right)^2 \leq \frac{1}{2},$$

and (5.12) follows from (5.15). Finally, (5.13) follows from (5.12) and (5.9).

The uniqueness of u_h is a direct consequence of (5.13), and existence follows, since u_h is defined as the solution of a finite-dimensional square linear system. $\square$

5.5. Pre-asymptotic error estimate

In the following, we derive a pre-asymptotic error estimate using the elliptic projection introduced in [21]. The projection $\mathcal{P}_h u$ is introduced in Lemma 5.4 where we derived error estimates for $u - \mathcal{P}_h u$. We emphasize that because of the singularities, L^2 error estimate (5.16) is different from the case of a smooth domain. This is the reason why our pre-asymptotic error estimate is only valid under the condition that $\omega^{\sigma+3}h^{1+\alpha}$ is small enough.

Lemma 5.4. *For $u, v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, we define*

$$a(u, v) = -i\omega\langle \mathscr{V}^{-1}u, v\rangle_{\Gamma_{\mathrm{Diss}}} + (\nabla u, \nabla v),$$

as well as

$$|u|_\star = \sqrt{|a(u, u)|},$$

so that $B(u, v) = -\omega^2(\mathscr{V}^{-2}u, v) + a(u, v)$. Then, we have

$$|a(u, v)| \leq |u|_\star |v|_\star, \forall u, v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega).$$

Furthermore, for each $u \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, we define its projection $\mathcal{P}_h u \in V_h$ as the unique solution to

$$a(v_h, \mathcal{P}_h u) = a(v_h, u), \quad \forall v_h \in V_h.$$

If $u_\phi \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ solves $B(u_\phi, v) = (\phi, v)$ for all $v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ for some $\phi \in L^2(\Omega)$, then we have

$$\omega^2\|u_\phi - \mathcal{P}_h u_\phi\|_{0,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}, \gamma)\left(\omega^{\sigma+3}h^{1+\alpha} + \omega^{\sigma+3/2+\alpha}h^{2\alpha}\right)\|\phi\|_{0,\Omega}, \tag{5.16}$$

and

$$|u_\phi - \mathcal{P}_h u_\phi|_\star \leq C(\Omega, \omega_0, \mathscr{V}, \gamma)\left(\omega^{\sigma-1/2}\omega^\alpha h^\alpha + \omega^{\sigma+1}h\right)\|\phi\|_{0,\Omega}. \tag{5.17}$$

Proof. By using Poincaré inequality, it is clear that the map

$$u \to |u|_\star = \sqrt{|a(u, u)|} = \sqrt{\omega|\mathscr{V}^{-1/2}u|^2_{0,\Gamma_{\mathrm{Diss}}} + |u|^2_{1,\Omega}}$$

is a norm on $H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$, equivalent to the usual $H^1(\Omega)$ norm (with a constant of equivalence depending on ω). As a result, a is a coercive and continuous sesquilinear form, and the existence and uniqueness of $\mathcal{P}_h u$ follow.

Furthermore, the multiplicative trace inequality shows that

$$|u|_\star \leq C(\Omega, \mathscr{V})|||u|||.$$

As a result, Céa's Lemma gives

$$|u_\phi - \mathcal{P}_h u_\phi|_\star \leq C(\Omega, \mathscr{V})|||u_\phi - \mathcal{I}_h u_\phi|||,$$

and we conclude that (5.17) holds with the help of (5.9).

We establish (5.16) using an Aubin-Nitsche trick. We introduce $\xi \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ solution to $a(\xi, v) = (u_\phi - p_h u_\phi, v)$ for all $v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. The existence and uniqueness of ξ follows from the properties of a, and we have

$$\begin{aligned}\|u_\phi - \mathcal{P}_h u_\phi\|^2_{0,\Omega} &= a(\xi, u_\phi - \mathcal{P}_h u_\phi, \xi) \\ &= a(\xi - \mathcal{P}_h\xi, u_\phi - \mathcal{P}_h u_\phi, \xi)\end{aligned}$$

$$\begin{aligned}
&\leq |\xi - \mathcal{P}_h \xi|_\star |u_\phi - \mathcal{P}_h u_\phi|_\star \\
&\leq C(\Omega, \mathscr{V}) |||\xi - \mathcal{P}_h \xi||| \cdot |u_\phi - \mathcal{P}_h u_\phi|_\star \\
&\leq C(\Omega, \mathscr{V}) h^\alpha |u_\phi - \mathcal{P}_h u_\phi|_{0,\Omega} |u_\phi - \mathcal{P}_h u_\phi|_\star
\end{aligned}$$

so that

$$\omega \|u_\phi - \mathcal{P}_h u_\phi\|_{0,\Omega} \leq C(\Omega, \mathscr{V}) \omega h^\alpha |u_\phi - \mathcal{P}_h u_\phi|_\star \leq C(\Omega, \mathscr{V}) \omega h^\alpha |||u_\phi - \mathcal{I}_h u_\phi|||,$$

and (5.16) follows from (5.9). $\square$

The elliptic projection and its approximation properties being introduced in Lemma 5.4, we can follow [21] to produce a preasymptotic error estimate in Theorem 5.5.

Theorem 5.5. *Assume that $\omega^{\sigma+3} h^{1+\alpha}$ is small enough, then there exists a unique solution $u_h \in V_h$ to problem (5.1) and it holds that*

$$|||u - u_h||| \leq C(\Omega, k, \gamma) \omega^\sigma \left(\omega^{-1/2} \omega^\alpha h^\alpha + \omega h + \omega^3 h^2 \right) \|f\|_{0,\Omega}. \tag{5.18}$$

Proof. The proof relies on an Aubin-Nitsche type argument. We thus introduce $\xi \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$ solution to $B(v, \xi) = (v, u - u_h)$ for all $v \in H^1_{\Gamma_{\mathrm{Dir}}}(\Omega)$. Then, we have

$$\|u - u_h\|^2_{0,\Omega} = B(u - u_h, \xi) = B(u - u_h, \xi - \mathcal{P}_h \xi).$$

Thanks to the properties of $\mathcal{P}_h$, we have

$$\begin{aligned}
B(u - u_h, \xi - \mathcal{P}_h \xi) &= -\omega^2 (\mathscr{V}^{-2}(u - u_h), \xi - \mathcal{P}_h \xi) + a(u - u_h, \xi - \mathcal{P}_h \xi) \\
&= -\omega^2 (\mathscr{V}^{-2}(u - u_h), \xi - \mathcal{P}_h \xi) + a(u - \mathcal{I}_h u, \xi - \mathcal{P}_h \xi),
\end{aligned}$$

so that

$$\begin{aligned}
\|u - u_h\|^2_{0,\Omega} &\leq \frac{\omega^2}{\mathscr{V}_{min^2}} \|u - u_h\|_{0,\Omega} \|\xi - \mathcal{P}_h \xi\|_{0,\Omega} + |a(u - \mathcal{I}_h u, \xi - \mathcal{P}_h \xi)| \\
&\leq \frac{\omega^2}{\mathscr{V}_{min^2}} \|u - u_h\|_{0,\Omega} \|\xi - \mathcal{P}_h \xi\|_{0,\Omega} + |u - \mathcal{I}_h u|_\star |\xi - \mathcal{P}_h \xi|_\star.
\end{aligned}$$

As Lemmas 5.2 and 5.4 yield

$$\begin{aligned}
&\omega^2 \|\xi - \mathcal{P}_h \xi\|_{0,\Omega} \leq C(\Omega, \omega_0, \mathscr{V}, \gamma) \omega^\sigma \left(\omega^3 h^{1+\alpha} + \omega^{3/2+\alpha} h^{2\alpha} \right) \|u - u_h\|_{0,\Omega}, \\
&|||\xi - \mathcal{P}_h \xi||| \leq C(\Omega, \omega_0, \gamma) \omega^\sigma \left(\omega h + \omega^{-1/2} \omega^\alpha h^\alpha \right) \|u - u_h\|_{0,\Omega},
\end{aligned}$$

and

$$|u - \mathcal{I}_h u|_\star \leq C(\Omega, \mathscr{V}) |||u - \mathcal{I}_h u||| \leq C(\Omega, \omega_0, \mathscr{V}, \gamma) \omega^\sigma \left(\omega h + \omega^{-1/2} \omega^\alpha h^\alpha \right) \|f\|_{0,\Omega},$$

it follows that

$$\|u-u_h\|_{0,\Omega}^2 \le C(\Omega,\omega_0,\mathscr{V},\gamma)\omega^\sigma \left\{ \left(\omega^3 h^{1+\alpha} + \omega^{3/2+\alpha}\omega^{2\alpha}\right) \|u-u_h\|_{0,\Omega}^2 + \left(\omega h + \omega^{-1/2}\omega^\alpha h^\alpha\right)^2 \|f\|_{0,\Omega}\|u-u_h\|_{0,\Omega} \right\},$$

and

$$\left\{1 - C(\Omega,\omega_0,\gamma)\omega^\sigma \left(\omega^3 h^{1+\alpha} + (\omega^{3+2\alpha}h^{4\alpha})^{1/2}\right)\right\} \|u-u_h\|_{0,\Omega} \le C(\Omega,\omega_0,\gamma)\omega^\sigma \left(\omega h + \omega^{\alpha-1/2}h^\alpha\right)^2 \|f\|_{0,\Omega}.$$

