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ERROR ANALYSIS OF A FVEM FOR FRACTIONAL ORDER EVOLUTION EQUATIONS WITH NONSMOOTH INITIAL DATA[☆]

Samir Karaa[1] and Amiya K. Pani[2,*]

Abstract. In this paper, a finite volume element (FVE) method is considered for spatial approximations of time fractional diffusion equations involving a Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ in time. Improving upon earlier results [Karaa *et al.*, *IMA J. Numer. Anal.* **37** (2017) 945–964], error estimates in $L^2(\Omega)$- and $H^1(\Omega)$-norms for the semidiscrete problem with smooth and mildly smooth initial data, *i.e.*, $v \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$ are established. For nonsmooth data, that is, $v \in L^2(\Omega)$, the optimal $L^2(\Omega)$-error estimate is shown to hold only under an additional assumption on the triangulation, which is known to be satisfied for symmetric triangulations. Superconvergence result is also proved and as a consequence, a quasi-optimal error estimate is established in the $L^\infty(\Omega)$-norm. Further, two fully discrete schemes using convolution quadrature in time generated by the backward Euler and the second-order backward difference methods are analyzed, and error estimates are derived for both smooth and nonsmooth initial data. Based on a comparison of the standard Galerkin finite element solution with the FVE solution and exploiting tools for Laplace transforms with semigroup type properties of the FVE solution operator, our analysis is then extended in a unified manner to several time fractional order evolution problems. Finally, several numerical experiments are conducted to confirm our theoretical findings.

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1. Introduction

Let Ω be a bounded, convex polygonal domain in $\mathbb{R}^2$ with a boundary $\partial\Omega$, $T > 0$, and let v be a given function (initial data) defined on Ω. We now consider the following time fractional diffusion problem: find u in $\Omega \times (0,T]$ such that

$$u'(x,t) + \partial_t^{1-\alpha} Au(x,t) = 0, \quad \text{in } \Omega \times (0,T], \tag{1.1a}$$

$$u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,T], \tag{1.1b}$$

$$u(x,0) = v(x), \quad \text{in } \Omega, \tag{1.1c}$$

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[1] Department of Mathematics and Statistics, Sultan Qaboos University, Al-Khod 123, Muscat, Oman.
[2] Department of Mathematics, Industrial Mathematics Group, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India.

[*] Corresponding author: akp@math.iitb.ac.in

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where $Au = -\Delta u$, u' is the partial derivative of u with respect to time, and $\partial_t^{1-\alpha} := {}^R\mathrm{D}^{1-\alpha}$ is the Riemann-Liouville fractional derivative in time defined for $0 < \alpha < 1$ by:

$$\partial_t^{1-\alpha}\varphi(t) := \frac{\mathrm{d}}{\mathrm{dt}}\mathcal{I}^{\alpha}\varphi(t) := \frac{\mathrm{d}}{\mathrm{dt}}\int_0^t \omega_\alpha(t-s)\varphi(s)\,\mathrm{d}s, \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{1.2}$$

Here, $\mathcal{I}^\alpha$ denotes the temporal Riemann-Liouville fractional integral operator of order α. This class of problems describes the model of an anomalous subdiffusion, see [9, 10, 27].

Over the last two decades, considerable attention from both practical and theoretical points of view has been given to fractional diffusion models due to their various applications. Several numerical techniques for the problem (1.1) have been proposed with different types of spatial discretizations. Although the numerical study of (1.1) has been discussed in a large number of papers, optimal error estimates with respect to the smoothness of the solution expressed through the initial data have been established only in few papers recently. This is mainly due to the limited smoothing properties of the problem, and hence, obtaining sharp error bounds under reasonable regularity assumptions on the exact solution has become a challenging task. In the literature, the finite element method (FEM) has, in particular, been given a special attention in approximating the solution of (1.1), see [2, 11–13, 16, 24–26, 28] and references, therein. Most recently, a FVE method was analyzed in [14] and *a priori* error estimates with respect to data regularity have been derived.

In the context of FEM, we begin by recalling some facts on the spatially semidiscrete standard Galerkin FE method for the problem (1.1) in the piecewise FE element space

$$V_h = \{\chi \in C^0(\overline{\Omega}) \ : \ \chi|_K \text{ is linear for all } K \in \mathcal{T}_h \text{ and } \chi|_{\partial\Omega} = 0\},$$

where $\{\mathcal{T}_h\}_{0<h<1}$ is a family of regular triangulations $\mathcal{T}_h$ of the domain Ω into triangles K and h is the maximum diameter of the triangles $K \in \mathcal{T}_h$. With $a(\cdot,\cdot)$ denoting the bilinear form associated with the operator A, and $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, the semidiscrete Galerkin FE method is to seek $u_h(t) \in V_h$ satisfying

$$(u_h', \chi) + a(\partial_t^{1-\alpha}u_h, \chi) = 0, \quad \forall \chi \in V_h, \quad t \in (0,T], \quad u_h(0) = v_h, \tag{1.3}$$

where $a(v,w) := (\nabla v, \nabla w)$ and $v_h \in V_h$ is an approximation of the initial data v. In [24], McLean and Thomée have established the following estimate for the Galerkin FE approximation to (1.1) using Laplace transformation technique: with $v_h = P_h v$, there holds for $t > 0$

$$\|u_h(t) - u(t)\| \le Ch^2 t^{-\alpha(2-q)/2}|v|_q, \quad 0 \le q \le 2, \tag{1.4}$$

where $\|v\|$ is the $L^2(\Omega)$-norm of v, $|v|_q = \|A^{q/2}v\|$ is a weighted norm defined on the space $\dot{H}^q(\Omega)$ to be described in Section 2 and $P_h : L^2(\Omega) \to V_h$ is the L^2-projection given by : $(P_h v - v, \chi) = 0$ for all $\chi \in V_h$. The estimate (1.4) extends results obtained for the standard parabolic problem, *i.e.*, $\alpha = 1$, which has been thoroughly studied, see [29]. In the recent work [2], an approach based on Laplace transform and semigroup type theory has been exploited to derive *a priori* error estimate of the type (1.4), and most recently, a delicate energy analysis has been developed in [15] to obtain similar estimate for the FE solution.

Regarding the optimal estimate in the gradient norm, the following result

$$\|\nabla(u_h(t) - u(t))\| \le Cht^{-\alpha(2-q)/2}|v|_q, \quad 0 \le q \le 2, \tag{1.5}$$

holds with $v_h = P_h v$ on quasi-uniform meshes. For the cases $q = 1, 2$, one can also choose $v_h = R_h v$, where $R_h : H_0^1(\Omega) \to V_h$ is the standard Ritz projection defined by the relation:

$$a(R_h v - v, \chi) = 0, \quad \forall \chi \in V_h. \tag{1.6}$$

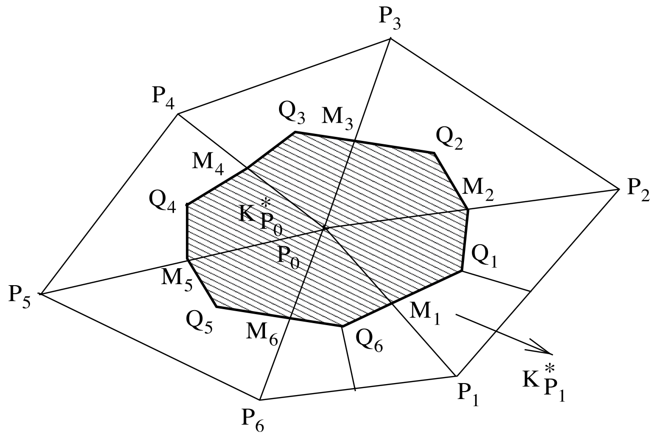


FIGURE 1. Control volume for interior node.

However, without the quasi-uniformity assumption on the mesh, the estimate (1.5) remains valid only for $0 \leq q \leq 1$, see [15]. Subsequently, optimal convergence rate up to a logarithmic factor in the stronger $L^\infty(\Omega)$-norm has been derived in [15, 25]. In [15], the following $L^\infty(\Omega)$-error estimate

$$\|u(t) - u_h(t)\|_{L^\infty(\Omega)} \leq C|\ln h|^{\frac{5}{2}} h^2 t^{-\alpha(3-q)/2}(|v|_q + \|v\|_{L^\infty(\Omega)}), \quad 1 \leq q \leq 2. \tag{1.7}$$

is established for $v \in \dot{H}^q(\Omega) \cap L^\infty(\Omega)$ and $v_h = P_h v$ on quasi-uniform meshes.

The main contribution of this work is to establish optimal (with respect to data regularity) error estimates for the FVE discretization of problem (1.1), and thus, provide improvements of the results derived in [14]. The FVM is very popular in engineering literature due its local conservation property, its flexibility in tackling domains with complex boundaries, and more importantly due to its easy implementation on both structured and unstructured meshes. For a review article, see, Lin *et al.* [18]. In this paper, the choice of the FV method for the problem under consideration is as used in Chatzipantelidis *et al.* [3], Ewing *et al.* [8], and Chou and Li [6] for linear parabolic problems. To describe the finite volume element formulation, we first introduce the dual mesh on the domain Ω. Let N_h be the set of nodes or vertices, that is,

$$N_h := \left\{P_i : P_i \quad \text{is a vertex of the element} \quad K \in \mathcal{T}_h \text{ and } P_i \in \overline{\Omega}\right\}$$

and let N_h^0 be the set of interior nodes in $\mathcal{T}_h$. Further, let $\mathcal{T}_h^*$ be the dual mesh associated with the primary mesh $\mathcal{T}_h$, which is defined as follows. With P_0 as an interior node of the triangulation $\mathcal{T}_h$, let P_i $(i = 1, 2, \ldots, m)$ be its adjacent nodes (see, Fig. 1 with $m = 6$). Let $M_i,\ i = 1, 2, \ldots, m$ denote the midpoints of $\overline{P_0 P_i}$ and let $Q_i,\ i = 1, 2, \ldots, m$, be the barycenters of the triangle $\triangle P_0 P_i P_{i+1}$ with $P_{m+1} = P_1$. The *control volume* $K^*_{P_0}$ is constructed by joining successively $M_1,\ Q_1, \ldots,,\ M_m,\ Q_m,\ M_1$. With Q_i $(i = 1, 2, \ldots, m)$ as the nodes of *control volume* $K^*_{p_i}$, let N_h^* be the set of all dual nodes Q_i. For a boundary node P_1, the control volume $K^*_{P_1}$ is shown in Figure 1. Note that the union of the control volumes forms a partition $\mathcal{T}_h^*$ of $\overline{\Omega}$.

The dual FVE space V_h^* on the dual mesh $\mathcal{T}_h^*$ is defined as

$$V_h^* = \{\chi \in L^2(\Omega) \ : \ \chi|_{K^*_{P_0}} \text{ is constant for all } K^*_{P_0} \in \mathcal{T}_h^* \text{ and } \chi|_{\partial\Omega} = 0\}.$$

The semidiscrete FVE formulation for (1.1) is to seek $\bar{u}_h(t) \in V_h$ such that

$$(\bar{u}_h', \chi) + a_h(\partial_t^{1-\alpha} \bar{u}_h, \chi) = 0, \quad \forall \chi \in V_h^*, \quad t > 0, \quad \bar{u}_h(0) = v_h, \tag{1.8}$$

where the bilinear form $a_h(\cdot,\cdot) : V_h \times V_h^* \longrightarrow \mathbb{R}$ is defined by

$$a_h(\psi, \chi) = -\sum_{P_i \in N_h^0} \chi(P_i) \int_{\partial K_{P_i}^*} \nabla\psi \cdot \mathbf{n}\, \mathrm{d}s, \quad \forall \psi \in V_h,\ \chi \in V_h^* \tag{1.9}$$

with $\mathbf{n}$ denoting the outward unit normal to the boundary of the control volume $K_{P_i}^*$.

The error equations associated with the proposed FV method involves a perturbation term, which is not easy to handle. In [14], we employed the FV elliptic projection to get rid of this term, but for this choice, high regularity assumptions on v are imposed. For instance, the following $L^2(\Omega)$-error estimate

$$\|\bar{u}_h(t) - u(t)\| \leq C\, h^2 \left(\|v\|_{H^3(\Omega)} + \int_0^t \|u'(s)\|_{H^3(\Omega)}\, \mathrm{d}s \right),$$

has been established requiring that $v \in \dot{H}^q(\Omega)$ with $q \geq 3$. The main objective of this study is then to improve the results derived in [14] and establish optimal estimates with respect to data regularity for the solution of the FVE semidiscrete problem (1.8). These improvements are made possible by combining known error estimates for the standard Galerkin FE solution stated above with new bounds for the difference $\xi(t) = \bar{u}_h(t) - u_h(t)$. Introduced in [4, 5], this approach is based on phrasing the FVE problem into a self-adjoint one and deriving an equation for $\xi(t)$. Then, by choosing an appropriate representation of $\xi(t)$, we establish the following error estimate for $0 < \alpha < 1$

$$\|\bar{u}_h(t) - u(t)\| + h\|\nabla(\bar{u}_h(t) - u(t))\| \leq Ch^2 t^{-\alpha(2-q)/2} |v|_q, \quad 0 \leq q \leq 2. \tag{1.10}$$

We shall derive this estimate for $q = 1$, 2 in Section 4.1 and for $q = 0$ in Section 4.2. For the latter case, we are only able to prove the *a priori* estimate under an additional hypothesis on $\mathcal{T}_h$, which is known to be satisfied for symmetric triangulations. Without any such condition, only sub-optimal order convergence is obtained, which is similar to the result proved in [5] for linear parabolic problems. For the stronger $L^\infty(\Omega)$-norm, a quasi-optimal error estimate analogous to (1.7) is established for $1 \leq q \leq 2$.