We see that

$$\omega^{3+2\alpha}h^{4\alpha} = h^\beta \left(\omega^3 h^{1+\alpha}\right)^{(3+2\alpha)/3},$$

with

$$\beta = \left(\frac{12\alpha}{3+2\alpha} - 1 - \alpha\right)\frac{3+2\alpha}{3} > 0,$$

for $1/2 \le \alpha \le 1$. Hence, assuming that $\omega^{\sigma+3}h^{1+\alpha}$ is small enough, we have

$$C(\Omega,\omega_0,\mathscr{V},\gamma)\omega^\sigma \left(\omega^3 h^{1+\alpha} + (\omega^{3+2\alpha}h^{4\alpha})^{1/2}\right) \le \frac{1}{2},$$

and the previous estimate yields

$$\|u-u_h\|_{0,\Omega} \le C(\Omega,\omega_0,\mathscr{V},\gamma)\omega^\sigma \left(\omega h + \omega^{-1/2}\omega^\alpha h^\alpha\right)^2 \|f\|_{0,\Omega}.$$

Since

$$\omega(\omega h + \omega^{-1/2}\omega^\alpha h^\alpha)^2 \le 2\left(\omega^3 h^2 + \omega^{2\alpha}h^{2\alpha}\right) \le 2\left(\omega^3 h^2 + \omega h\right),$$

we obtain

$$\omega\|u-u_h\|_{0,\Omega} \le C(\Omega,\omega_0,\gamma)\omega^\sigma \left(\omega^3 h^2 + \omega h\right)\|f\|_{0,\Omega}.$$

Finally, we have

$$\begin{aligned} |||u-u_h|||^2 &\le \frac{2\omega^2}{\mathscr{V}_{\min}^2}\|u-u_h\|_{0,\Omega}^2 + \operatorname{Re} B(u-u_h, u-u_h) \\ &= \frac{2\omega^2}{\mathscr{V}_{\min}^2}\|u-u_h\|_{0,\Omega}^2 + \operatorname{Re} B(u-u_h, u-\mathcal{I}_h u) \\ &\le C(\Omega,\omega_0,\mathscr{V})\left(\omega^2\|u-u_h\|_{0,\Omega}^2 + |||u-u_h|||\cdot|||u-\mathcal{I}_h u|||\right), \end{aligned}$$

and using the algebraic inequality, we obtain

$$|||u-u_h||| \le C(\Omega,\omega_0,\mathscr{V})\left(\omega\|u-u_h\|_{0,\Omega} + |||u-\mathcal{I}_h u|||\right).$$

Then, the result follows since

$$|||u - \mathcal{I}_h u||| \leq C(\Omega, \omega_0, \gamma)\omega^\sigma \left(\omega h + \omega^{-1/2}\omega^\alpha h^\alpha\right) \|f\|_{0,\Omega}.$$

□

Error estimate (5.18) is called pre-asymptotic because it is valid in the range $\omega^{\sigma+3}h^{1+\alpha} \leq C$ which (in general) is larger than the asymptotic range $\omega^{\sigma+2}h \leq C$. In error estimate (5.18), we see that the pollution term $\omega^{\sigma+3}h^2$ is added to the best approximation error.

The validity range of (5.18) depends on α. The authors believe this is not sharp, and the dependence on α is due to the particular proof techniques. Focusing on domains for which $\sigma = 0$, in the limit case $\alpha = 1/2$, the condition $\omega^3 h^{1+\alpha} \leq C$ is equivalent to $\omega^2 h \leq C$, so that the result is equivalent to asymptotic error estimate of Theorem 5.3. On the other hand, in the limit case $\alpha = 1$, $\omega^3 h^{1+\alpha} = \omega^3 h^2$ and we recover the usual validity condition of smooth domains [21]. In the general case where $1/2 < \alpha < 1$, we obtain a pre-asymptotic error estimate valid under a condition less restrictive than the quasi-optimality condition $\omega^2 h \leq C$, but more restrictive than the validity condition $\omega^3 h^2 \leq C$ of smooth domains.

6. Numerical examples

6.1. A model problem with an analytical solution

We first present a model problem that we will use below for which an analytical solution is available. The idea is to consider a disc sector, as presented in Section 2. However, in order to avoid curved elements, we consider a square with a re-entrant corner at the origin, namely the domain of computation is defined by

$$\Omega_\alpha = \left\{ \mathbf{x} = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid |\mathbf{x}_1| \leq 1,\ |\mathbf{x}_2| \leq 1,\ 0 \leq \theta \leq \frac{\pi}{\alpha} \right\},$$

with $\frac{1}{2} < \alpha < 1$. The boundary of Ω_α is split up as

$$\Gamma_{\mathrm{Diss}} = \{\mathbf{x} \in \partial\Omega \mid |\mathbf{x}_1| = 1 \text{ or } |\mathbf{x}_2| = 1\}, \quad \Gamma_{\mathrm{Dir}} = \left\{(r\cos\theta, r\sin\theta) \in \partial\Omega \mid \theta = 0 \text{ or } \theta = \frac{\pi}{\alpha}\right\}.$$

We consider this problem for three different values of α: 4/5, 4/6, 4/7. These values correspond to domains that are easily meshed, so that we can used structured meshes to solve the problem, as shown in Figure 3. These meshes are not refined near the singularity. We also point out that our definition of Ω_α is consistent with Assumption 2.2.

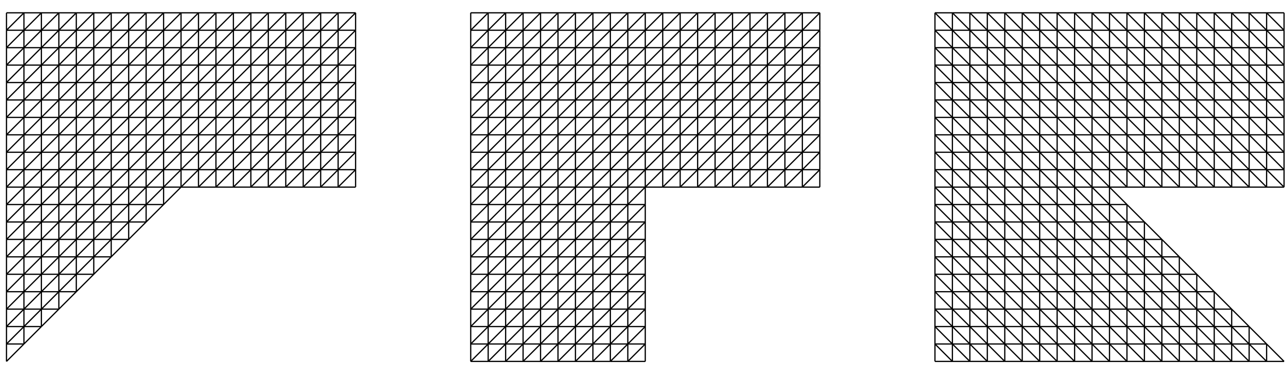


Figure 3. Structured meshes.

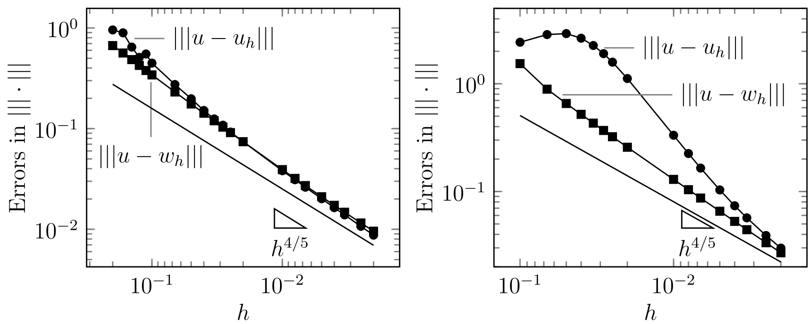


FIGURE 4. Convergence curves for $\alpha = 4/5$ with $\omega = 2\pi$ (*left*) and $\omega = 10\pi$ (*right*).

Furthermore, following Section 2, we will consider as analytical solutions slight variations of the function

$$\psi_\omega(\mathbf{x}) = \omega^{-1/2} J_\alpha(\omega r)\sin(\alpha\theta) = \omega^{-1/2} s_\omega(\mathbf{x}).$$

Note that ψ_ω satisfies the homogeneous problem inside Ω_α and exhibits a singularity near the origin.

6.2. Asymptotic error estimates

The aim of this first numerical experiment is to illustrate asymptotic error-estimate (5.13) developed in Theorem 5.3. Specifically, we consider the case where $\sigma = 0$ and demonstrate that the asymptotic convergence rate, and the "resolution condition" $\omega^2 h \le C$ are sharp.

We investigate the test-case presented above with the analytical solution

$$\phi_\omega(\mathbf{x}) = \psi_\omega(\mathbf{x})\chi(|\mathbf{x}|),$$

where $\chi \in C^1(\mathbb{R})$ is a cut-off function such that $\chi = 1$ if $0 \le |\mathbf{x}| \le 0.2$, $\chi = 0$ if $|\mathbf{x}| > 0.9$, and $\chi \in \mathcal{P}_3$ if $0.2 \le |\mathbf{x}| \le 0.9$.

Then, we solve

$$\left\{\begin{array}{rcll} -\omega^2 u - \Delta u &=& f & \text{in } \Omega_\alpha, \\ \nabla u\cdot\mathbf{n} - i\omega u &=& 0 & \text{on } \Gamma_{\text{Diss}}, \\ u &=& 0 & \text{on } \Gamma_{\text{Dir}}, \end{array}\right.$$

with $f = 2\nabla\chi\cdot\nabla\psi_\omega + \Delta\chi\psi_\omega$. Because $\|J_\alpha(\omega r)\|_{1,\Omega_\alpha} \simeq \omega^{1/2}$, we see that $\|f\|_{L^2(\Omega_\alpha)} \simeq 1$.

Figures 4 and 5 illustrate the convergence curves of the finite element error $|||u-u_h|||$ and of the best approximation error $|||u-w_h|||$, where the best approximation $w_h \in V_h$ is obtained as the solution to

$$\omega^2(w_h,\xi_h) + (\nabla w_h,\nabla\xi_h) = \omega^2(u,\xi_h) + (\nabla u,\nabla\xi_h), \quad \forall \xi_h \in V_h.$$

As expected, we see that the asymptotic convergence rate is in $\mathcal{O}(h^\alpha)$. Moreover, we see that the asymptotic convergence rate is achieved faster for lower frequencies. Also, Figures 4 and 5 clearly depict the pollution effect: for large mesh sizes, there is a gap between the finite-element error and the best approximation error. Furthermore, this gap is more important for higher frequencies.

Next, we validate that the condition $\omega^2 h \le C$ is necessary for the finite-element solution to be quasi-optimal. To this end, we first compute convergence curves as in Figures 4 and 5 for different values of ω (ranging from

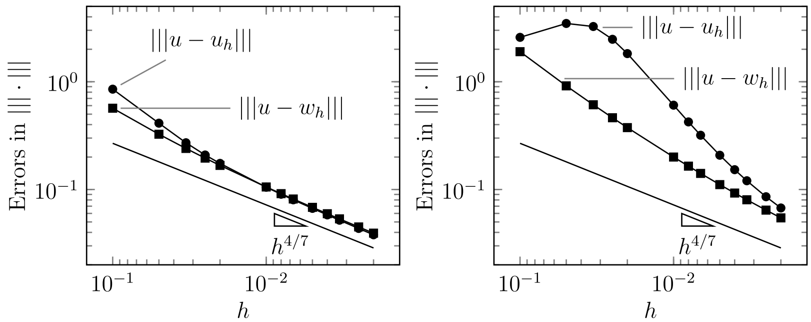


FIGURE 5. Convergence curves for $\alpha = 4/7$ with $\omega = 2\pi$ (*left*) and $\omega = 10\pi$ (*right*).

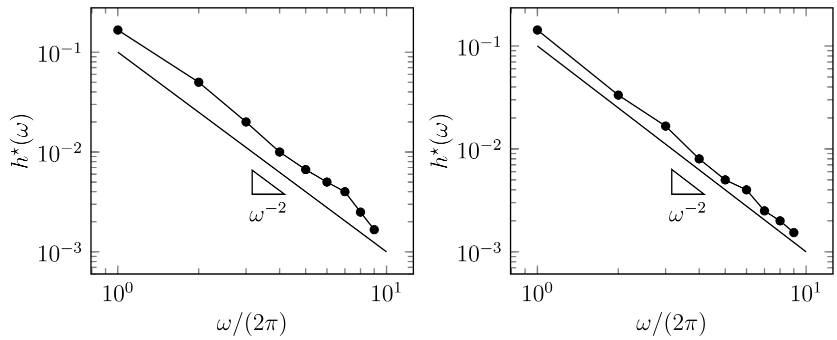


FIGURE 6. $h^\star(\omega)$ for $\alpha = 4/5$ (*left*) and $\alpha = 4/7$ (*right*).

2π to 20π). Next, for each convergence curve, we define the value $h^\star(\omega)$ as the largest value such that

$$|||u - u_h||| \leq 2|||u - w_h|||, \quad \forall h \leq h^\star(\omega). \tag{6.1}$$

Equation (6.1) means that finite element solution is quasi-optimal (uniformly in frequency), with the arbitrary constant 2. Theorem 5.3 shows that a sufficient condition for (6.1) to hold is $\omega^2 h \leq C$ which would correspond to $h^\star(\omega) \simeq \omega^{-2}$.

Figure 6 depicts the dependence of $h^\star(\omega)$ on ω. We see that $h^\star(\omega)$ indeed behaves as ω^{-2}. As a result, we have numerically observed that the condition on $\omega^2 h$ to be sufficiently small is actually necessary for the finite element solution to be quasi-optimal, and we conclude that Theorem 5.3 is sharp.

6.3. Pre-asymptotic error-estimates

We present two other experiments that focus on preasymptotic error estimates. Our aim is to investigate if the condition

$$\omega^{2p+1} h^{2p} \leq C \tag{6.2}$$

is sufficient to ensure that the $hp-$finite element error remains bounded independently of ω. This condition is known to be optimal in the case of smooth non-trapping domains ($\sigma = 0$). A proof is available for 1D problems [28], and 2D and 3D problems with Cartesian grids have been analyzed using dispersion analysis [3].

In the analysis presented above, we "almost" show that condition (6.2) is sufficient for the case of linear elements. Indeed, error estimate (5.18) clearly shows that the finite element error is bounded independently