Our second objective is to analyze two fully discrete schemes for the semidiscrete problem (1.8) based on convolution quadrature in time generated by the backward Euler and the second-order backward difference methods. Error estimates with respect to the data regularity are provided in Theorems 5.4 and 5.7. For instance, it is shown that the discrete solution U_h^n obtained by the backward Euler method with a time step size τ satisfies the following *a priori* error estimate

$$\|U_h^n - \bar{u}_h(t_n)\| \leq C(\tau t_n^{-1+\alpha q/2} + h^2 t_n^{-\alpha(1-q/2)}) |v|_q, \quad q = 0, 1, 2, \ \ 0 < \alpha < 1.$$

When $q = 0$, an additional restriction on the triangulation is imposed. A similar type of error bound is shown to hold for the second-order backward difference scheme in Section 5.2.

Our third objective is to generalize our results on FVE method for both smooth and nonsmooth initial data to other classes of fractional order evolution equations in Section 6. Say for example, we can extend our FVE analysis to the following class of time fractional problems:

$$u'(x,t) + \mathcal{J}^\alpha A u(x,t) = 0, \quad \text{in } \Omega \times (0,T], \tag{1.11}$$

with homogeneous Dirichlet boundary conditions and initial condition $u(x, 0) = v(x)$ for $x \in \Omega$. When $\mathcal{J}^\alpha = \mathcal{I}^\alpha$, this class of problems is known as fractional diffusion-wave equation or evolution equation with positive memory, see [22, 24], and references, therein. The case $\mathcal{J}^\alpha = I + \mathcal{I}^\alpha$ corresponds to the partial integro-differential equation with singular kernel, refer to [23]. Now if $\mathcal{J}^\alpha = I + \partial_t^{1-\alpha}$, then this class of problems is known as the Rayleigh-Stokes problems for generalized second grade fluid, see [2]. Even our FVE analysis can be directly applied to the following time fractional order diffusion problem:

$$ {}^{C}\partial_t^{\alpha} u(x,t) + Au(x,t) = 0, \tag{1.12} $$

where ${}^{C}\partial_t^{\alpha} v(t) := \mathcal{I}^{1-\alpha} v'(t)$ is the fractional Caputo derivative of order $0 < \alpha < 1$. For the semidiscrete FE analysis of (1.12), we refer to Jin *et al.* [12]. The unifying analysis of all these classes of evolution problems is based on comparing the FVE solution with the corresponding FE solution and exploiting the Laplace transform technique along with semigroup type properties of the FVE solution operator. To the best of our knowledge, the FV error analysis of either problem (1.11) or (1.12) is discussed for the first time in this article.

The rest of the paper is organized as follows. In the next section, we introduce notation, recall the solution representation for the continuous problem (1.1) and some smoothing properties of the solution operator, which play an important role in our subsequent error analysis. Section 3 deals with a brief description of the spatially semidiscrete FVE scheme and their properties. In Section 4, we derive error estimates for the semidiscrete FVE scheme for smooth and nonsmooth initial data $v \in \dot{H}^q$, $q = 0, 1, 2$ in Sections 4.1 and 4.2. For $q = 0$, *i.e.*, $v \in L^2(\Omega)$, we show an optimal error bound under an additional assumption on the triangulation. Superconvergence result is proved in Section 4.3 and as a consequence, a quasi-optimal error estimate is established in the $L^\infty(\Omega)$-norm. In Section 5, two fully discrete schemes based on convolution quadrature approximation of the fractional derivative are presented and error estimates are established. Section 6 focuses on possible generalization of the present FVE error analysis to various types of time fractional evolution problems. Finally, in Section 7, we present numerical results to confirm our theoretical findings.

Throughout the paper, C denotes a generic positive constant that may depend on α and T, but is independent of the spatial mesh element size h and the time step τ.

2. Representation of exact solution and properties

We first introduce some notations. Let $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ be the Dirichlet eigenpairs of the selfadjoint and positive definite operator A, with $\{\phi_j\}_{j=1}^{\infty}$ being an orthonormal basis in $L^2(\Omega)$. For $r \geq 0$, we denote by $\dot{H}^r(\Omega) \subset L^2(\Omega)$ the Hilbert space induced by the norm

$$ |v|_r^2 = \|A^{r/2} v\|^2 = \sum_{j=1}^{\infty} \lambda_j^r (v, \phi_j)^2, $$

with $(\cdot,\cdot)$ being the inner product on $L^2(\Omega)$. Then, it follows that $\dot{H}^r(\Omega) = \{\chi \in H^r(\Omega);\, A^j\chi = 0 \text{ on } \partial\Omega, \text{ for } j < r/2\}$, see Lemma 3.1 of [29]. In particular, $|v|_0 = \|v\|$ is the norm on $L^2(\Omega)$, $|v|_1 = \|\nabla v\|$ is also the norm on $H_0^1(\Omega)$ and $|v|_2 = \|Av\|$ is the equivalent norm in $H^2(\Omega) \cap H_0^1(\Omega)$. Note that $\{\dot{H}^r(\Omega)\}$, $r \geq 0$, form a Hilbert scale of interpolation spaces. Motivated by this, we denote by $\|\cdot\|_{H_0^r(\Omega)}$ the norm on the interpolation scale between $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$ for r in the interval $[0, 2]$. Then, the $\dot{H}^r(\Omega)$ and $H_0^r(\Omega)$ norms are equivalent for any $r \in (1/2, 2]$ and for $r \in [0, 1/2]$, $\dot{H}^r(\Omega) = H^r(\Omega)$ by interpolation.

For $\delta > 0$ and $\theta \in (\pi/2, \pi)$, we introduce the contour $\Gamma_{\theta,\delta} \subset \mathbb{C}$ defined by

$$ \Gamma_{\theta,\delta} = \{\rho e^{\pm i\theta} : \rho \geq \delta\} \cup \{\delta e^{i\psi} : |\psi| \leq \theta\}, $$

oriented with an increasing imaginary part. Further, we denote by Σ_θ the sector

$$ \Sigma_\theta = \{z \in \mathbb{C},\, z \neq 0,\, |\arg z| < \theta\}. $$

For $z \in \Sigma_\theta$, it is clear that $z^\alpha \in \Sigma_\theta$ as $\alpha \in (0,1)$. Since the operator A is selfadjoint and positive definite, its resolvent $(z^\alpha I + A)^{-1} : L^2(\Omega) \to L^2(\Omega)$ satisfies the bound

$$\|(z^\alpha I + A)^{-1}\| \le M_\theta |z|^{-\alpha}, \quad \forall z \in \Sigma_\theta, \tag{2.1}$$

where $M_\theta = 1/\sin(\pi - \theta)$. We now make use of the Laplace transform $\hat{u} := \mathcal{L}(u)$ of the solution u defined by

$$\hat{u}(x,z) = \int_0^\infty e^{-zt} u(x,t)\,\mathrm{d}t.$$

The boundary condition $u(x,t) = 0$ on $\partial\Omega$ transforms into $\hat{u}(x,z) = 0$ on $\partial\Omega$. Taking Laplace transforms in (1.1a), we, then, arrive at

$$(zI + z^{1-\alpha}A)\hat{u}(z) = v, \tag{2.2}$$

and hence,

$$\hat{u}(z) = \hat{E}(z)v, \quad \hat{E}(z) := z^{\alpha-1}(z^\alpha I + A)^{-1}. \tag{2.3}$$

In view of (2.1) and (2.3), $\hat{E}(z)$ satisfies the following bound

$$\|\hat{E}(z)\| \le M_\theta |z|^{-1}, \quad \forall z \in \Sigma_\theta. \tag{2.4}$$

From (2.3), the Laplace inversion formula yields an integral representation for the solution of (1.1) as

$$u(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{zt}\hat{E}(z)v\,\mathrm{d}z, \quad t > 0, \tag{2.5}$$

where the contour of integration $\mathcal{C}$, known as Bromwich contour, is any line in the right-half plane parallel to the imaginary axis and with $\mathrm{Im}z$ increasing. Since $\hat{E}(z)$ is analytic in Σ_θ and satisfies the bound (2.4), the path of integration may, therefore, be deformed into the curve $\Gamma_{\theta,\delta}$ so that the integrand has an exponential decay property.

In the next lemma, we present some smoothing properties of the operator $\hat{E}(z)$ which play a key role in our error analysis. The estimates are proved for instance in [7], Lemma 2.2. Note that the first estimate (2.6) given below is obtained by interpolation technique.

Lemma 2.1. *The following estimates hold:*

$$\|A\hat{E}(z)\chi\| \le C_\theta |z|^{\alpha(1-p/2)-1}|\chi|_p, \quad \forall z \in \Sigma_\theta, \quad 0 \le p \le 2, \tag{2.6}$$
$$\|\nabla\hat{E}(z)\chi\| \le C_\theta |z|^{\alpha/2-1}\|\chi\|, \quad \forall z \in \Sigma_\theta, \tag{2.7}$$

where C_θ depends only on θ.

In the next section, we introduce the semidiscrete finite volume element scheme.

3. Semidiscrete FVE scheme and its properties

In this section, we first recall the semidiscrete FVE scheme (1.8) and discuss some associated properties. Now, a use of Green's formula applied to (1.9) yields for $w \in H^2(\Omega)$ and $\chi \in V_h^*$

$$(Aw, \chi) = a_h(w, \chi).$$

To rewrite the Petrov-Galerkin method (1.8) as a Galerkin method in V_h, we introduce the interpolation operator $\Pi_h^*: C^0(\bar{\Omega}) \longrightarrow V_h^*$ by

$$\Pi_h^* \chi = \sum_{P_i \in N_h^0} \chi(P_i)\eta_i(x),$$

where η_i is the characteristic function of the control volume $K_{P_i}^*$. The operator Π_h^* is selfadjoint and positive definite, see [6], and hence, the following relation

$$(\psi, \chi)_h = (\psi, \Pi_h^* \chi), \quad \forall \psi, \chi \in V_h$$

defines an inner product on V_h. Also, the corresponding norm $(\chi, \chi)_h^{1/2}$ is equivalent to the $L^2(\Omega)$-norm on V_h, uniformly in h, see [17]. Furthermore, from the following identity [1, 8]

$$a_h(\chi, \Pi_h^* v) = (\nabla \chi, \nabla v), \quad \forall \chi, v \in V_h,$$

the bilinear form $a_h(.,.)$ is symmetric and $a_h(\chi, \Pi_h^* \chi) = \|\nabla \chi\|^2$ for $\chi \in V_h$.

With this notation, the Petrov-Galerkin method (1.8) can be rewritten in the Galerkin form as

$$(\bar{u}_h', \chi)_h + a(\partial_t^{1-\alpha} \bar{u}_h, \chi) = 0, \quad \forall \chi \in V_h, \quad t > 0, \quad \bar{u}_h(0) = v_h. \tag{3.1}$$

We now introduce the discrete operator $\bar{A}_h : V_h \to V_h$ corresponding to the inner product $(\cdot, \cdot)_h$ by

$$(\bar{A}_h \psi, \chi)_h = (\nabla \psi, \nabla \chi), \quad \forall \psi, \chi \in V_h.$$

Then, the FVE method (3.1) is written in an operator form as

$$\bar{u}_h'(t) + \partial_t^{1-\alpha} \bar{A}_h \bar{u}_h(t) = 0, \quad t > 0, \quad \bar{u}_h(0) = v_h. \tag{3.2}$$

An appropriate modification of arguments in [5, 12] yields the following discrete analogue of Lemma 2.1 and therefore, we skip the proof.

Lemma 3.1. *Let $\hat{E}_h(z) = z^{\alpha-1}(z^\alpha I + \bar{A}_h)^{-1}$. With $\chi \in V_h$, the following estimates hold:*

$$\|\bar{A}_h \hat{E}_h(z)\chi\| \le C_\theta |z|^{\alpha(1-p/2)-1} \, \|\bar{A}_h^{p/2} \chi\|, \quad \forall z \in \Sigma_\theta, \quad 0 \le p \le 2, \tag{3.3}$$

$$|\hat{E}_h(z)\chi|_1 \le C_\theta |z|^{\alpha/2-1} \|\chi\|, \quad \forall z \in \Sigma_\theta, \tag{3.4}$$

where C_θ is independent of the mesh size h.

In the context of FEM, we introduce the discrete operator $A_h : V_h \to V_h$ defined by

$$(A_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall \psi, \chi \in V_h,$$

then the semidiscrete FE scheme (1.3) is rewritten in an operator form as

$$u_h'(t) + \partial_t^{1-\alpha} A_h u_h(t) = 0, \quad t > 0, \quad u_h(0) = v_h. \tag{3.5}$$

The analogue of Lemma 3.1 holds then for $\hat{F}_h(z) := z^{\alpha-1}(z^\alpha I + A_h)^{-1}$, when we replace $\hat{E}_h(z)$ in Lemma 3.1 by $\hat{F}_h(z)$.