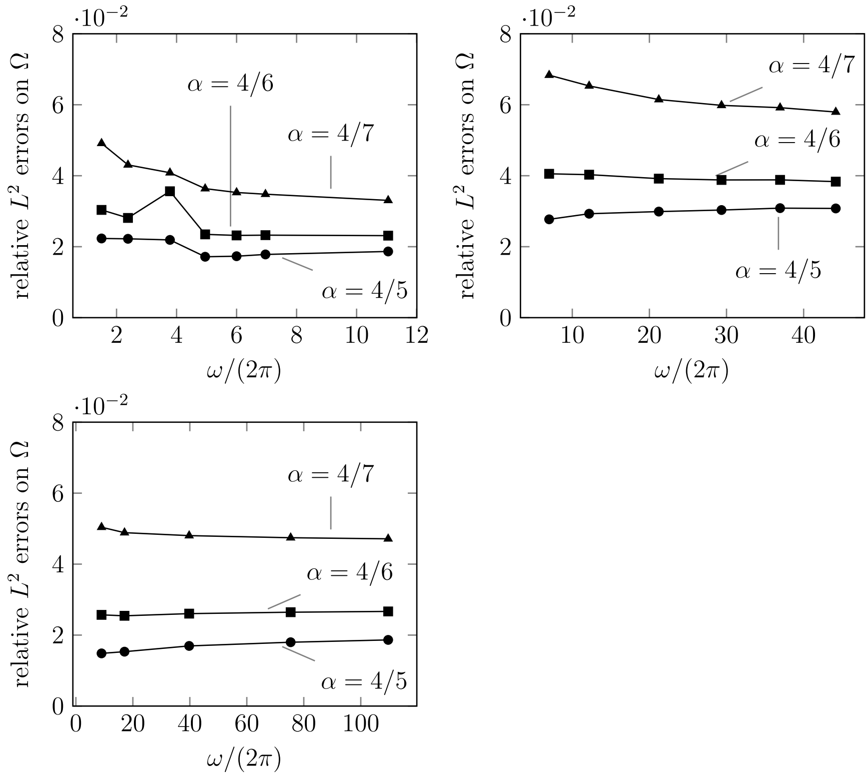


FIGURE 7. *Top left panel*: $\mathcal{P}_1$ elements, $\omega^3 h^2 = C$. *Top right panel*: $\mathcal{P}_2$ elements, $\omega^5 h^4 = C$. *Bottom left panel*: $\mathcal{P}_6$ elements, $\omega^{13} h^{12} = C$.

of ω as soon as $\omega^3 h^2$ is bounded. Unfortunately, we are only able to prove (5.18) under the more restrictive condition that $\omega^3 h^{1+\alpha}$ is small enough.

For each experiment, we start by selecting an initial frequency ω_0. We empirically find a mesh size h_0 so that the relative L^2 finite element error is about 5%, when solving problem (5.1) for the frequency ω_0. Then, we validate condition (6.2) by solving problem (5.1) for increasing values of ω, the mesh size h being constraint by $\omega^{2p+1} h^{2p} = \omega_0^{2p+1} h_0^{2p}$, and checking that the error remains bounded.

In the two experiments, the relative $L^2(\Omega)$ error

$$\frac{\|u - u_h\|_{0,\Omega}}{\|u\|_{0,\Omega}}$$

is measured.

6.3.1. Analytical solution

We investigate a test-case with an analytical solution, as depicted previously. We solve

$$\begin{cases} -\omega^2 u - \Delta u &= 0 \quad \text{in } \Omega_\alpha, \\ \nabla u \cdot \mathbf{n} - i\omega u &= g \quad \text{on } \Gamma_{\text{Diss}}, \\ u &= 0 \quad \text{on } \Gamma_{\text{Dir}}, \end{cases}$$

with $g = \nabla \psi_\omega \cdot \mathbf{n} - i\omega\psi_\omega$, so that the exact solution is $u = \psi_\omega$. Because $|J_\alpha(\omega r)| \leq C(\Omega, \omega_0)\omega^{-1/2}$ on Γ_{Diss} when ω is large, we see that $\|g\|_{L^2(\Gamma_{\text{Diss}})} \leq C(\Omega, \omega_0)$ for all $\omega \geq \omega_0$.

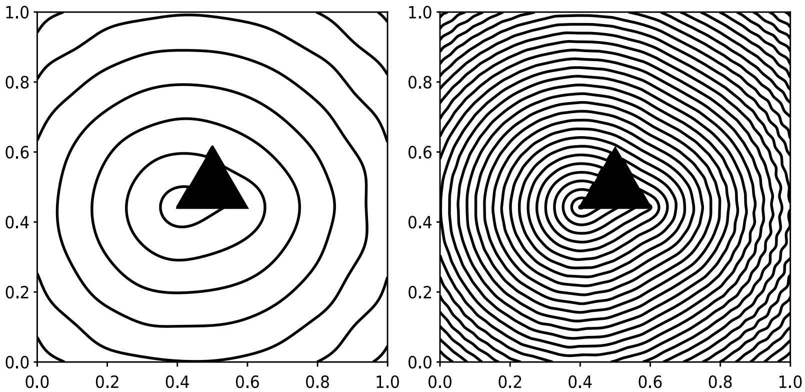


FIGURE 8. Zero-level sets of the real part of the solution of the scattering problem for $\omega = 10\pi$ (*left*) and 20π (*right*).

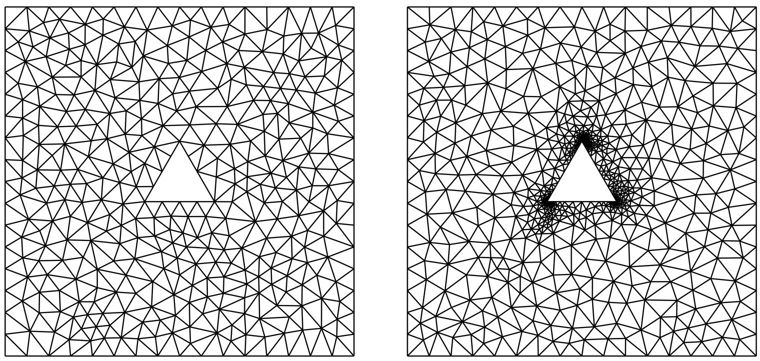


FIGURE 9. Uniform (*left*) and refined (*right*) meshes for the scattering problem.

As we explained before, we solve the problem for different values of ω by starting with an initial guess (ω_0, h_0) (fixed heuristically) and then impose the mesh size for higher values of ω so that $\omega^{2p+1}h^{2p} = \omega_0^{2p+1}h_0^{2p}$. We employ three different values of p: 1, 2 and 6 and for this experiment, the heuristically determined values of (ω_0, h_0) are given by $(3\pi, 1/50)$,$(14\pi, 1/50)$ and $(18\pi, 1/10)$ for $p = 1, 2$ and 6.

We present the dependence of the relative $L^2(\Omega)$ error with respect to ω on Figure 7. As shown there, the error is bounded independently of ω under the condition that $\omega^{2p+1}h^{2p} \leq C$. As observed above, for the case $p = 1$, this is almost consistent with the pre-asymptotic error estimate derived in Theorem 5.5.

Figure 7 also shows that the error is more important for smaller values of α. This is not surprising, since in this case the solution is more singular and furthermore, the domain of computation is wider. However, the error is only increased by a constant factor that is about 3 between the largest and smallest considered values of α. In particular, as predicted by our analysis for the linear case, the stability of the scheme is not affected by the value of α.

6.3.2. Scattering by a triangle

The problem of scattering we consider reads

$$\begin{cases} -\omega^2 u - \Delta u &= 0 \quad \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} - i\omega u &= 0 \quad \text{on } \Gamma_{\text{Diss}}, \\ u &= e_\phi \quad \text{on } \Gamma_{\text{Dir}}, \end{cases}$$

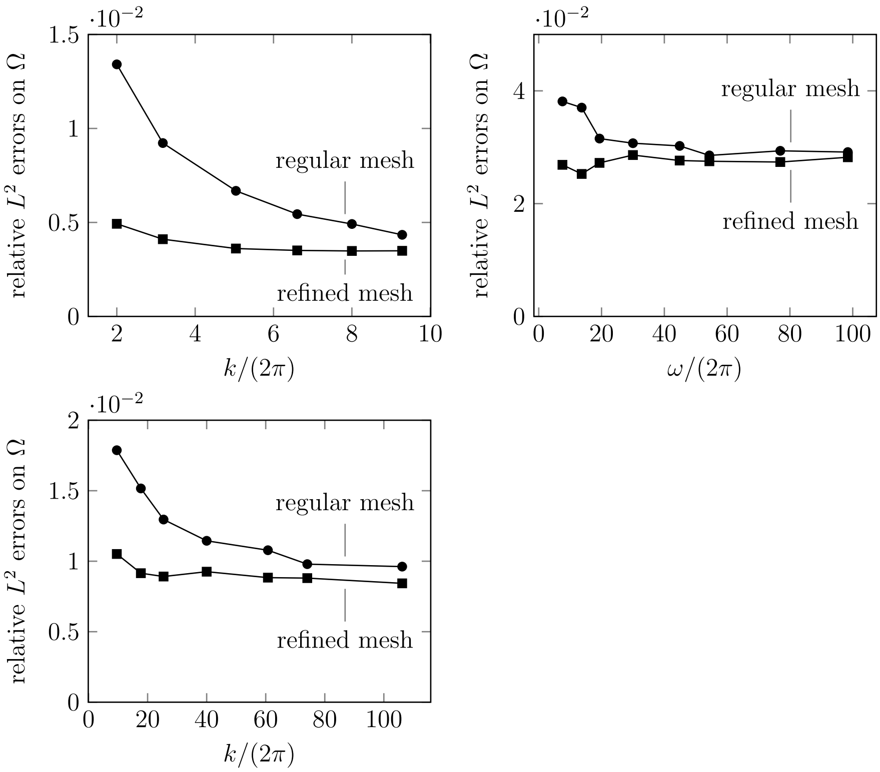


FIGURE 10. *Top left panel*: $\mathcal{P}_1$ elements, $\omega^3 h^2 = C$. *Top right panel*: $\mathcal{P}_3$ elements, $\omega^5 7^6 = C$. *Bottom left panel*: $\mathcal{P}_4$ elements, $\omega^9 h^8 = C$.

where $e_\phi(\mathbf{x}) = \exp(i\omega\nu \cdot \mathbf{x})$, with $\nu = (\cos\phi, \sin\phi)$ and $\phi = \pi/3$. The numerical solutions obtained for different frequencies are depicted on Figure 8.