4. Error analysis

This section deals with *a priori* optimal error estimates for the semidiscrete FVE scheme (3.1) with initial data $v \in \dot{H}^q(\Omega)$, $q = 0, 1, 2$. To do so, we first introduce the quadrature error $Q_h : V_h \to V_h$ defined by

$$(\nabla Q_h \chi, \nabla \psi) = \epsilon_h(\chi, \psi) := (\chi, \psi)_h - (\chi, \psi), \quad \forall \psi \in V_h. \tag{4.1}$$

The operator Q_h, introduced in [4] for the lumped mass FE element, represents the quadrature error in a special way. It satisfies the following error estimates, see [4, 5].

Lemma 4.1. *Let Q_h be defined by (4.1). Then, there holds*

$$\|\nabla Q_h \chi\| + h\|\bar{A}_h Q_h \chi\| \le Ch^{p+1}\|\nabla^p \chi\|, \quad \forall \chi \in V_h, \quad p = 0, 1. \tag{4.2}$$

Note that, by Lemma 4.1, and without additional assumptions on the mesh, the following estimate holds:

$$\|Q_h \chi\| \le C\|\nabla Q_h \chi\| \le Ch\|\chi\|, \quad \forall \chi \in V_h.$$

This estimate cannot be improved in general, see [4, 5] for some counter examples. However, on some special meshes, one can derive a better approximation. For instance, if the mesh is symmetric (see [4, 5] for the definition and examples), the operator Q_h is shown to satisfy

$$\|Q_h \chi\| \le Ch^2\|\chi\|, \quad \forall \chi \in V_h. \tag{4.3}$$

To derive optimal error estimates for the FVE solution $\bar{u}_h$, we split the error $\bar{e}(t) := \bar{u}_h(t) - u(t)$ into $\bar{e}(t) := (u_h(t) - u(t)) + \xi(t)$, where $\xi(t) = \bar{u}_h(t) - u_h(t)$ and u_h being the standard Galerkin FE solution. Then, from the definitions of $u_h(t)$, $\bar{u}_h(t)$ and Q_h, $\xi(t)$ satisfies

$$\xi_t(t) + \partial_t^{1-\alpha} \bar{A}_h \xi(t) = -\bar{A}_h Q_h u_{ht}(t), \quad t > 0, \quad \xi(0) = 0. \tag{4.4}$$

4.1. Error estimates for smooth initial data

In the following theorem, optimal error estimates are derived for smooth initial data $v \in \dot{H}^q(\Omega)$ with $q \in [1, 2]$.

Theorem 4.2. *Let u and $\bar{u}_h$ be the solutions of (1.1) and (3.1), respectively, with $v \in \dot{H}^q(\Omega)$ for $q \in [1, 2]$ and $v_h = R_h v$, where $R_h v$ is defined by (1.6). Then, there is a positive constant C, independent of h, such that*

$$\|\bar{u}_h(t) - u(t)\| + h\|\nabla(\bar{u}_h(t) - u(t))\| \le C\, t^{-\alpha(2-q)/2}\, h^2 |v|_q, \quad t > 0. \tag{4.5}$$

Proof. Since the estimates for $u_h - u$ are given in (1.4) and (1.5), it is sufficient to show

$$\|\xi(t)\| + h\|\nabla \xi(t)\| \le C\, t^{-\alpha(2-q)/2}\, h^2 |v|_q, \;\; q \in [1, 2]. \tag{4.6}$$

By taking Laplace transforms in (4.4) and following the analysis in Section 2, we represent $\xi(t)$ by

$$\xi(t) = -\frac{1}{2\pi i} \int_\Gamma e^{zt} \hat{E}_h(z) \bar{A}_h Q_h \widehat{u_{ht}}(z)\, \mathrm{d}z. \tag{4.7}$$

Here and also throughout this article, Γ is the particular contour chosen as $\Gamma = \Gamma_{\theta,\delta}$ with $\delta = 1/t$. From (4.7), it follows that

$$\|\xi(t)\| + h\|\nabla \xi(t)\| \le \frac{1}{2\pi} \int_\Gamma |e^{zt}| \Big(\|\hat{E}_h(z) \bar{A}_h Q_h \widehat{u_{ht}}(z)\| + h\|\nabla \hat{E}_h(z) \bar{A}_h Q_h \widehat{u_{ht}}(z)\| \Big)\, |\mathrm{d}z|. \tag{4.8}$$

To complete the proof of the estimate, we need to bound the terms under the integral sign on the right of side of (4.8). Now, we discuss two cases for $q = 2$ and $q = 1$ separately.

When $q = 2$, that is, $v \in \dot{H}^2(\Omega)$, apply (3.3) with $p = 1$ and (3.4) in Lemma 3.1 to obtain

$$\|\hat{E}_h(z)\bar{A}_h Q_h \widehat{u_{ht}}(z)\| \leq C|z|^{\alpha/2-1}\|\nabla Q_h \widehat{u_{ht}}(z)\|, \tag{4.9}$$

and

$$\|\nabla \hat{E}_h(z)\bar{A}_h Q_h \widehat{u_{ht}}(z)\| \leq C|z|^{\alpha/2-1}\|\bar{A}_h Q_h \widehat{u_{ht}}(z)\|. \tag{4.10}$$

Then, by (4.2), it follows that

$$\|\hat{E}_h(z)\bar{A}_h Q_h \widehat{u_{ht}}(z)\| + h\|\nabla \hat{E}_h(z)\bar{A}_h Q_h \widehat{u_{ht}}(z)\| \leq Ch^2|z|^{\alpha/2-1}\|\nabla \widehat{u_{ht}}(z)\|. \tag{4.11}$$

Since

$$\widehat{u_{ht}}(z) = -z^{1-\alpha}A_h \hat{u}_h(z) = -z^{1-\alpha}A_h \hat{F}_h(z)v_h,$$

an estimate analogous to (3.4) yields

$$\|\nabla \widehat{u_{ht}}(z)\| = |z|^{1-\alpha}\|\nabla \hat{F}_h(z)A_h v_h\| \leq C|z|^{1-\alpha}\,|z|^{\alpha/2-1}\|A_h v_h\| \leq C|z|^{-\alpha/2}\|A_h v_h\|. \tag{4.12}$$

On substitution of (4.11) and (4.12) in (4.8), we use (4.7) to obtain

$$\begin{aligned}\|\xi(t)\| + h\|\nabla\xi(t)\| &\leq Ch^2\left(\int_\Gamma |e^{zt}|\,|z|^{-1}\,|\mathrm{d}z|\right)\|A_h v_h\| \\ &\leq Ch^2\left(\int_{1/t}^\infty e^{\rho t\cos\theta}\rho^{-1}d\rho + \int_{-\theta}^{\theta} e^{\cos\psi}d\psi\right)\|A_h v_h\| \\ &\leq Ch^2\|A_h v_h\|. \end{aligned}\tag{4.13}$$

Now, by the identity $A_h R_h = P_h A$, we have

$$\|A_h R_h v\| = \|P_h A v\| \leq \|Av\| = |v|_2,$$

which shows the estimate (4.6) for $q = 2$.

For the case $q = 1$, that is, $v \in \dot{H}^1(\Omega)$, consider (4.11) and the identity

$$\widehat{u_{ht}}(z) = z\hat{u}_h(z) - v_h$$

to obtain using (2.4)

$$\|\nabla \hat{u}_{ht}(z)\| = \|\nabla(z\hat{F}_h(z)v_h - v_h)\| \leq (M+1)\|\nabla v_h\|. \tag{4.14}$$

From the estimate (4.8), using (4.11) and (4.14) with $\|\nabla v_h\| = \|\nabla R_h v\| \leq \|\nabla v\|$, we deduce that

$$\begin{aligned}\|\xi(t)\| + h\|\nabla\xi(t)\| &\leq Ch^2 \left(\int_\Gamma |e^{zt}|z|^{\alpha/2-1}\,|\mathrm{d}z|\right)|v|_1 \\ &\leq Ch^2 \left(\int_{1/t}^{\infty} e^{\rho t\cos\theta}\rho^{\alpha/2-1}d\rho + \int_{-\theta}^{\theta} e^{\cos\psi}t^{-\alpha/2}d\psi\right)|v|_1 \\ &\leq Ct^{-\alpha/2}h^2|v|_1.\end{aligned}$$

This completes the proof for the case $q = 1$.

Since estimates for $q = 1$ and $q = 2$ are known, then interpolation technique provides result for $q \in [1, 2]$. This concludes the rest of the proof. □

Remark 4.3. Note that the estimate (4.5) in Theorem 4.2 remains valid when $v_h = P_h v$. Indeed, for $q = 2$, let $\tilde{u}_h$ denote the solution of (3.1) with $v_h = P_h v$. Then $\zeta := \tilde{u}_h - \bar{u}_h$ satisfies

$$\zeta_t + \partial_t^{1-\alpha}\bar{A}_h\zeta = 0, \quad t > 0, \quad \zeta(0) = P_h v - R_h v.$$

Since

$$\zeta(t) = -\frac{1}{2\pi i}\int_\Gamma e^{zt}\hat{E}_h(z)(P_h v - R_h v)\,\mathrm{d}z,$$

we deduce

$$\|\zeta(t)\| \leq C\,\|P_h v - R_h v\| \int_\Gamma |e^{zt}|\,|z|^{-1}|d\,z| \leq Ch^2|v|_2.$$

Thus, the estimate (4.5) with $q = 2$ follows by the triangle inequality. If the inverse inequality $\|\nabla\chi\| \leq Ch^{-1}\|\chi\|$ holds, which is the case if the mesh is quasi-uniform, then the estimate in the gradient norm follows directly for $v_h = P_h v$.

If the $L^2(\Omega)$-projection operator P_h is stable in $\dot{H}^1(\Omega)$, *i.e.*, $\|\nabla P_h w\| \leq C|w|_1$, then the estimate (4.5) holds for the case $q = 1$ and the choice $v_h = P_h v$. A sufficient condition for such stability of P_h is the quasi-uniformity of the mesh. Now, by interpolation the estimate (4.5) holds for $q \in [1, 2]$ and $v_h = P_h v$.

4.2. Error estimates for nonsmooth initial data

In this subsection, we establish optimal error estimates for the semidiscrete FVE scheme (3.1) for nonsmooth initial data $v \in L^2(\Omega)$.

Theorem 4.4. *Let u and $\bar{u}_h$ be the solution of (1.1) and (3.1), respectively, with $v \in L^2(\Omega)$ and $v_h = P_h v$. Then, there exists a positive constant C, independent of h, such that*

$$\|\bar{u}_h(t) - u(t)\| + \|\nabla(\bar{u}_h(t) - u(t))\| \leq Cht^{-\alpha}\|v\|, \quad t > 0. \tag{4.15}$$

Furthermore, if the quadrature error operator Q_h satisfies (4.3), then the following optimal error estimate holds:

$$\|\bar{u}_h(t) - u(t)\| \leq Ch^2t^{-\alpha}\|v\|, \quad t > 0. \tag{4.16}$$

Proof. As before, it is sufficient to prove estimates for ξ. We first apply (3.3) with $p = 0$ to arrive at

$$\|\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}\| \leq C|z|^{\alpha-1}\|Q_h\widehat{u_{ht}}\|.$$

Then, the following bound follows from the integral representation (4.7):

$$\|\xi(t)\| \le C \int_\Gamma |e^{zt}||z|^{\alpha-1}\|Q_h\widehat{u_{ht}}(z)\|\,|\mathrm{d}z|. \tag{4.17}$$

To estimate the gradient of ξ, we note that

$$\|\nabla \hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}\| \le C|z|^{\alpha-1}\|\nabla Q_h\widehat{u_{ht}}\|,$$

and hence,

$$\|\nabla\xi(t)\| \le C \int_\Gamma |e^{zt}||z|^{\alpha-1}\|\nabla Q_h\widehat{u_{ht}}(z)\|\,|\mathrm{d}z|. \tag{4.18}$$

Note that $\|Q_h\widehat{u_{ht}}\| \le Ch\|\widehat{u_{ht}}\|$ holds on a general mesh, and $\|\nabla Q_h\widehat{u_{ht}}\| \le Ch\|\widehat{u_{ht}}\|$ by (4.2). Since $\|\widehat{u_{ht}}(z)\| = \|z\hat{F}_h(z)v_h - v_h\| \le C\|v_h\|$ by (2.4), a substitution into (4.17) and (4.18) yields the first estimate (4.15). Finally, if (4.3) holds, then (4.16) follows immediately from (4.17), which completes the proof. □

4.3. $L^\infty(\Omega)$-error estimates

In the following, we obtain a superconvergence result for the gradient of ξ in the $L^2(\Omega)$-norm. As a consequence, assuming $v \in L^\infty(\Omega)$ and the quasi-uniformity on the mesh, a quasi-optimal error estimate in the stronger $L^\infty(\Omega)$-norm is derived for the semidiscrete FVE solution $\bar{u}_h$. We first prove the following lemma by refining some of the estimates derived in the proof of Theorem 4.2.

Lemma 4.5. *For $1 \le q \le 2$, and with $v_h = R_h v$, where $R_h v$ is defined by (1.6), there is a positive constant C, independent of h, such that*

$$\|\nabla\xi(t)\| \le Ch^2 t^{-\alpha(3-q)/2}|v|_q, \quad t > 0.$$

The estimate is still valid for $v_h = P_h v$ on quasi-uniform meshes.