Instead of computing a lifting for e_ϕ, we weakly impose the inhomogeneous Dirichlet condition with a penalization method [34, 35]. Hence, we modify the sesquilinear form B as

$$B_h(u_h, v_h) = B(u_h, v_h) + \int_{\Gamma_{\mathrm{Dir}}} \nabla u_h \cdot \mathbf{n}\overline{v_h} + \int_{\Gamma_{\mathrm{Dir}}} u_h \nabla\overline{v_h} \cdot \mathbf{n} + \frac{p^2}{h}\int_{\Gamma_{\mathrm{Dir}}} u_h \overline{v_h},$$

and we solve

$$B_h(u_h, v_h) = \frac{p^2}{h}\int_{\Gamma_{\mathrm{Dir}}} e_\phi \overline{v_h}, \quad \forall v_h \in V_h^p. \tag{6.3}$$

Though problem (6.3) is not directly covered by our analysis, similar error estimates can be obtained with slight modifications of our arguments.

The domain of computations as well as the used meshes are depicted at Figure 9. The meshes are obtained using the software `triangle` [39]. The mesh size is imposed as an area condition ($|K| \leq h^2/2$) and the meshes satisfy a minimal angle condition of 33 degrees. We also produce "refined" meshes by forcing the mesh to include three additional points. Each of the three points is placed at a distance $h/1000$ of one vertex of the triangle. In that way, the local mesh size at the singular points is 1000 times finer than the global mesh size in refined meshes.

Following our methodology, we impose the condition that $\omega^{2p+1}h^{2p} = \omega_0^{2p+1}h_0^{2p+1}$. We use three different values of p: 1, 3 and 4, and the associated couples (ω_0, h_0) are $(4\pi, 1/50)$, $(14\pi, 1/10)$ and $(18\pi, 1/10)$. In order to evaluate the $L^2(\Omega)$-norm error a "reference solution" is computed with a $p+1$ finite element method on the same mesh. Then, each solution is evaluated onto a 1024×1024 grid, and the $L^2(\Omega)$ error is computed as the l^2-norm of this discrete vector.

We present the results on Figure 10. The error is bounded independently of the frequency for both uniform and refined meshes. For low frequency simulations, refined meshes improve the precision of the finite element method (up to a factor 3). However, we see that this improvement is greatly reduced for higher frequencies. This is in agreement with our analysis, where we pointed out that the singular part of the solution is "less important" for high frequencies.

7. Conclusion

In this work, we have analyzed the acoustic Helmholtz problem set in domains Ω where a Dirichlet boundary condition is imposed on a part $\Gamma_{\rm Dir}$ of its boundary and an absorbing boundary condition is prescribed on the remaining part $\Gamma_{\rm Diss}$. Our main assumption on the domain is that the solution depends continuously on the datum, with a stability constant that grows as $\mathcal{O}(\omega^\sigma)$ where ω is the frequency and σ is a fixed exponent. As we have illustrated, our assumptions are rather general, and handle a number of applications, including scattering by a sound-soft (trapping or not) obstacle, or by a cavity.

Since the boundary $\Gamma_{\rm Dir}$ can feature re-entrant corners, the solution might become singular. We have proposed a splitting of the solution of the Helmholtz problem with a regular part in $H^2(\Omega)$ and a singular function for each corner of K. The regularity as well as the high frequency behaviour of each component of the splitting have been rigorously analyzed. Our main conclusion is that in some sense, as the frequency increases, the "amplitude" of the singularities vanishes before the amplitude of the regular part.

We have taken advantage of this splitting to derive sharp error estimates for finite element discretizations. The different behaviours of the regular and singular parts in terms of frequency is visible in these error estimates. The main conclusion is that if the frequency is high, numerical discretizations do not "see" the singularities unless the mesh size is "small".

Numerical experiments that focuses on non-trapping domains ($\sigma = 0$) have been presented to illustrate the above-mentioned error estimates. First we have checked that our resolution condition $\omega^2 h \leq C$ for a small enough constant C is necessary for the quasi-optimality of the $\mathcal{P}_1$ finite element method. Secondly, in smooth non-trapping domains, it is known that the condition $\omega^{2p+1}h^{2p} \leq C$ is optimal to ensure that the finite element error remains bounded independently of the frequency. We have numerically investigated if this condition is also sufficient in our setting (in particular for the case of non convex domains with re-entrant corners). We conclude that this condition is indeed sufficient. Furthermore, we have analyzed the dependence of the error with respect to the singular exponent α, and we conclude that if the error does increase when α gets closer to $1/2$, this increase never exceeds one order of magnitude.

Future works will be guided towards edge and corner singularities of 3D scattering problems. Specifically, corner singularities close to a conical point can be handled by replacing the singular function $J_\alpha(\omega r)\sin(\alpha\theta)$ by $r^{-1/2}J_\lambda(\omega r)\psi(\nu)$ where r is the radial variable, $\nu = (\theta, \psi)$ is the angular position and the couple (λ, ψ) is an eigenpair of the Laplace-Beltrami operator on the cone [15, 22]. In addition, edge singularities can be analyzed by combining the present analysis with Fourier transform in the direction tangential to the edge, as performed in [23] for the Laplace operator. Finally, the analysis of other wave operators in 2D, like the time-harmonic elastodynamic system (using the approach from [24]), would be considered.

Appendix A. Bessel functions

Here, $\nu \in (1/2, 1)$ is an arbitrary real number. Bessel functions of first and second kind are defined by

$$J_{\pm\nu}(\rho) = \left(\frac{\rho}{2}\right)^{\pm\nu} \sum_{l=0}^{+\infty} \frac{1}{l!\Gamma(\pm\nu + l + 1)} \left(-\frac{\rho^2}{4}\right)^l$$

and

$$Y_\nu(\rho) = \frac{J_\nu(\rho)\cos(\nu\pi) - J_{-\nu}(\rho)}{\sin(\nu\pi)}.$$

Hereafter, we list well-known properties of Bessel functions that can be bound in Chapter 9 of [1]. For $0 < \rho \le 1$, and $\nu > 0$, it holds that

$$|J_\nu(\rho)| \le \frac{1}{\Gamma(\nu+1)}\left(\frac{\rho}{2}\right)^{\nu},$$
$$|Y_\nu(\rho)| \le \frac{1}{\Gamma(\nu+1)}\left(\frac{\rho}{2}\right)^{-\nu}.$$

The following expansions hold for large ρ

$$J_\nu(\rho) = \sqrt{\frac{2}{\pi\rho}}\cos\left(\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}),$$
$$J'_\nu(\rho) = -\sqrt{\frac{2}{\pi\rho}}\sin\left(\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}),$$
$$Y_\nu(\rho) = \sqrt{\frac{2}{\pi\rho}}\sin\left(\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}),$$
$$Y'_\nu(\rho) = \sqrt{\frac{2}{\pi\rho}}\cos\left(\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}).$$

With the above properties, one easily shows:

Lemma A.1. *For all $\nu \in (1/2, 1)$, there exists a constant $C(\nu)$ such that*

$$|J_\nu(\rho)| \le C(\nu)\rho^{\nu}, \quad |Y_\nu(\rho)| \le C(\nu)\rho^{-\nu},$$

for all $\rho \in (0,1)$, and

$$|J_\nu(\rho)| \le C(\nu)\rho^{-1/2}, \quad |Y_\nu(\rho)| \le C(\nu)\rho^{-1/2},$$

for all $\rho \ge 1$.