Proof. By using bounds (3.3) and (4.2), we obtain instead of (4.10) the following estimate

$$\|\nabla \hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}(z)\| \le C|z|^{\alpha-1}\|\nabla Q_h\widehat{u_{ht}}(z)\| \le Ch^2|z|^{\alpha-1}\|\nabla\widehat{u_{ht}}(z)\|.$$

Since $\|\nabla\widehat{u_{ht}}(z)\| \le c|z|^{-\alpha/2}\|A_h v_h\|$ by (4.12), we note from the representation (4.7) that

$$\|\nabla\xi(t)\| \le Ch^2|v|_2 \int_\Gamma |e^{zt}||z|^{\alpha/2-1}\,|\mathrm{d}z| \le Ct^{-\alpha/2}h^2|v|_2.$$

Similarly, taking into account (4.14), we obtain

$$\|\nabla\xi(t)\| \le Ch^2|v|_1 \int_\Gamma |e^{zt}||z|^{\alpha-1}\,|\mathrm{d}z| \le Ct^{-\alpha}h^2|v|_1.$$

Now, the desired estimate (4.5) for $q \in [1,2]$ follows by interpolation which completes the proof. □

Note that for 2D-problems, the Sobolev inequality

$$\|\chi\|_{L^\infty(\Omega)} \le C\,|\ln h|\,\|\nabla\chi\|, \quad \forall\chi \in V_h,$$

and Lemma 4.2 imply for $q \in [1,2]$ that

$$\|\xi(t)\|_{L^\infty(\Omega)} \le C\,|\ln h|\,\|\nabla\xi(t)\| \le C|\ln h|\,h^2 t^{-\alpha(3-q)/2}|v|_q. \tag{4.19}$$

As a consequence, we obtain the following quasi-optimal $L^\infty(\Omega)$-error estimate by combining the results in (4.19) and (1.7).

Theorem 4.6. *Let u and $\bar u_h$ be the solution of (1.1) and (3.1), respectively, with $v_h = P_h v$. Assume that $v \in \dot H^q(\Omega) \cap L^\infty(\Omega)$ for $1 \le q \le 2$. Then, under the quasi-uniformity condition on the mesh, there holds*

$$\|\bar u_h(t) - u(t)\|_{L^\infty(\Omega)} \le C|\ln h|^{\frac52} h^2 t^{-\alpha(3-q)/2}\Big(|v|_q + \|v\|_{L^\infty(\Omega)}\Big), \quad 1 \le q \le 2.$$

5. Fully discrete schemes

In this section, we analyze two fully discrete schemes for the semidiscrete problem (3.1) using the framework of convolution quadrature developed in [7, 22], which has been initiated in [19, 20]. To describe this framework, we first divide the time interval $[0,T]$ into N equal subintervals with a time step size $\tau = T/N$, and let $t_j = j\tau$. Then, the convolution quadrature [19] refers to an approximation of any function of the form $k * \varphi$ as

$$(k*\varphi)(t_n) := \int_0^{t_n} k(t_n - s)\varphi(s)\,\mathrm{d}s \approx \sum_{j=0}^{n} \beta_{n-j}(\tau)\varphi(t_j),$$

where the convolution weights $\beta_j = \beta_j(\tau)$ are computed from the Laplace transform $\hat k(z)$ of k rather than the kernel $k(t)$. This method provides, in particular, an interesting tool for approximating the Riemann-Liouville fractional integral of order α, $\partial_t^{-\alpha}\varphi := \omega_\alpha * \varphi$, where $\omega_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$. Here, $\hat k(z) = \hat\omega_\alpha(z) = z^{-\alpha}$.

With ∂_t being time differentiation, we define $\hat k(\partial_t)$ as the operator of (distributional) convolution with the kernel k: $\hat k(\partial_t)\varphi = k * \varphi$ for a function $\varphi(t)$ with suitable smoothness. A convolution quadrature approximates $\hat k(\partial_t)\varphi$ by a discrete convolution $\hat k(\bar\partial_\tau)\varphi$ at $t = t_n$ as

$$\hat k(\bar\partial_\tau)\varphi(t_n) = \sum_{j=0}^{n} \beta_{n-j}(\tau)\varphi(t_j),$$

where the quadrature weights $\{\beta_j(\tau)\}_{j=0}^\infty$ are determined by the generating power series

$$\sum_{j=0}^{\infty} \beta_j(\tau)\xi^j = \hat k(\delta(\xi)/\tau)$$

with $\delta(\xi)$ being a rational function, chosen as the quotient of the generating polynomials of a stable and consistent linear multistep method. In this paper, we consider the backward Euler (BE) and the second-order backward difference (SBD) methods, for which $\delta(\xi) = 1 - \xi$ and $\delta(\xi) = (1-\xi) + (1-\xi)^2/2$, respectively. For the BE method, the convolution quadrature formula for approximating the fractional integral $\partial_t^{-\alpha}\varphi$ is given by

$$\bar\partial_\tau^{-\alpha}\varphi(t_n) = \sum_{j=0}^{n} \beta_{n-j}\varphi(t_j), \text{ where } \sum_{j=0}^{\infty} \beta_j \xi^j = [(1-\xi)/\tau]^{-\alpha}, \quad \beta_j = \tau^\alpha(-1)^j \begin{pmatrix} -\alpha \\ j \end{pmatrix},$$

while for the SBD method, the quadrature weights are provided by the formula [19]:

$$\beta_j = \tau^{\alpha}(-1)^j \left(\frac{2}{3}\right)^{\alpha} \sum_{l=0}^{j} 3^{-l} \begin{pmatrix} -\alpha \\ j-l \end{pmatrix} \begin{pmatrix} -\alpha \\ l \end{pmatrix}.$$

An important property of the convolution quadrature is that it maintains some relations of the continuous convolution. For instance, the associativity of convolution is valid for the convolution quadrature [21] such as

$$\hat{k}_1(\bar{\partial}_\tau)\hat{k}_2(\bar{\partial}_\tau) = \hat{k}_1\hat{k}_2(\bar{\partial}_\tau) \quad \text{and} \quad \hat{k}_1(\bar{\partial}_\tau)(k * \varphi) = (\hat{k}_1(\bar{\partial}_\tau)k) * \varphi. \tag{5.1}$$

In the following lemma, we state an interesting result on the error of the convolution quadrature, see [20], Theorem 4.1 and [21], Theorem 2.2.

Lemma 5.1. *Let $G(z)$ be analytic in the sector Σ_θ and such that*

$$\|G(z)\| \leq M|z|^{-\mu}, \quad \forall z \in \Sigma_\theta,$$

for some real μ and M. Assume that the linear multistep method is strongly A-stable and of order $p \geq 1$. Then, for $\varphi(t) = ct^{\nu-1}$, the convolution quadrature satisfies

$$\|G(\partial_t)\varphi(t) - G(\bar{\partial}_\tau)\varphi(t)\| \leq \begin{cases} Ct^{\mu-1+\nu-p}\tau^p, & \nu \geq p, \\ Ct^{\mu-1}\tau^\nu, & 0 < \nu \leq p. \end{cases} \tag{5.2}$$

5.1. Error analysis for the BE method

In this subsection, we specify the construction of a fully discrete scheme based on the BE method for the semidiscrete problem (3.1). Then, we derive $L^2(\Omega)$-error estimates for smooth and nonsmooth initial data.

After integrating in time from 0 to t, the semidiscrete scheme (3.2) takes the form

$$\bar{u}_h + \partial_t^{-\alpha}\bar{A}_h\bar{u}_h = v_h. \tag{5.3}$$

The second term on the left-hand side is a convolution, and then, it can be approximated at $t_n = n\tau$ with U_h^n by

$$U_h^n + \bar{\partial}_\tau^{-\alpha}\bar{A}_h U_h^n = v_h. \tag{5.4}$$

The symbol $\bar{\partial}_\tau^{-\alpha}$ refers to the relevant convolution quadrature generated by the BE method.

Thus, with $U_h^0 = v_h$, the fully discrete solution can be represented by

$$U_h^n = \left(I + \beta_0\bar{A}_h\right)^{-1}\left(U_h^0 - \sum_{j=0}^{n-1}\beta_{n-j}\bar{A}_h U^j\right), \quad \text{for } n \geq 1. \tag{5.5}$$

We notice that the term corresponding to $j = 0$ in the formula can be omitted without affecting the convergence rate of the scheme [22].

In view of (5.3) and (5.4), we can write the error $U_h^n - \bar{u}_h(t_n)$ at $t = t_n$ as

$$U_h^n - \bar{u}_h(t_n) = \left(G(\bar{\partial}_\tau) - G(\partial_t)\right)v_h,$$

where $G(z) = (I + z^{-\alpha}\bar{A}_h)^{-1}$. Using the identity

$$(I + z^{-\alpha}\bar{A}_h)^{-1} = I - (z^{\alpha}I + \bar{A}_h)^{-1}\bar{A}_h,$$

and denoting $\bar{G}(z) = -(z^{\alpha}I + \bar{A}_h)^{-1}$, the error can be represented as

$$U_h^n - \bar{u}_h(t_n) = \left(\bar{G}(\bar{\partial}_\tau) - \bar{G}(\partial_t)\right)\bar{A}_h v_h. \tag{5.6}$$

Using Lemma 5.1, we now derive the following error estimates.

Lemma 5.2. *Let $\bar{u}_h$ and U_h^n be the solutions of problems (3.1) and (5.4), respectively, with $U_h^0 = v_h$. Then, the following estimates hold:*

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then

$$\|U_h^n - \bar{u}_h(t_n)\| \le C\tau t_n^{\alpha-1}|v|_2. \tag{5.7}$$

(b) If $v \in L^2(\Omega)$ and $v_h = P_h v$, then

$$\|U_h^n - \bar{u}_h(t_n)\| \le C\tau t_n^{-1}\|v\|. \tag{5.8}$$

Proof. For the estimate (5.7), we recall that, by (2.1), $\|\bar{G}(z)\| \le M_\theta |z|^{-\alpha}$ $\forall z \in \Sigma_\theta$. An application of Lemma 5.1 (with $\mu = \alpha$, $\nu = 1$ and $p = 1$) to (5.6) yields

$$\|U_h^n - \bar{u}_h(t_n)\| \le C\tau t_n^{\alpha-1}\|\bar{A}_h v_h\|.$$

Now, we introduce a projection operator $\bar{P}_h : L^2(\Omega) \to V_h$ defined by

$$(\bar{P}_h w, \chi)_h = (w, \chi), \quad \forall \chi \in V_h.$$

Then, $\bar{P}_h$ is stable in $L^2(\Omega)$ and the identity $\bar{A}_h R_h = \bar{P}_h A$ holds, since

$$(\bar{A}_h R_h w, \chi)_h = (\nabla R_h w, \nabla \chi) = (\nabla w, \nabla \chi) = (Aw, \chi) = (\bar{P}_h A w, \chi)_h, \quad \forall \chi \in V_h.$$

As $v_h = R_h v$, it follows that

$$\|\bar{A}_h v_h\| = \|\bar{A}_h R_h v\| = \|\bar{P}_h A v\| \le C\|Av\| = C|v|_2,$$

which shows (5.7).

For the estimate (5.8), we notice that $\|G(z)\| = |z|^{\alpha}\|(z^{\alpha}I + \bar{A}_h)^{-1}\| \le M_\theta$ $\forall z \in \Sigma_\theta$. Then, by applying Lemma 5.1 (with $\mu = 0$, $\nu = 1$ and $p = 1$) to (5.1), we obtain

$$\|U_h^n - \bar{u}_h(t_n)\| \le C\tau t_n^{-1}\|v_h\|.$$

Now, the estimate follows from the $L^2(\Omega)$-stability of P_h. This completes the rest of the proof. □

Remark 5.3. For $v \in \dot{H}^2(\Omega)$, we can choose $v_h = P_h v$. Let $\tilde{U}_h^n$ be the solution of the fully discrete scheme (5.4) with $v_h = P_h v$. Then, by the stability of the scheme, a direct consequence of Lemma 5.2, we have $\|U_h^n - \tilde{U}_h^n\| \le \|R_h v - P_h v\| \le Ch^2|v|_2$, showing that

$$\|U_h^n - \bar{u}_h(t_n)\| \le C(\tau t_n^{\alpha-1} + h^2)|v|_2. \tag{5.9}$$

Hence, by interpolating (5.8) and (5.9) it follows that for $v_h = P_h v$,

$$\|U_h^n - \bar{u}_h(t_n)\| \le C(\tau t_n^{-1})^{1/2}(\tau t_n^{\alpha-1} + h^2)^{1/2}|v|_1. \tag{5.10}$$

As a consequence of Lemma 5.2, we obtain error estimates for the fully discrete scheme (5.5) with smooth and nonsmooth initial data.