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. From Lemma A.1, we have

$$\begin{aligned}\omega^2\int_0^R |Y_\alpha(\omega r)|^2 r\mathrm{d}r &= \int_0^{\omega R} |Y_\alpha(\rho)|^2\rho d\rho \\ &= \int_0^1 |Y_\alpha(\rho)|^2\rho d\rho + \int_1^{\omega R} |Y_\alpha(\rho)|^2\rho d\rho\end{aligned}$$

$$
\begin{aligned}
&\leq C(\alpha)\left(\int_0^1 |\rho^{-\alpha}|^2 \rho d\rho + \int_1^{\omega R} |\rho^{-1/2}|^2 \rho d\rho\right) \\
&\leq C(\alpha)\left(\frac{1}{2-2\alpha} + \omega R - 1\right) \\
&\leq C(\alpha)\left\{R + \left(\frac{1}{2-2\alpha} - 1\right)\omega_0^{-1}\right\}\omega.
\end{aligned}
$$

The same estimate holds for J_α, since $\rho^\alpha \leq \rho^{-\alpha}$ for $\rho \leq 1$ and (3.1)–(3.2) directly follow. In order to establish the lower bound, we first write that

$$
\int_0^{\omega R} |H_\alpha^{(2)}(\rho)|^2 \rho d\rho \geq \int_{\omega R/2}^{\omega R} |H_\alpha^{(2)}(\rho)|^2 \rho d\rho.
$$

Then, we have

$$
H_\alpha^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}} \exp\left\{-i\left(\rho - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\right\} + \mathcal{O}\left(\rho^{-3/2}\right)
$$

so that

$$
|H_\alpha^{(2)}(\rho)|^2 \geq \frac{2}{\pi\rho} - M(\alpha, R, \omega_0)\rho^{-3}, \forall \rho \geq \omega R/2.
$$

As a result, we have

$$
\begin{aligned}
\int_{\omega R/2}^{\omega R} |H_\alpha^{(2)}(\rho)|^2 \rho d\rho &\geq \frac{2}{\pi}\int_{\omega R/2}^{\omega R} d\rho - M(\alpha, \omega_0)\int_{\omega R/2}^{\omega R} \rho^{-2} d\rho \\
&\geq \frac{\omega R}{\pi} - M(\alpha, \omega_0)(\omega R)^{-1} \\
&\geq C(\alpha, R, \omega_0)\omega,
\end{aligned}
$$

assuming that ω_0 is sufficiently large.

We now prove (3.3). We start by writing

$$
\kappa = \rho - \frac{\nu\pi}{2} - \frac{\pi}{4},
$$

so that

$$
\begin{aligned}
J_\nu'(\rho) + iJ_\nu(\rho) &= \sqrt{\frac{2}{\pi\rho}}\left(-\sin\kappa + i\cos\kappa\right) + \mathcal{O}\left(\rho^{-3/2}\right) \\
&= i\sqrt{\frac{2}{\pi\rho}}\left(\cos\kappa + i\sin\kappa\right) + \mathcal{O}\left(\rho^{-3/2}\right) \\
&= i\sqrt{\frac{2}{\pi\rho}} e^{i\kappa} + \mathcal{O}\left(\rho^{-3/2}\right),
\end{aligned}
$$

and

$$\begin{aligned} Y_\nu'(\rho) + iY_\nu(\rho) &= \sqrt{\frac{2}{\pi\rho}}\,(\cos\kappa + i\sin\kappa) + \mathcal{O}\left(\rho^{-3/2}\right) \\ &= \sqrt{\frac{2}{\pi\rho}}e^{-i\kappa} + \mathcal{O}\left(\rho^{-3/2}\right). \end{aligned}$$

Then, we have

$$\frac{Y_\nu'(\rho) + iY_\nu(\rho)}{J_\nu'(\rho) + iJ_\nu(\rho)} = \frac{1}{i} + \mathcal{O}\left(\rho^{-3/2}\right),$$

and the result follows.

Finally, (3.4) is just the Wronskian of J_α and Y_α that is given by

$$J_\nu(\rho)Y_\nu'(\rho) - J_\nu'(\rho)Y_\nu(\rho) = \frac{2}{\pi\rho}, \forall\rho > 0.$$

□

Appendix B. Checking the stability property

Before investigating some special configurations, we may notice that the variational formulation (2.2) directly yields

Lemma B.1. *We have*

$$\|\nabla u\|_{0,\Omega}^2 \le \frac{\mathscr{V}_{\min}^2}{4\omega^2}\|f\|_{0,\Omega}^2 + \frac{2\omega^2}{\mathscr{V}_{\min}^2}\|u\|_{0,\Omega}^2, \tag{B.1}$$

and

$$\omega\int_{\Gamma_{\mathrm{Diss}}} \mathscr{V}^{-1}|u|^2 \le \|f\|_{0,\Omega}\|u\|_{0,\Omega}. \tag{B.2}$$

Indeed this follows by taking $v = u$ as test function in (2.2), and using Cauchy-Schwarz's inequality on the real and imaginary part.

B.1 Trapping cavities

Here we consider a cavity. We assume that $\Omega \subset (0,1) \times (0,-L)$, where L denote the depth of the cavity. The part of its boundary $\Gamma_{\mathrm{Diss}} = (0,1) \times \{0\}$ represents the "entrance" of the cavity. Furthermore, we assume that the remaining part of the boundary of the cavity Γ_{Dir} can be divided into two parts Γ_{Dir}^h and Γ_{Dir}^v that are respectively made of horizontal and vertical segments. In addition, we assume that $\mathbf{n}_2 = (0,-1)$ on Γ_{Dir}^h, see Figure 1 left for an illustration. Finally, we assume that

$$\frac{\partial\mathscr{V}}{\partial\mathbf{x}_2} \le 0.$$

Under these assumptions, we have the

Theorem B.1. *For all $f \in L^2(\Omega)$ and $\omega > 0$, there exists a unique solution $u \in H^1_{\Gamma_{\rm Dir}}(\Omega)$ to (2.2). Furthermore, we have*

$$\|u\|_{0,\Omega} \leq \frac{L^2}{2}\left(3 + \frac{\omega L}{\mathscr{V}_{\min}}\right)\|f\|_{0,\Omega}, \tag{B.3}$$

$$\left\|\frac{\partial u}{\partial \mathbf{x}_1}\right\|_{0,\Omega} \leq L\left\{\frac{1}{2}\left(3 + \frac{\omega L}{\mathscr{V}_{\min}}\right) + \frac{\omega^2 L^2}{4\mathscr{V}_{\min}^2}\left(3 + \frac{\omega L}{\mathscr{V}_{\min}}\right)^2\right\}^{1/2}\|f\|_{0,\Omega}, \tag{B.4}$$

and

$$\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|_{0,\Omega} \leq \frac{L}{2}\left(3 + \frac{\omega L}{\mathscr{V}_{\min}}\right)\|f\|_{0,\Omega}. \tag{B.5}$$

In particular, Assumption 2.1 holds with $\sigma = 2$.

Proof. Using the identity $2\,\mathrm{Re}\,\phi\partial_{\mathbf{x}_j}\overline{\phi} = \partial_{\mathbf{x}_j}|\phi|^2$ that holds for sufficiently smooth complex valued functions ϕ, and integration by parts, we derive:

$$\begin{aligned}
2\,\mathrm{Re}\int_\Omega f\left((\mathbf{x}_2 + L)\frac{\partial \bar{u}}{\partial \mathbf{x}_2} + \frac{1}{2}\bar{u}\right) &= 2\,\mathrm{Re}\int_\Omega\left(-\frac{\omega^2}{\mathscr{V}^2}u - \Delta u\right)\left((\mathbf{x}_2 + L)\frac{\partial \bar{u}}{\partial \mathbf{x}_2} + \frac{1}{2}\bar{u}\right)\\
&= -2\omega^2\int_\Omega(\mathbf{x}_2 + L)\mathscr{V}^{-3}\frac{\partial \mathscr{V}}{\partial \mathbf{x}_2}|u|^2 + 2\int_\Omega\left|\frac{\partial u}{\partial \mathbf{x}_2}\right|^2\\
&\quad + \int_{\Gamma^h_{\rm Dir}}(\mathbf{x}_2 + L)\left|\frac{\partial u}{\partial \mathbf{x}_2}\right|^2 - \omega^2 L\int_{\Gamma_{\rm Diss}}\mathscr{V}^{-2}|u|^2.
\end{aligned}$$