Theorem 5.4. *Let u and U_h^n be the solutions of problems (1.1) and (5.4), respectively, with $U_h^0 = v_h$. Then, the following error estimates hold:*

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then

$$\|U_h^n - u(t_n)\| \le C(h^2 + \tau t_n^{\alpha-1})|v|_2. \tag{5.11}$$

(b) If $v \in \dot{H}^1(\Omega)$, $v_h = P_h v$ and the mesh is quasi-uniform, then

$$\|U_h^n - u(t_n)\| \le C(h^2 t_n^{-\alpha/2} + \tau t_n^{-1+\alpha/2})|v|_1. \tag{5.12}$$

(c) If $v \in L^2(\Omega)$, $v_h = P_h v$ and Q_h satisfies (4.3), then

$$\|U_h^n - u(t_n)\| \le C(h^2 t_n^{-\alpha} + \tau t_n^{-1})\|v\|. \tag{5.13}$$

Proof. The first estimate (5.11) follows from (4.5), (5.7) and the triangle inequality, while the third estimate (5.13) follows from (4.16) and (5.8). By combining (4.5) (with $q = 1$) which holds for $v_h = P_h v$ and (5.10), we deduce

$$\|U_h^n - u(t_n)\| \le C(h^2 t_n^{-\alpha/2} + \tau t_n^{-1+\alpha/2} + \tau^{1/2} t_n^{-1/2} h)|v|_1.$$

An inspection of the three terms between brackets shows that the square of the third term equals the product of the first two terms, which proves the estimate (5.12). This concludes the proof. ☐

5.2. Error analysis for the SBD method

Now we consider the time discretization of (3.1) constructed with the convolution quadrature based on the second-order backward difference formula. From Lemma 5.1, it is obvious that one can get only a first-order error bound if, for instance, φ is constant (*i.e.*, $\nu = 1$). In order to overcome this difficulty, a correction of the scheme is needed. Below, we present modifications of the convolution quadrature based on the strategy in [7, 22]. By noting the identity

$$(I + \partial_t^{-\alpha}\bar{A}_h)^{-1} = I - (I + \partial_t^{-\alpha}\bar{A}_h)^{-1}\partial_t^{-\alpha}\bar{A}_h,$$

it turns out from (5.3) that the semidiscrete solution $\bar{u}_h$ can be rewritten as

$$\bar{u}_h = v_h - (I + \partial_t^{-\alpha}\bar{A}_h)^{-1}\partial_t^{-\alpha}\bar{A}_h v_h.$$

This leads to the modified convolution quadrature [7]

$$U_h^n = v_h - (I + \bar{\partial}_\tau^{-\alpha}\bar{A}_h)^{-1}\partial_t^{-\alpha}\bar{A}_h v_h, \tag{5.14}$$

where the exact contribution $\partial_t^{-\alpha}\bar{A}_h v_h = \omega_{\alpha+1}(t)\bar{A}_h v_h$ is kept in the new formula (5.14) in order to improve the time accuracy. The symbol $\bar{\partial}_\tau^{-\alpha}$ refers to the convolution quadrature generated by the SBD method. Unfortunately, this correction would not yield optimal time accuracy. A second choice for the modified convolution quadrature which will be considered here is based on the approximation [22]

$$U_h^n = v_h - (I + \bar{\partial}_\tau^{-\alpha} \bar{A}_h)^{-1} \bar{\partial}_\tau^{1-\alpha} \partial_t^{-1} \bar{A}_h v_h, \tag{5.15}$$

where the term ∂_t^{-1} is kept to achieve second-order time accuracy. The advantages of both numerical methods (5.14) and (5.15) are described in [7].

For the numerical implementation, it is essential to write (5.15) as a time stepping algorithm. Let $1_\tau = (0, 3/2, 1, \ldots)$ so that $1_\tau = \bar{\partial}_\tau \partial_t^{-1} 1$ at grid point t_n. Then by applying the operator $(I + \bar{\partial}_\tau^{-\alpha} \bar{A}_h)$ to both sides of (5.15) and using the associativity of convolution in (5.1), we arrive at the equivalent form

$$(I + \bar{\partial}_\tau^{-\alpha} \bar{A}_h)(U_h^n - v_h) = -\bar{\partial}_\tau^{-\alpha} \bar{A}_h 1_\tau v_h.$$

By applying again the operator $\bar{\partial}_\tau$, we obtain

$$\bar{\partial}_\tau (U_h^n - v_h) + \bar{\partial}_\tau^{1-\alpha} \bar{A}_h (U_h^n - v_h) = -\bar{\partial}_\tau^{1-\alpha} \bar{A}_h 1_\tau v_h. \tag{5.16}$$

By noting that $1 v_h - 1_\tau v_h = (v_h, -1/2 v_h, 0, \ldots)$, we thus define the time stepping scheme as: with $U_h^0 = v_h$, find U_h^n such that

$$\frac{3}{2} \tau^{-1} (U_h^1 - U_h^0) + \tilde{\partial}_\tau^{1-\alpha} \bar{A}_h U_h^1 = 0,$$

and for $n \geq 2$

$$\bar{\partial}_\tau U_h^n + \tilde{\partial}_\tau^{1-\alpha} \bar{A}_h U_h^n = 0,$$

where the modified convolution quadrature $\tilde{\partial}_\tau^{1-\alpha}$ is given by [22]

$$\tilde{\partial}_\tau^{1-\alpha} \varphi^n = \left(\sum_{j=1}^{n} \beta_{n-j}^{(1-\alpha)} \varphi^j + \frac{1}{2} \beta_{n-1}^{(1-\alpha)} \varphi^0 \right),$$

with the weights $\{\beta_j^{(1-\alpha)}\}$ being generated by the SBD method.

Now using Lemma 5.1, we derive the following error bounds for smooth and nonsmooth initial data.

Lemma 5.5. *Let $\bar{u}_h$ and U_h^n be the solutions of problems (3.1) and (5.16), respectively, and set $U_h^0 = v_h$. Then, the following estimates hold:*

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then

$$\|U_h^n - \bar{u}_h(t_n)\| \leq C \tau^2 t_n^{\alpha - 2} |v|_2. \tag{5.17}$$

(b) If $v \in L^2(\Omega)$ and $v_h = P_h v$, then

$$\|U_h^n - \bar{u}_h(t_n)\| \leq C \tau^2 t_n^{-2} \|v\|. \tag{5.18}$$

Proof. For the estimate (5.17), we set

$$\bar{G}(z) = z^{1-\alpha} (I + z^{-\alpha} \bar{A}_h)^{-1}$$

and write the error as

$$U_h^n - \bar{u}_h(t_n) = \left(\bar{G}(\bar{\partial}_\tau) - \bar{G}(\partial_t)\right) \partial_t^{-1} \bar{A}_h v_h. \tag{5.19}$$

Since $\|\bar{G}(z)\| \le M_\theta |z|^{1-\alpha}$ $\forall z \in \Sigma_\theta$ by (2.1), (5.19) and Lemma 5.1 (with $\mu = \alpha - 1$, $\nu = 2$ and $p = 2$) imply

$$\|U_h^n - \bar{u}_h(t_n)\| \le c\tau^2 t_n^{\alpha-2} \|\bar{A}_h v_h\|.$$

Then, the desired estimate (5.17) follows from the identity $\bar{A}_h R_h = \bar{P}_h A$.

For the estimate (5.18), we note with

$$\bar{G}(z) = z^{1-\alpha}(I + z^{-\alpha}\bar{A}_h)^{-1}\bar{A}_h$$

and using (5.15) that

$$U_h^n - \bar{u}_h(t_n) = \left(\bar{G}(\bar{\partial}_\tau) - \bar{G}(\partial_t)\right) \partial_t^{-1} v_h. \tag{5.20}$$

Since $\|\bar{G}(z)\| \le M_\theta |z|$ $\forall z \in \Sigma_\theta$, a use of (5.20), Lemma 5.1 (with $\mu = -1$, $\nu = 2$ and $p = 2$) and the $L^2(\Omega)$-stability of P_h yield the estimate (5.18). This completes the rest of the proof. ☐

Remark 5.6. By the stability of the scheme, a direct consequence of Lemma 5.5, and the arguments in Remark 5.3, the following error estimate holds for $v_h = P_h v$

$$\|U_h^n - \bar{u}_h(t_n)\| \le C(\tau^2 t_n^{\alpha-2} + h^2)|v|_2. \tag{5.21}$$

Then, by interpolation of (5.18) and (5.21) we get for $v_h = P_h v$

$$\|U_h^n - \bar{u}_h(t_n)\| \le C(\tau^2 t_n^{-2})^{1/2}(\tau t_n^{\alpha-2} + h^2)^{1/2}|v|_1.$$

Using the estimates derived in Sections 4.1 and 4.2 for the semidiscrete problem, and following the arguments in the proof of Theorem 5.4, we can now state the error estimates for the fully discrete scheme (5.16) with smooth and nonsmooth initial data.

Theorem 5.7. *Let u and U_h^n be the solutions of problems (1.1) and (5.16), respectively, with $U_h^0 = v_h$. Then the following error estimates hold:*

(a) If $v \in \dot{H}^2(\Omega)$ and $v_h = R_h v$, then

$$\|U_h^n - u(t_n)\| \le C(h^2 + \tau^2 t_n^{\alpha-2})|v|_2.$$

(b) If $v \in \dot{H}^1(\Omega)$, $v_h = P_h v$ and the mesh is quasi-uniform, then

$$\|U_h^n - u(t_n)\| \le C(h^2 t_n^{-\alpha/2} + \tau^2 t_n^{\alpha/2-2})|v|_1.$$

(c) If $v \in L^2(\Omega)$, $v_h = P_h v$ and Q_h satisfies (4.3), then

$$\|U_h^n - u(t_n)\| \le C(h^2 t_n^{-\alpha} + \tau^2 t_n^{-2})\|v\|.$$

6. On extensions

In this section, we discuss the extension of our analysis to other type of problems including those with more general linear elliptic operator and other time fractional evolution problems. We only concentrate on the error analysis of the semidiscrete FVE method. Completely discrete schemes can be discussed in a similar way by choosing appropriate convolution quadratures and following the analysis in Section 5.

6.1. Problems with more general elliptic operators

More precisely, we consider problem (1.3) with

$$Au = -\nabla \cdot (\kappa(x)\nabla u) + c(x)u,$$

where $\kappa(x)$ is a symmetric, positive definite 2×2 matrix function on $\bar{\Omega}$ with smooth entries and $c(x) \in L^\infty(\Omega)$ and $c(x) \geq c_0 > 0$. The corresponding bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ becomes

$$a(w, \chi) = (\kappa(x)\nabla w, \nabla \chi) + (c(x)w, \chi), \quad \forall \chi \in H_0^1(\Omega).$$

The natural generalization of the finite volume element method (1.8) yields

$$a_h(w, \chi) = \sum_{P_i \in N_h^0} \chi(P_i) \left(-\int_{\partial K_{P_i}^*} (\kappa \nabla w) \cdot \mathbf{n} \, \mathrm{d}s + \int_{K_{P_i}^*} c(x) w \, \mathrm{d}x \, \mathrm{d}s \right), \quad \forall w \in V_h,\ \chi \in V_h^*.$$

In general, the bilinear form $a_h(w, \Pi_h^* \chi)$, $\chi \in V_h$, is not symmetric on V_h. However, if κ and c are constant over each element of the triangulation $\mathcal{T}_h$, then the bilinear form takes the form, see [1],

$$a_h(w, \Pi_h^* \chi) = (\kappa(x)\nabla w, \nabla \chi) + (c(x)w, \Pi_h^* \chi), \quad \forall w, \chi \in V_h,$$

which is symmetric since $(c(x)w, \Pi_h^*\chi) = (c(x)\chi, \Pi_h^* w)$. As symmetry is important in our analysis, we shall consider the modified bilinear form, see [5],

$$\tilde{a}_h(w, \chi) = \sum_{P_i \in N_h^0} \chi(P_i) \left(-\int_{\partial K_{P_i}^*} (\tilde{\kappa}(x) \nabla w) \cdot \mathbf{n} \, \mathrm{d}s + \int_{K_{P_i}^*} \tilde{c}(x) w \, \mathrm{d}x \, \mathrm{d}s \right), \quad \forall w \in V_h,\ \chi \in V_h^*,$$

where, for each $x \in K$, $K \in \mathcal{T}_h$, $\tilde{\kappa}(x) = \kappa(x_K)$ and $\tilde{c}(x) = c(x_K)$, with x_K being the barycenter of the element K. Now, the FVE method reads: find $\tilde{u}_h(t) \in V_h$ such that

$$(\tilde{u}_h', \chi)_h + \tilde{a}_h(\partial_t^{1-\alpha} \tilde{u}_h, \Pi_h^* \chi) = 0, \quad \forall \chi \in V_h, \quad t \in (0, T], \quad \tilde{u}_h(0) = v_h. \tag{6.1}$$

Introducing the discrete operator $\tilde{A}_h : V_h \to V_h$ by

$$(\tilde{A}_h w, \chi)_h = \tilde{a}_h(w, \Pi_h^* \chi), \quad \forall w, \chi \in V_h, \tag{6.2}$$

we rewrite (6.1) as

$$\tilde{u}_h'(t) + \partial_t^{1-\alpha} \tilde{A}_h \tilde{u}_h(t) = 0, \quad t > 0, \quad \tilde{u}_h(0) = v_h. \tag{6.3}$$

Following our analysis in Section 4, with $\xi(t) = \tilde{u}_h(t) - u_h(t)$, we split the error $\tilde{u}_h(t) - u(t) = (u_h(t) - u(t)) + \xi(t)$, where it is well known that $u_h(t) - u(t)$ and $\nabla(u_h(t) - u(t))$ are estimated by the analogues of

(1.4)–(1.5). It is, therefore, sufficient to derive estimates for ξ, which satisfies for $t \geq 0$

$$(\xi', \chi)_h + \tilde{a}(\partial_t^{1-\alpha}\xi, \Pi_h^*\chi) = -\epsilon_h(u_{ht}, \chi) - \tilde{\epsilon}_h(u_h, \chi), \quad \forall \chi \in V_h, \quad \tilde{u}_h(0) = v_h, \tag{6.4}$$

where $\epsilon_h(\cdot,\cdot)$ is defined in (4.1) and $\tilde{\epsilon}_h(\cdot,\cdot)$ is given by

$$\tilde{\epsilon}_h(w, \chi) = \tilde{a}_h(w, \Pi_h^*\chi) - a(w, \chi), \quad \forall w, \chi \in V_h. \tag{6.5}$$

Upon introducing the quadrature error operators $Q_h : V_h \to V_h$ and $\tilde{Q}_h : V_h \to V_h$ defined by

$$\tilde{a}_h(Q_h w, \Pi_h^*\chi) = \epsilon_h(\chi, \psi) \quad \text{and} \quad \tilde{a}_h(\tilde{Q}_h w, \Pi_h^*\chi) = \tilde{\epsilon}_h(\chi, \psi), \quad \forall w, \chi \in V_h, \tag{6.6}$$

the equation (6.4) can be rewritten in the operator form as

$$\xi_t(t) + \partial_t^{1-\alpha}\tilde{A}_h\xi(t) = -\tilde{A}_h Q_h u_{ht}(t) - \tilde{A}_h\tilde{Q}_h u_h(t), \quad t > 0, \quad \xi(0) = 0. \tag{6.7}$$

To derive estimates for ξ, we need the following bound, see [5] for a proof.

Lemma 6.1. *Let $\tilde{A}_h$, Q_h and $\tilde{Q}_h$ be the operators defined in (6.2) and (6.6). Then*

$$\|\nabla Q_h\chi\| + h\|\tilde{A}_h Q_h\chi\| \leq Ch^{p+1}\|\nabla^p\chi\|, \quad \forall \chi \in V_h, \quad p = 0, 1, \tag{6.8}$$

and similar result holds for the operator $\tilde{Q}_h$.

Now, we show the following estimates.

Theorem 6.2. *For the error ξ defined by (6.7), there is a positive constant C, independent of h, such that for $t > 0$,*

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \leq C\max\{t^{1-\alpha/2}, t^{1-\alpha}\}h^2\|A_h v_h\|, \tag{6.9}$$

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \leq Ct^{1-\alpha/2}h^2\|\nabla v_h\|, \tag{6.10}$$

and

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \leq Ct^{1-\alpha}h\|v_h\|. \tag{6.11}$$

If $\tilde{Q}_h$ satisfies $\|\tilde{Q}_h\chi\| \leq Ch^2\|\chi\|$ $\forall \chi \in V_h$, then

$$\|\xi(t)\| \leq Ct^{1-\alpha}h^2\|v_h\|. \tag{6.12}$$

Proof. By taking Laplace transforms in (6.7), we represent $\xi(t)$ by

$$\begin{aligned}\xi(t) = &-\frac{1}{2\pi i}\int_\Gamma e^{zt}\hat{E}_h(z)\tilde{A}_h Q_h\hat{u}_{ht}(z)\,\mathrm{d}z \\ &-\frac{1}{2\pi i}\int_\Gamma e^{zt}\hat{E}_h(z)\tilde{A}_h\tilde{Q}_h\hat{u}_h(z)\,\mathrm{d}z =: \xi_1 + \xi_2,\end{aligned} \tag{6.13}$$

where $\hat{E}_h(z) = z^{\alpha-1}(z^\alpha I + \tilde{A}_h)^{-1}$. The first term ξ_1 is bounded as in the proofs of Theorems 4.2 and 4.4 using Lemma 6.1 instead of Lemma 4.1. To bound the second term ξ_2, we notice that, similar to (4.11), we arrive at

$$\|\hat{E}_h(z)\tilde{A}_h\tilde{Q}_h\hat{u}_h(z)\| + h\|\nabla\hat{E}_h(z)\tilde{A}_h\tilde{Q}_h\hat{u}_h(z)\| \leq Ch^2|z|^{\alpha/2-1}\|\nabla\hat{u}_h(z)\|. \tag{6.14}$$

Using the identity

$$\hat{E}_h(z) = z^{-1}[I - \hat{E}_h(z)\tilde{A}_h]$$

and (2.7), it follows that

$$\begin{aligned}\|\nabla \hat{E}_h(z)v_h\| &\le |z|^{-1}[\|\nabla v_h\| + \|\nabla \hat{E}_h(z)\tilde{A}_h v_h\|]\\ &\le C|z|^{-1}[\|\tilde{A}_h v_h\| + |z|^{\alpha/2-1}\|\tilde{A}_h v_h\|].\end{aligned} \tag{6.15}$$

Substituting (6.15) in (6.14) and using the integral representation of ξ_2 in (6.13), we obtain the estimate (6.9). To derive (6.10), a use of (2.4) yields

$$\|\nabla \hat{E}_h(z)v_h\| \le C|z|^{-1}\|\nabla v_h\|.$$

Then, the bound follows immediately. For the last cases (6.11) and (6.12), we apply (2.6) to get

$$\|\hat{E}_h(z)\bar{A}_h\tilde{Q}_h\hat{u}_h\|_p \le C|z|^{\alpha-1}\|\tilde{Q}_h\hat{u}_h\|_p, \quad p = 0, 1.$$

Then, the left-hand side in (6.14) is bounded by

$$C|z|^{\alpha-1}(\|\tilde{Q}_h\hat{u}_h(z)\| + h\|\nabla\tilde{Q}_h\hat{u}_h(z)\|).$$

Using Lemma 6.1 and the fact that $\|\hat{u}_h(z)\| \le |z|^{-1}\|v_h\|$, we obtain the desired results by following the arguments in the proof of Theorem 4.4. This completes the proof of the theorem. □

6.2. Other time fractional evolution problems

Our analysis can be applied to obtain optimal FVE error estimates for other type of time fractional evolution problems. This may include, for instance, evolution equations with memory terms of convolution type:

$$u'(x,t) + \mathcal{I}^\alpha Au(x,t) = 0, \quad \alpha \in (0,1), \tag{6.16}$$

see [22], which is also called fractional diffusion-wave equation, the following parabolic integro-differential equation with singular kernel of the type

$$u'(x,t) + (I + \mathcal{I}^\alpha)Au(x,t) = 0, \quad \alpha \in (0,1), \tag{6.17}$$

see, [23], and the Rayleigh-Stokes problem described by the time fractional differential equation

$$u'(x,t) + (I + \gamma\partial_t^\alpha)Au(x,t) = 0, \quad \alpha \in (0,1), \tag{6.18}$$

which has been considered in [2]. Here γ is a positive constant. In order to unify problems (6.16)–(6.18), we define $\mathcal{J}^\alpha$ denoting a time integral/differential operator and consider the unified problem by

$$u'(x,t) + \mathcal{J}^\alpha Au(x,t) = 0. \tag{6.19}$$

Now an application of Laplace transforms in (6.19) yields

$$z\hat{u} + h(z)A\hat{u} = v,$$

with some function $h(z)$ depending on α. Hence, we formally have, $\hat{u} = (z + h(z)A)^{-1}v =: \hat{E}_h(z)v$.

Let $\bar{A}_h$ and Q_h be the operators defined in Sections 3 and 4, respectively. Then, the FVE method reads: find $\bar{u}_h(t) \in V_h$ such that

$$\bar{u}_h' + \mathcal{J}^\alpha \bar{A}_h \bar{u}_h = 0, \quad t \in (0, T], \quad \bar{u}_h(0) = v_h. \tag{6.20}$$

Again using the corresponding FE solution u_h, we split $\bar{u}_h - u := (u_h - u) + (\bar{u}_h - u_h) =: (u_h - u) + \xi$, where ξ satisfies the similar representation formula

$$\xi(t) = -\frac{1}{2\pi i} \int_{\bar{\Gamma}_\theta} e^{zt} \hat{E}_h(z) \bar{A}_h Q_h \hat{u}_{ht}(z) \, \mathrm{d}z. \tag{6.21}$$

Note that in this case the operator $\hat{E}_h(z)$ is given by

$$\hat{E}_h(z) = \beta(z)(z\beta(z)I + \bar{A}_h)^{-1}, \tag{6.22}$$

and $\beta(z) = h(z)^{-1}$. For the problem (6.16), we observe that $\beta(z) = z^\alpha$, for the problem (6.17), $\beta(z) = z^\alpha/(1 + z^\alpha)$, and for the problem (6.18), $\beta(z) = 1/(1 + \gamma z^\alpha)$. We assume that one can properly choose θ in $(\pi/2, \pi)$ such that $z\beta(z) \in \Sigma_{\theta'}$ for all $z \in \Sigma_\theta$ where the angle $\theta' \in (\pi/2, \pi)$. This is indeed possible in all given examples. With this, the resolvent estimate yields

$$\|(z\beta(z)I + \bar{A}_h)^{-1}\| \le \frac{M_{\theta'}}{|z\beta(z)|}, \quad \forall z \in \Sigma_\theta, \tag{6.23}$$

where $M_{\theta'} = 1/\sin(\pi - \theta')$. Therefore, from (6.22),

$$\|\hat{E}_h(z)\| \le M_{\theta'} |z|^{-1}, \quad \forall z \in \Sigma_\theta. \tag{6.24}$$

Following arguments from [22], we deduce that

$$\|\bar{A}_h \hat{E}_h(z)\| \le C_{\theta'} |\beta(z)| \quad \forall z \in \Sigma_\theta. \tag{6.25}$$

Now, we can prove the analogue of Lemma 2.1.

Lemma 6.3. *Let $\hat{E}_h(z)$ be given by (6.22). With $\chi \in V_h$, the following estimates hold:*

$$\|\bar{A}_h \hat{E}_h(z)\chi\| \le C_{\theta'} |\beta(z)|^{1-p/2} |z|^{-p/2} \, \|\bar{A}_h^{p/2}\chi\|, \quad \forall z \in \Sigma_\theta, \quad 0 \le p \le 2, \tag{6.26}$$

$$|\hat{E}_h(z)\chi|_1 \le C_{\theta'} |\beta(z)|^{1/2} |z|^{-1/2} \|\chi\|, \quad \forall z \in \Sigma_\theta, \tag{6.27}$$

where $C_{\theta'}$ is independent of the mesh size h.

Proof. We obtain the first estimate (6.26) by interpolating (6.24) and (6.25). The second estimate follows from the fact that

$$\|\nabla(z\beta(z)I + \bar{A}_h)^{-1}\chi\| \le C|z\beta(z)|^{-1/2}\|\chi\|, \quad \forall \chi \in V_h,$$

see (2.13) in [7]. □

In the following theorem, optimal error estimates are obtained for smooth and nonsmooth initial data $v \in \dot{H}^q(\Omega)$, $q = 0, 1, 2$.

Theorem 6.4. *For the error ξ defined by (6.21), there is a positive constant C, independent of h, such that for $t > 0$,*

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \leq Ch^2\|\bar{A}_h v_h\|. \tag{6.28}$$

If $|\beta(z)| \leq C|z|^\mu$ $\forall z \in \Sigma_\theta$ for some real $\mu < 1$, then

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \leq Ct^{-(\mu+1)/2}h^2\|\nabla v_h\|. \tag{6.29}$$

If $|\beta(z)| \leq C|z|^\mu$ $\forall z \in \Sigma_\theta$ and $\bar{Q}$ satisfies (4.3), then

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \leq Ct^{-(\mu+1)}h^2\|v_h\|. \tag{6.30}$$

Proof. We will only prove the estimate in the $L^2(\Omega)$-norm. The estimate in the gradient norm is derived in a similar way. We shall make use of the estimate (4.8) obtained in the proof of Theorem 4.2.

When $q = 2$, that is, $v \in \dot{H}^2(\Omega)$, apply (6.26) with $p = 1$ and (6.27) in Lemma 6.3 to get

$$\|\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}(z)\| \leq C|\beta(z)|^{1/2}|z|^{-1/2}\|\nabla Q_h\widehat{u_{ht}}(z)\|,$$

and

$$\|\nabla\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}(z)\| \leq C|\beta(z)|^{1/2}|z|^{-1/2}\|\bar{A}_h Q_h\widehat{u_{ht}}(z)\|.$$

Then, by (4.2) in Lemma 4.1, we deduce

$$\|\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}(z)\| + h\|\nabla\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}(z)\| \leq Ch^2|\beta(z)|^{1/2}|z|^{-1/2}\|\nabla\widehat{u_{ht}}(z)\|. \tag{6.31}$$

Since

$$\widehat{u_{ht}}(z) = -h(z)\bar{A}_h\hat{u}_h(z) = -h(z)\bar{A}_h\hat{F}_h(z)v_h,$$

an estimate analogous to (6.27) yields

$$\begin{aligned}\|\nabla\widehat{u_{ht}}(z)\| &= |h(z)|\|\nabla\hat{F}_h(z)\bar{A}_h v_h\| \\ &\leq C|h(z)|\,|\beta(z)|^{1/2}|z|^{-1/2}\|\bar{A}_h v_h\| \\ &\leq C|\beta(z)|^{-1/2}|z|^{-1/2}\|\bar{A}_h v_h\|.\end{aligned}$$

Thus, the left-hand side in (6.31) is bounded by $|z|^{-1}\|\bar{A}_h v_h\|$. Now, substitution in (4.8) gives the desired estimate.