Then, since $0 \leq \mathbf{x}_2 + L \leq L$ and $\partial_{\mathbf{x}_2}\mathscr{V} \leq 0$, we have

$$2\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|^2_{0,\Omega} \leq 2L\|f\|_{0,\Omega}\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|_{0,\Omega} + \|f\|_{0,\Omega}\|u\|_{0,\Omega} + \frac{\omega^2 L}{\mathscr{V}_{\min}}\int_{\Gamma_{\rm Diss}}\mathscr{V}^{-1}|u|^2,$$

and using (B.2), we obtain

$$2\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|^2_{0,\Omega} \leq 2L\|f\|_{0,\Omega}\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|_{0,\Omega} + \left(1 + \frac{\omega L}{\mathscr{V}_{\min}}\right)\|f\|_{0,\Omega}\|u\|_{0,\Omega}.$$

If we denote by $\tilde{u}$ the extension of u by 0 to $\tilde{\Omega} = (0,1)\times(0,-L)$, we have $\tilde{u} \in H^1(\tilde{\Omega})$ and $\tilde{u} = 0$ on $(0,1)\times\{-L\}$. For such a $\tilde{u}$ the Poincaré inequality

$$\|u\|_{0,\Omega} = \|\tilde{u}\|_{0,\tilde{\Omega}} \leq L\left\|\frac{\partial \tilde{u}}{\partial \mathbf{x}_2}\right\|_{0,\tilde{\Omega}} = L\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|_{0,\Omega}, \tag{B.6}$$

holds, so that

$$2\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|^2_{0,\Omega} \leq L\left(3 + \frac{\omega L}{\mathscr{V}_{\min}}\right)\|f\|_{0,\Omega}\left\|\frac{\partial u}{\partial \mathbf{x}_2}\right\|_{0,\Omega},$$

and (B.5) follows. In addition, (B.3) directly follows from (B.5) and (B.6). Finally, using (B.1), we have

$$\left\|\frac{\partial u}{\partial \mathbf{x}_1}\right\|^2 \leq \|\nabla u\|_{0,\Omega}^2 \leq \|f\|_{0,\Omega}\|u\|_{0,\Omega} + \frac{\omega^2}{\mathscr{V}_{min^2}}\|u\|_{0,\Omega}^2,$$

and (B.4) follows from (B.3). □

B.2 Non-trapping obstacles with smoothly decreasing wavenumber

Theorem B.2. *Assume that $\nabla\mathscr{V}\cdot\mathbf{x} \leq (1-\delta)\mathscr{V}$ for a fixed $\delta \in (0,1)$. Furthermore, assume that $\mathbf{x}\cdot\mathbf{n} \leq 0$ on Γ_{Dir} and $\mathbf{x}\cdot\mathbf{n} \geq \gamma|\mathbf{x}|$ on Γ_{Diss}, for some $\gamma > 0$. Then, Assumption 2.1 holds with $\sigma = 0$.*

Proof. Using Green's formula and Rellich's identity, we derive:

$$\begin{aligned}
2\operatorname{Re}\int_\Omega f\mathbf{x}\cdot\nabla\bar{u} &= 2\operatorname{Re}\int_\Omega\left(-\frac{\omega^2}{\mathscr{V}^2}u - \Delta u\right)\mathbf{x}\cdot\nabla\bar{u} \\
&= \omega^2\int_\Omega \nabla\cdot\left(\mathscr{V}^{-2}\mathbf{x}\right)|u|^2 + \int_{\Gamma_{\mathrm{Diss}}}|\nabla u|^2\mathbf{x}\cdot\mathbf{n} - \int_{\Gamma_{\mathrm{Dir}}}|\nabla u\cdot\mathbf{n}|^2\mathbf{x}\cdot\mathbf{n} \\
&\quad -\omega^2\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-2}|u|^2\mathbf{x}\cdot\mathbf{n} - 2\operatorname{Re} i\omega\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-1}u\mathbf{x}\cdot\nabla\bar{u}.
\end{aligned}$$

Then, we denote by

$$m = \inf_{\mathbf{x}\in\Gamma_{\mathrm{Diss}}}|\mathbf{x}|, \quad M = \sup_{\mathbf{x}\in\Omega}|\mathbf{x}|,$$

and we remark that $m > 0$. In addition, we have

$$\nabla\cdot\left(\mathscr{V}^{-2}\mathbf{x}\right) = 2\mathscr{V}^{-3}\left(\mathscr{V} - \nabla\mathscr{V}\cdot\mathbf{x}\right) \geq 2\delta\mathscr{V}^{-2} \geq 2\delta\mathscr{V}_{\max}^{-2},$$

by assumption on $\mathscr{V}$.

Recalling that $\mathbf{x}\cdot\mathbf{n} \leq 0$ on Γ_{Dir} and $\mathbf{x}\cdot\mathbf{n} \geq \gamma|\mathbf{x}| \geq \gamma m$ on Γ_{Diss}, it follows

$$\begin{aligned}
2\delta\mathscr{V}_{\max}^{-2}\omega^2\|u\|_{0,\Omega}^2 + \gamma m\|\nabla u\|_{0,\Gamma_{\mathrm{Diss}}}^2 \leq 2M\|f\|_{0,\Omega}|u|_{1,\Omega} + M\omega^2\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-2}|u|^2 \\
+2M\omega\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-1}|u||\nabla u|.
\end{aligned}$$

We then employ Young's inequality to get

$$2M\omega\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-1}|u||\nabla u| \leq \frac{M^2\omega^2}{\gamma m}\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-2}|u|^2 + \gamma m\int_{\Gamma_{\mathrm{Diss}}}|\nabla u|^2,$$

so that

$$\begin{aligned}
2\delta\mathscr{V}_{\max}^{-2}\omega^2\|u\|_{0,\Omega}^2 &\leq 2M\|f\|_{0,\Omega}|u|_{1,\Omega} + M\omega^2\left(1+\frac{M}{\gamma m}\right)\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-2}|u|^2 \\
&\leq 2M\|f\|_{0,\Omega}|u|_{1,\Omega} + \frac{M\omega^2}{\mathscr{V}_{\min}}\left(1+\frac{M}{\gamma m}\right)\int_{\Gamma_{\mathrm{Diss}}}\mathscr{V}^{-1}|u|^2.
\end{aligned}$$

$$
\begin{aligned}
&\leq 2M\|f\|_{0,\Omega}|u|_{1,\Omega} + \frac{M\omega}{\mathscr{V}_{\min}}\left(1+\frac{M}{\gamma m}\right)\|f\|_{0,\Omega}\|u\|_{0,\Omega},\\
&\leq \left(\frac{M^2}{\epsilon_1} + \frac{M^2\omega^2}{\mathscr{V}_{\min}^2\epsilon_2}\left(1+\frac{M}{\gamma m}\right)^2\right)\|f\|_{0,\Omega}^2 + \epsilon_1|u|_{1,\Omega}^2 + \epsilon_2\|u\|_{0,\Omega}^2,
\end{aligned}
$$

for all $\epsilon_1, \epsilon_2 > 0$, where we have used (B.2) and again Young's inequality. Furthermore, using (B.1), we have

$$
2\delta\mathscr{V}_{\max}^{-2}\omega^2\|u\|_{0,\Omega}^2 \leq \left(\frac{\epsilon_1\mathscr{V}_{\min}^2}{4\omega^2} + \frac{M^2}{\epsilon_1} + \frac{M^2\omega^2}{\mathscr{V}_{\min}^2\epsilon_2}\left(1+\frac{M}{\gamma m}\right)^2\right)\|f\|_{0,\Omega}^2 + \left(\frac{2\epsilon_1}{\omega^2}\mathscr{V}_{\min}^2 + \epsilon_2\right)\|u\|_{0,\Omega}^2,
$$

so that

$$
2\delta\mathscr{V}_{\max}^{-2}\left\{1 - \frac{\mathscr{V}_{\max}^2}{2\delta}\left(\frac{2\epsilon_1}{\omega^2}\mathscr{V}_{\min}^2 + \epsilon_2\right)\right\}\omega^2\|u\|_{0,\Omega}^2 \leq \left(\frac{\epsilon_1\mathscr{V}_{\min}^2}{4\omega^2} + \frac{M^2}{\epsilon_1} + \frac{M^2\omega^2}{\mathscr{V}_{\min}^2\epsilon_2}\left(1+\frac{M}{\gamma m}\right)^2\right)\|f\|_{0,\Omega}^2.
$$