For $q = 1$, we notice that in view of (6.24), the bound (4.14) holds, and therefore substitution in (6.31) gives the new upper bound $Ch^2|z|^{\mu/2-1/2}\|\nabla v_h\|$ in (6.31). The estimate (6.29) follows then by integration.

Finally, for $q = 0$, we have by (6.25),

$$\|\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}\| \leq C|\beta(z)|\,\|Q_h\widehat{u_{ht}}\| \leq C|z|^\mu\|Q_h\widehat{u_{ht}}\|.$$

In view of (6.24), we have $\|\widehat{u_{ht}}(z)\| = \|z\hat{F}_h(z)v_h - v_h\| \leq C\|v_h\|$. Therefore, if (4.3) is satisfied then $\|\hat{E}_h(z)\bar{A}_h Q_h\widehat{u_{ht}}\| \leq Ch^2|z|^\mu\|\widehat{u_{ht}}\| \leq Ch^2|z|^\mu\|v_h\|$. Now, (6.30) follows by integration and this concludes the rest of the proof. $\square$

By interpolating (6.28) and (6.30) we obtain for $q \in [0, 2]$

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \le Ct^{-(\mu+1)(2-q)/2}h^2\|\bar{A}_h^{q/2}v_h\|, \quad t > 0.$$

Notice that $\mu = \alpha$ for problems (6.16) and (6.17), while $\mu = -\alpha$ for the Rayleigh-Stokes problem (6.18). Hence, for the Rayleigh-Stokes problem the previous estimate reads:

$$\|\xi(t)\| + h\|\nabla\xi(t))\| \le Ct^{-(1-\alpha)(2-q)/2}h^2\|\bar{A}_h^{q/2}v_h\|, \quad t > 0,$$

provided (4.3) is satisfied.

We finally consider the following class of time fractional order diffusion problems:

$$^C\partial_t^\alpha u(x, t) + Au(x, t) = 0, \tag{6.32}$$

where $^C\partial_t^\alpha$ is the fractional Caputo derivative of order $\alpha \in (0, 1)$. For this class of equations, optimal error estimates for the semidiscrete FE method have been established in [12]. The FVE method applied to (6.32) is to seek $\bar{u}_h \in V_h$ such that

$$^C\partial_t^\alpha \bar{u}_h + \bar{A}_h\bar{u}_h = 0, \quad t \in (0, T], \quad \bar{u}_h(0) = v_h.$$

Again a comparison between the FE solution and FVE solution along with Laplace transformation techniques and semigroup type properties as has been done in Section 4 yields *a priori* FVE error estimates for the fractional order evolution problem (6.32) for both smooth and nonsmooth initial data. Since the proof technique is similar to the tool used in Section 4, we skip the details.

Remark 6.5. Note that the estimates and their proofs obtained in this section are analogous to those of the lumped mass FE method as the operators $\bar{A}_h$ and Q_h have similar properties to those of the corresponding operators for the lumped mass method.

7. Numerical experiments

In this section, we present some numerical tests to validate our theoretical results. We choose $\Omega = (0, 1) \times (0, 1)$ and perform the computation on two families of symmetric and nonsymmetric triangular meshes. The symmetric meshes are uniform with mesh size $h = \sqrt{2}/M$, where M is the number of equally spaced subintervals in both the x- and y-directions, see Figure 2a. For the nonsymmetric meshes, we choose M subintervals in the x-direction and $3M/4$ equally spaced subintervals in the y-direction with the assumption that M is divisible by 4. The intervals in the x-direction are of lengths $4/3M$ and $2/3M$ and distributed such that they form an alternating series as shown in Figure 2b. One can notice that the nonsymmetric mesh defines a triangulation that is not symmetric at any vertex, see Section 5 from [5] for more details.

We consider three numerical examples with smooth and nonsmooth initial data. By separation of variables, the exact solution of problem (1.1) can represented by a rapidly converging Fourier series

$$u(x, y, t) = \sum_{m,n=1}^{\infty} (v, \phi_{mn})E_\alpha(-\lambda_{mn}t^\alpha)\phi_{mn}(x, y), \tag{7.1}$$

where $E_\alpha(t) := \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(\alpha p+1)}$ is the Mittag-Leffler function and

$$\phi_{mn}(x, y) = 2\sin(m\pi x)\sin(n\pi y) \quad \text{and} \quad \lambda_{mn} = (m^2 + n^2)\pi^2 \quad \text{for} \quad m, n = 1, 2, \ldots$$

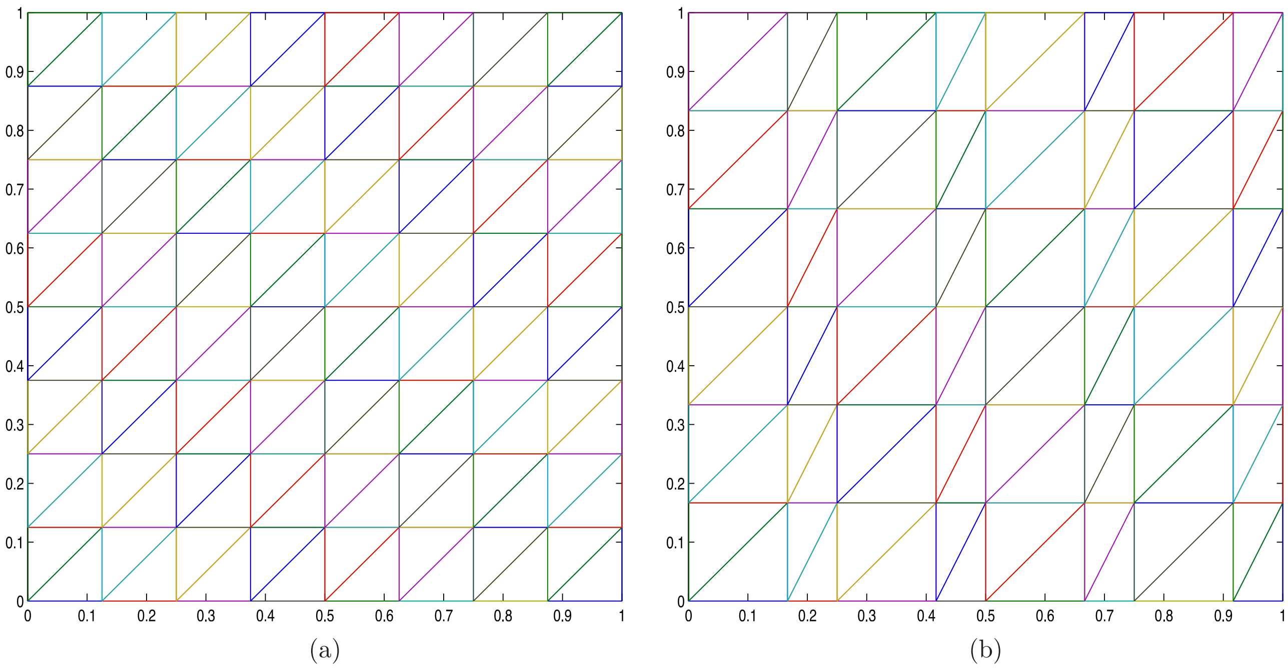


FIGURE 2. Triangular meshes with $M = 8$, (a) symmetric mesh and (b) nonsymmetric mesh.

are the orthonormal eigenfunctions and corresponding eigenvalues of $-\Delta$ subject to homogeneous Dirichlet boundary conditions. In our computation, we evaluate the exact solution by truncating the Fourier series in (7.1) after 60 terms.

We consider the following initial data to illustrate the convergence theory.

(a) With $v = xy(1-x)(1-y)$, its Fourier sine coefficients become

$$(v, \phi_{mn}) = 8(1-(-1)^m)(1-(-1)^n)(mn\pi^2)^{-3}, \quad \text{for} \;\; m, n = 1, 2, \ldots .$$

This example represents the smooth case as $v \in \dot{H}^2(\Omega)$.

(b) For this example, choose $v = g(x)g(y)$ where $g(z) = z$ on $[0, 1/2)$ and $g(z) = 1 - z$ on $(1/2, 1]$. This initial data is less smooth compared to the previous case. One can verify that its Fourier coefficients are given by

$$(v, \phi_{mn}) = 2(1-(-1)^m)(1-(-1)^n)(mn\pi^2)^{-2}(-1)^{mn}, \quad \text{for} \;\; m, n = 1, 2, \ldots .$$

Note that $v \in \dot{H}^{1+\epsilon}(\Omega)$ for $0 \le \epsilon < 1/2$.

(c) With $v = \chi_{(0,1/2[\times(0,1)}(x, y)$, its Fourier sine coefficients become

$$(v, \phi_{mn}) = 2(1-\cos(m\pi/2))(1-(-1)^n)(mn\pi^2)^{-1}, \quad \text{for} \;\; m, n = 1, 2, \ldots .$$

Here, $v \in \dot{H}^{1/2-\epsilon}(\Omega)$ with $\epsilon > 0$.

To examine the temporal accuracy of the proposed schemes, we employ a uniform temporal mesh with a time step $\tau = T/N$, where $T = 0.5$ is the time of interest in all numerical experiments. We fix the mesh size h at $h = 1/400$ so that the error incurred by spatial discretization is negligible, which enable us to examine the temporal convergence rate. The computation is performed on symmetric meshes. We measure the error

TABLE 1. L^2-Error for cases (a)–(c) on symmetric meshes, $\alpha = 0.75$, $h = 1/400$.

N	BE	Rate	SBD	Rate
Case (a)				
5	4.89e−3		1.32e−3	
10	2.19e−3	1.16	3.16e−4	2.06
20	1.04e−3	1.08	7.26e−5	2.12
40	5.05e−4	1.04	1.69e−5	2.10
80	2.49e−4	1.02	3.69e−6	2.18
Case (b)				
5	4.83e−3		1.39e−3	
10	2.16e−3	1.16	3.33e−4	2.06
20	1.02e−3	1.07	7.70e−5	2.11
40	5.00e−4	1.03	1.77e−5	2.19
80	2.48e−4	1.02	3.68e−6	2.27
Case (c)				
5	2.97e−3		8.24e−4	
10	1.33e−3	1.16	2.05e−4	2.01
20	6.32e−4	1.07	4.73e−5	2.11
40	3.09e−4	1.03	1.09e−5	2.11
80	1.53e−4	1.01	2.43e−6	2.17

TABLE 2. Errors for cases (a)–(c) on symmetric meshes, $\alpha = 0.75$, $\tau = 1/500$.

M	L^2-Error	Rate	L^∞-Error	Rate
Case (a)				
8	1.46e−3		1.06e−4	
16	3.74e−4	1.96	2.74e−5	1.95
32	9.33e−5	2.00	6.86e−6	2.00
64	2.25e−5	2.05	1.68e−6	2.03
128	4.82e−6	2.23	3.81e−7	2.14
Case (b)				
8	8.93e−4		2.04e−4	
16	2.29e−4	1.96	5.54e−5	1.88
32	5.73e−5	2.00	1.43e−5	1.95
64	1.38e−5	2.05	3.56e−6	2.01
128	2.98e−6	2.21	8.04e−7	2.15
Case (c)				
8	7.19e−4		2.70e−3	
16	1.81e−4	1.99	8.74e−4	1.63
32	4.51e−5	2.01	2.72e−4	1.69
64	1.10e−5	2.03	7.62e−5	1.83
128	2.66e−6	2.05	2.05e−5	1.90

$e^n =: u(t_n) - U^n$ by the normalized $L^2(\Omega)$-norm $\|e^n\|_{L^2(\Omega)}/\|v\|_{L^2(\Omega)}$. The numerical results are presented in Table 1 for the three proposed cases (a)–(c). In the table, BE and SBD denote the convolution quadrature generated by the backward Euler and the second-order backward difference methods, respectively. The `rate` refers to the empirical convergence rate, when the time step size τ halves. From the Table 1, a convergence rate

TABLE 3. L^2-Error for cases (a)–(c) as $t \to 0$; $\alpha = 0.75$, $h = 1/64$, $N = 10^3$.

	Method	1e−2	1e−3	1e−4	1e−5	1e−6	Rate
(a)	BE	3.03e−4	4.74e−4	5.06e−4	5.14e−4	5.16e−4	−0.00 (0)
	SBD	3.83e−4	4.86e−4	5.08e−4	5.14e−4	5.16e−4	−0.00 (0)
(b)	BE	1.85e−4	3.13e−4	3.89e−4	5.01e−4	6.66e−4	−0.12 ($-\alpha/4$)
	SBD	2.23e−4	3.19e−4	3.90e−4	5.00e- 4	6.56e−4	−0.11 ($-\alpha/4$)
(c)	BE	1.72e−4	6.55e−4	2.22e−3	7.74e−3	2.44e−2	−0.52 ($-3\alpha/4$)
	SBD	2.34e−4	6.99e−4	2.33e−3	8.69e−3	3.60e−2	−0.59 ($-3\alpha/4$)

TABLE 4. Errors for case (c) on nonsymmetric meshes, $\alpha = 0.75$, $\tau = 1/500$.

M	L^2-Error	Rate	L^∞-Error	Rate
	FVEM			
8	1.12e−3		4.17e−3	
1.6	2.78e−4	2.01	1.37e−3	1.61
32	6.80e−5	2.03	4.20e−4	1.71
64	1.65e−5	2.04	1.11e−4	1.92
128	3.97e−6	2.06	3.04e−5	1.88
	Lumped mass FEM			
8	1.167e−3		4.15e−3	
16	3.125e−4	1.90	1.37e−3	1.60
32	8.228e−5	1.92	4.15e−4	1.72
64	2.14e−5	1.94	1.11e−4	1.90
128	5.80e−6	1.88	3.35e−5	1.73

of order $O(\tau)$ and $O(\tau^2)$ is observed for the BE and SBD schemes, respectively, and clearly both schemes exhibit a very steady behavior for both smooth and nonsmooth data, which agree well with our convergence theory. Additional numerical experiments with different values of the fractional order α have shown similar convergence rates. It was, in particular, observed that the error decreases as the fractional order α increases. More details on the behavior of errors from BE and SBD methods combined with a Galerkin FE discretization in space can be found in [11].

To check the spatial discretization error, we fix the time step $\tau = 1/500$ and use the SBD scheme so that the temporal discretization error is negligible. We carry out the computation on symmetric meshes. In Table 2, we list the normalized $L^2(\Omega)$-norm and $L^\infty(\Omega)$-norms of the error for the cases (a)–(c). The numerical results show a convergence rate $O(h^2)$ for the $L^2(\Omega)$-norm of the error for smooth and nonsmmoth initial data. A similar convergence rate is obtained in the $L^\infty(\Omega)$-norm (ignoring a logarithmic factor). The results fully confirm the predicted rates on symmetric meshes. They also show the validity of the convergence rate in Theorem 4.6 for case (c) where $0 < q < 1$.

To investigate the behavior of the error as $t \to 0$, we check the (spatial) prefactors in Theorems 5.4 and 5.7. In Table 3, we present the numerical results obtained as $t \to 0$ with the meshsize h and the number of time steps N being fixed. The results indicate that the error essentially stays unchanged in the smooth case, whereas it deteriorates as $t \to 0$ in the other two cases. From Theorems 5.4 and 5.7, the error is expected to grow like $O(t^{-\alpha(2-q)/2})$. We observe that the empirical convergence rate in the table agrees well with the theoretical rate (given between brackets). In case (c), for instance, the initial data $v \in \dot{H}^{1/2-\epsilon}(\Omega)$ with $\epsilon > 0$, and the numerical results show a growth $O(t^{-3\alpha/4})$ as $t \to 0$.

TABLE 5. Errors for case (d) on nonsymmetric meshes, $\alpha = 0.75$, $\tau = 1/500$.

M	L^2-Error	Rate	L^∞-Error	Rate
	FVEM			
8	9.92e−5		3.85e−4	
16	2.31e−5	2.10	1.12e−4	1.77
32	1.15e−5	1.01	4.85e−5	1.21
64	5.12e−6	1.17	2.05e−5	1.24
128	2.56e−6	1.00	9.72e−6	1.08
	Lumped mass FEM			
8	4.49e−4		1.74e−3	
16	1.04e−4	2.11	5.07e−4	1.78
32	5.18e−5	1.01	2.18e−4	1.21
64	2.30e−5	1.17	9.25e−5	1.24
128	1.15e−5	1.00	4.37e−5	1.08

For nonsymmetric meshes, we are especially interested in spatial errors for nonsmooth initial data as the convergence theory suggests. In Table 4, we display the $L^2(\Omega)$- and $L^\infty(\Omega)$-norms of the error for case (c) using the FVE and the lumped mass FE discretizations on nonsymmetric meshes. The numerical results reveal that both discretizations exhibit a convergence rate of order $O(h^2)$, which may be seen as an unexpected result. However, as the initial data $v \in \dot{H}^{1/2-\epsilon}(\Omega)$ for any $\epsilon > 0$, v has some smoothness, and hence, the numerical results do not contradict our theoretical findings. In addition, we notice that as the convergence rate is $O(h^2)$ for initial data in $\dot{H}^1(\Omega)$, by interpolation in $[0, 1]$, a convergence rate of order $O(h^{3/2})$ is expected for $v \in \dot{H}^{1/2}(\Omega)$. In our case, the smoothness of the particular initial data v could then have a positive effect on the convergence rate.

In [5], the authors considered the nonsymmetric partition shown in Figure 2b and provided an initial data for which the optimal L^2-convergence does not hold. They proved that the best possible error bound in this case is of order 1, see Proposition 5.1 of [5]. Earlier in [4], the same authors have established a one-dimensional example for which the $O(h^2)$ nonsmooth data error does not hold for the lumped mass FE method. We, then, carried out our computation based on the example in [5], Proposition 5.1. The numerical results are presented in Table 5 using the SBD scheme. The error reported in the table represents the quantity $\xi(t)$ which measures the difference between the Galerkin FE solution and the FVE solution for the first set of numerical results and between the Galerkin FE solution and the lumped mass FE solution for the second set. As the nonsmooth data error from the standard Galerkin FE is always $O(h^2)$, the error from the considered methods is dominated by $\xi(t)$. From Table 5, an order $O(h)$ of convergence rate is observed for both methods, which agrees well with the results in [5] and confirms our theoretical analysis.

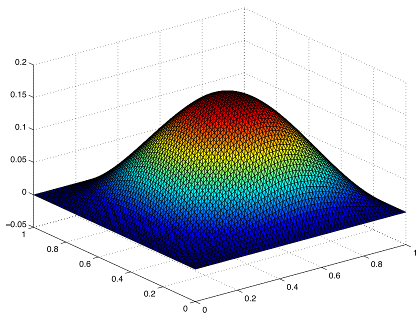
For completeness, we extend our numerical study to examine some of the problems presented in Section 6, namely; the subdiffusion problem (6.32) with a fractional Caputo derivative and the wave-diffusion problem (6.16). The numerical solution in each case is obtained by using the FVE method in space and a convolution quadrature in time generated by the second-order backward difference method. We run both examples with the initial data v given in case (c).

For the first test problem, we employ the second-order time discretization scheme derived in [11], formula (2.16). The computed errors are presented in Table 6 and are clearly identical to the results in Table 2. Even though it is known that the two representations (6.32) and (1.1a) are equivalent, the numerical methods obtained for each representation are in general different. However, in the current case, the fact that the time discrete schemes are equivalent is due to the feature of the convolution quadrature, in particular, to the properties given in (5.1).

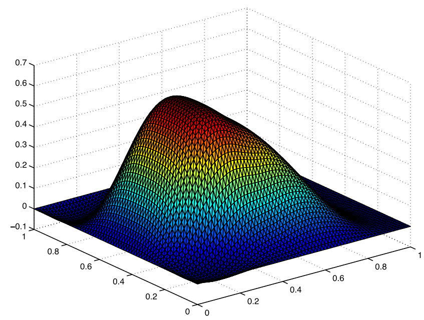
For the wave-diffusion problem, the numerical results are listed in Table 7 for $\alpha = 0.5$. We observe a $O(h^2)$ convergence for the $L^2(\Omega)$- and $L^\infty(\Omega)$-norm of the errors which confirms our predictions. It is known that the

TABLE 6. Numerical results for problem (6.32), $\alpha = 0.75$, $\tau = 1/500$.

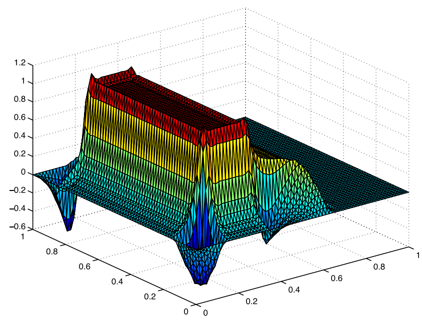
M	L^2-Error	Rate	L^∞-Error	Rate
8	7.19e−4		2.70e−3	
16	1.81e−4	1.99	8.74e−4	1.63
32	4.52e−5	2.01	2.72e−4	1.69
64	1.10e−5	2.03	7.62e−5	1.83
128	2.66e−6	2.05	2.05e−5	1.90



(a) $\alpha = 0.1$



(b) $\alpha = 0.5$



(c) $\alpha = 0.9$

FIGURE 3. The profile of solutions of problem (6.16) at $t = 0.1$ with different values of α.

TABLE 7. Numerical results for problem (6.16), $\alpha = 0.5$, $\tau = 1/500$.

M	L^2-Error	Rate	L^∞-Error	Rate
8	5.75e−3		1.10e−2	
16	1.44e−3	2.00	2.80e−3	1.97
32	3.57e−4	2.01	7.26e−4	1.94
64	8.55e−5	2.06	1.96e−4	1.89
128	1.98e−5	2.11	5.14e−5	1.93

model (6.16) interpolates the heat and wave equations when the fractional order α increases from zero to one. This transition is observed numerically. In Figure 3, we display the profile of the numerical solutions to case (c) at time $t = 0.1$ with different values of α. We observe that, the closer α is to zero, the slower is the decay. Furthermore, the oscillations in Figure 3a are inherited from the L^2-projection $P_h v$ which is oscillatory. This reflects, in particular, the wave feature of the model (6.16).

REFERENCES

[1] R.E. Bank and D.J. Rose, Some error estimates for the box method. *SIAM J. Numer. Anal.* **24** (1987) 777–787.

[2] E. Bazhlekova, B. Jin, R. Lazarov and Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid. *Numer. Math.* **131** (2016) 1–31.

[3] P. Chatzipantelidis, R.D. Lazarov and V. Thomée, Error estimates for a finite volume element method for parabolic equations in convex polygonal domains. *Numer. Methods Part. Differ. Equ.* **20** (2004) 650–674.

[4] P. Chatzipantelidis, R.D. Lazarov and V. Thomée, Some error estimates for the lumped mass finite element method for a parabolic problem. *Math. Comput.* **81** (2012) 1–20.

[5] P. Chatzipantelidis, R.D. Lazarov and V. Thomée, Some error estimates for the finite volume element method for a parabolic problem. *Comput. Methods Appl. Math.* **13** (2013) 251–279.

[6] S.H. Chou and Q. Li, Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems: a unified approach. *Math. Comput.* **69** (2000) 103–120.

[7] E. Cuesta, C. Lubich and C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations. *Math. Comput.* **75** (2006) 673–696.

[8] R.E. Ewing, R.D. Lazarov and Y. Lin, Finite volume element approximations of nonlocal reactive flows in porous media. *Numer. Methods Part. Differ. Equ.* **16** (2000) 285–311.

[9] R. Gorenflo, F. Mainardi, D. Moretti and P. Paradisi, Time fractional diffusion: a discrete random walk approach. *Nonlinear Dyn.* **29** (2002) 129–143.

[10] B.I. Henry and S.L. Wearne, Fractional reaction-diffusion. *Physica A* **276** (2000) 448–455.

[11] B. Jin, R. Lazarov and Z. Zhou, Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. *SIAM J. Sci. Comput.* **38** (2016) 146–170.

[12] B. Jin, R. Lazarov and Z. Zhou, Error estimates for a semidiscrete finite element method for fractional order parabolic equations. *SIAM J. Numer. Anal.* **51** (2013) 445–466.

[13] B. Jin, R. Lazarov, J. Pascal and Z. Zhou, Error analysis of semidiscrete finite element methods for inhomogeneous time-fractional diffusion. *IMA J. Numer. Anal.* **35** (2015) 561–582.

[14] S. Karaa, K. Mustapha and A.K. Pani, Finite volume element method for two-dimensional fractional subdiffusion problems. *IMA J. Numer. Anal.* **37** (2017) 945–964.

[15] S. Karaa, K. Mustapha and A.K. Pani, Optimal error analysis of a FEM for fractional diffusion problems by energy arguments. *J. Sci. Comput.* **74** (2018) 519–535.

[16] K. Mustapha, FEM for time-fractional diffusion equations, novel optimal error analyses. *Math. Comput.* **87** (2018) 2259–2272.

[17] R.H. Li, Z.Y. Chen and W. Wu, Generalized Difference Methods for Differential Equations. Marcel Dekker, New York (2000).

[18] Y.Lin, J. Liu and M. Yang, Finite volume element methods: an overview on recent developments. *IJNAM Int. J. Numer. Anal. Modeling Ser. B* **4** (2013) 14–34.

[19] C. Lubich, Discretized fractional calculus. *SIAM J. Math. Anal.* **17** (1986) 704–719.

[20] C. Lubich, Convolution quadrature and discretized operational calculus-I. *Numer. Math.* **52** (1988) 129–145.

[21] C. Lubich, Convolution quadrature revisited. *BIT Numer. Math.* **44** (2004) 503–514.

[22] C. Lubich, I.H. Sloan and V. Thomée, Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comput.* **65** (1996) 1–17.

[23] W. McLean, I.H. Sloan and V. Thomée, Time discretization via Laplace transformation of an integro-differential equation of parabolic type. *Numer. Math.* **102** (2006) 497–522.

[24] W. McLean and V. Thomée, Numerical solution via Laplace transforms of a fractional order evolution equation. *J. Integr. Equ. Appl.* **22** (2010) 57–94.

[25] W. McLean and V. Thomée, Maximum-norm error analysis of a numerical solution via Laplace transformation and quadrature of a fractional order evolution equation. *IMA J. Numer. Anal.* **30** (2010) 208–230.

[26] W. McLean and V. Thomée, Time discretization of an evolution equation via Laplace transforms. *IMA J. Numer. Anal.* **24** (2004) 439–463.

[27] R. Metzler and J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339** (2000) 1–77.

[28] K. Mustapha and W. McLean, Piecewise-linear, discontinuous Galerkin method for a fractional diffusion equation. *Numer. Algorithms* **56** (2011) 159–184.

[29] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Berlin (2006).