We select

$$
\epsilon_1 = \frac{\omega_0^2\delta}{4\mathscr{V}_{\max}^2\mathscr{V}_{\min}^2}, \quad \epsilon_2 = \frac{\delta}{2\mathscr{V}_{\max}^2},
$$

so that

$$
\frac{\mathscr{V}_{\max}^2}{2\delta}\left(\frac{2\epsilon_1}{\omega^2}\mathscr{V}_{\min}^2 + \epsilon_2\right) \leq \frac{1}{2},
$$

and deduce that

$$
\omega\|u\|_{0,\Omega} \leq C(\Omega,\omega_0,\mathscr{V})\|f\|_{0,\Omega},
$$

with

$$
C(\Omega,\omega_0,\mathscr{V}) = \frac{\mathscr{V}_{\max}}{\sqrt{\delta}}\left(\frac{\epsilon_1\mathscr{V}_{\min}^2}{4\omega^2} + \frac{M^2}{\epsilon_1} + \frac{M^2\omega^2}{\mathscr{V}_{\min}^2\epsilon_2}\left(1+\frac{M}{\gamma m}\right)^2\right)^{1/2}.
$$

Then, it follows from (B.1) that

$$
|u|_{1,\Omega}^2 \leq C(\Omega,\omega_0,\mathscr{V})\|f\|_{0,\Omega}^2,
$$

and the result follows. □

B.3 Parabolic trapping obstacles

Inspired from [10], we here consider the case where the part of the boundary where we impose Dirichlet boundary condition corresponds to a parabolic trapping obstacles. More precisely, according to [10], Definition 1.1, for two fixed positive real numbers R_0, R_1 such that $R_0 < R_1$, we say that a domain Ω_- of $\mathbb{R}^2$ with boundary Γ is an (R_0, R_1) obstacle if

$$
Z(x)\cdot n_-(x) \geq 0, \text{ for almost all } x \in \Gamma_{\mathrm{Dir}},
$$

where n_- is the unit outward normal vector in Ω_- along Γ and the multiplier Z is defined by

$$Z(x) = \chi(r)(0,1)^\top x_2 + (1-\chi(r))x, \quad \forall x \in \mathbb{R}^2,$$

where r is the distance from x to 0 and χ is a $C^{1,1}$ cut-off function defined by

$$\chi(r) = \frac{1}{2}\left(1 + \psi\left(\frac{2r-(R_0+R_1)}{R_1-R_0)}\right)\right), \quad \forall r \geq 0,$$

the function ψ being defined by

$$\psi(t) = \begin{cases} (1-t)^3 - 1 & \text{if } 0 \leq t \leq 1, \\ -1 & \text{if } t > 1, \\ -\psi(-t) & \text{if } t < 0. \end{cases}$$

Examples of (R_0, R_1) obstacles from [10] are drawn in Figure 1.

In accordance with Theorem 1.7 of [10], we prove the next stability property.

Theorem B.3. *Let Ω_- be a (R_0, R_1) obstacle such that $R_1 \geq 121 R_0$ and fix $R > R_1$ large enough such that $\bar{\Omega}_- \subset B(0,R)$. Let O be any star-shaped domain (with respect to the origin) such that $B(0,R) \subset O$ and consider problem (2.1) in $\Omega = O \setminus \bar{\Omega}_-$ with $\Gamma_{\text{Diss}} = \partial O \setminus \Gamma$, $\Gamma_{\text{Dir}} = \Gamma$, and $\mathscr{V} = 1$. Then the stability estimate (2.3) holds with $\sigma = 2$.*

Proof. First, similarly to [10], Lemma 3.1, we use the Morawetz-type identity (see [10], Lem. 2.1) to the solution u of problem (2.1) in Ω with

$$\mathcal{Z}u = Z \cdot \nabla u - i\omega R u + \alpha u,$$

and

$$2\alpha = \nabla Z - \frac{1}{48}(1-\chi(r)).$$

This yields (compare with the identity (3.6) from [10])

$$\begin{aligned} &\operatorname{Re} \int_\Omega \left[2\partial_i Z_j \partial_i u \partial_j \bar{u} + 2\bar{u}\nabla\alpha \cdot \nabla u - (2\alpha - \nabla \cdot Z)(\omega^2|u|^2 - \nabla u|^2)\right] \mathrm{d}x + \int_\Gamma (Z\cdot n)|\partial_n u|^2 \,\mathrm{d}\sigma \\ &\quad = 2\operatorname{Re}\int_\Omega \overline{\mathcal{Z}u} f \mathrm{d}x + I_{\text{Diss}}, \end{aligned} \tag{B.7}$$

where

$$I_{\text{Diss}} = \int_{\Gamma_{\text{Diss}}} \left[(Z\cdot n)(2\omega^2|u|^2 - |\nabla_S u|^2) + 2\omega^2 R|u|^2 - 2\operatorname{Re}(Z\cdot\nabla_S\bar{u} + \alpha\bar{u})i\omega u\right] \,\mathrm{d}\sigma.$$

We now estimate the right-hand side of this identity. Indeed by our assumption on O, there exist four positive constants C_0, C_1, C_2 and C_3 such that

$$I_{\text{Diss}} \leq \int_{\Gamma_{\text{Diss}}} \left[(2\omega^2(R+C_0) + C_1\omega)|u|^2 - C_2|\nabla_S u|^2 + 2C_3\omega|\nabla_S u||u|\right] \,\mathrm{d}\sigma.$$

Hence by Young's inequality we find that

$$I_{\text{Diss}} \leq \int_{\Gamma_{\text{Diss}}} \left[(2\omega^2(R+C_0)+C_1\omega)|u|^2 - C_2|\nabla_S u|^2 + \varepsilon C_3\omega|\nabla_S u|^2 + \frac{1}{\varepsilon}C_3\omega|u|^2\right] \mathrm{d}\sigma,$$

for all $\varepsilon > 0$. Chosing ε such that $\varepsilon C_3\omega = C_2$, we obtain

$$I_{\text{Diss}} \leq \int_{\Gamma_{\text{Diss}}} \left[2\omega^2(R+C_0)+C_1\omega + \frac{C_3^2\omega^2}{C_2}\right]|u|^2 \,\mathrm{d}\sigma.$$

Hence for ω large enough, we have proved that

$$I_{\text{Diss}} \leq C_4\omega^2 \int_{\Gamma_{\text{Diss}}} |u|^2 \,\mathrm{d}\sigma,$$

for some positive constant C_4 independent of ω. Now the definition of $\mathcal{Z}$ leads to

$$2\,\mathrm{Re}\int_0 \overline{\mathcal{Z}u} f \,\mathrm{d}x \leq C_5\|f\|_\Omega(\omega\|u\|_\Omega + |u|_{1,\Omega}), \tag{B.8}$$

for some positive constant C_5 independent of ω. Now exactly as in the proof of Lemma 3.4 of [10], we have

$$\begin{aligned}&\mathrm{Re}\int_0 \left[2\partial_i Z_j \partial_i u \partial_j \bar{u} + 2\bar{u}\nabla\alpha\cdot\nabla u - (2\alpha - \nabla\cdot Z)(\omega^2|u|^2 - \nabla u|^2)\right]\mathrm{d}x + \int_\Gamma (Z\cdot n)|\partial_n u|^2 \,\mathrm{d}\sigma \\ &\geq \frac{1}{96R_0^2}\int_\Omega |u|^2 \,\mathrm{d}x,\end{aligned}$$

for ω large enough. This estimate and (B.8) in (B.7) allow to obtain

$$\|u\|_\Omega^2 \leq C_6(\omega\|u\|_\Omega + |u|_{1,\Omega})\|f\|_\Omega,$$

for some positive constant C_6 independent of ω. By (B.1), the previous estimate becomes

$$\|u\|_\Omega^2 \leq C_7(\omega\|u\|_\Omega + \|f\|_\Omega)\|f\|_\Omega,$$

for $\omega \geq 1$ and some positive constant C_7 independent of ω. Young's inequality then yields

$$\frac{1}{2}\|u\|_\Omega^2 \leq \left(C_7 + \frac{C_7^2\omega^2}{2}\right)\|f\|_\Omega^2.$$

This leads to the conclusion due to (B.1). $\square$

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