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QUADRATIC CONVERGENCE OF LEVENBERG-MARQUARDT METHOD FOR ELLIPTIC AND PARABOLIC INVERSE ROBIN PROBLEMS

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Abstract. We study the Levenberg-Marquardt (L-M) method for solving the highly nonlinear and ill-posed inverse problem of identifying the Robin coefficients in elliptic and parabolic systems. The L-M method transforms the Tikhonov regularized nonlinear non-convex minimizations into convex minimizations. And the quadratic convergence of the L-M method is rigorously established for the nonlinear elliptic and parabolic inverse problems for the first time, under a simple novel adaptive strategy for selecting regularization parameters during the L-M iteration. Then the surrogate functional approach is adopted to solve the strongly ill-conditioned convex minimizations, resulting in an explicit solution of the minimisation at each L-M iteration for both the elliptic and parabolic cases. Numerical experiments are provided to demonstrate the accuracy, efficiency and quadratic convergence of the methods.

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1. Introduction

We are concerned in this work with the determination of the Robin coefficient in both stationary elliptic and time-dependent parabolic systems from noisy measurement data on a partial boundary. This is a highly nonlinear and ill-posed inverse problem and arises in many applications of practical importance. The Robin coefficient may characterize the thermal properties of conductive materials on the interface or certain physical processes near the boundary, *e.g.*, it represents the corrosion damage profile in corrosion detection [10, 15], and indicates the thermal property in quenching processes [27].

For the description of the model problems that are considered in this work, we let $\Omega \subset R^d$ $(d \leq 3)$ be an open bounded and connected domain, with a C^2-smooth boundary $\partial\Omega$, which consists of two relatively smooth

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disjointed parts Γ_i and Γ_a, *i.e.*, $\partial\Omega = \Gamma_i \cup \Gamma_a$. Γ_i and Γ_a are respectively the part of the boundary that is inaccessible and accessible to experimental measurements. Then we shall consider the inverse Robin problems associated with the elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (a(\mathbf{x})\nabla u) + c(\mathbf{x})u = f(\mathbf{x}) & \text{in } \Omega\,, \\ a(\mathbf{x})\frac{\partial u}{\partial n} + \gamma(\mathbf{x})u = g(\mathbf{x}) & \text{on } \Gamma_i\,, \\ a(\mathbf{x})\frac{\partial u}{\partial n} = h(\mathbf{x}) & \text{on } \Gamma_a\,, \end{cases} \tag{1.1}$$

and the parabolic initial boundary value problem

$$\begin{cases} \partial_t u - \nabla \cdot (a(\mathbf{x})\nabla u) = f(\mathbf{x},t) & \text{in } \Omega \times [0,T]\,, \\ a(\mathbf{x})\frac{\partial u}{\partial n} + \gamma(\mathbf{x})u = g(\mathbf{x},t) & \text{on } \Gamma_i \times [0,T]\,, \\ a(\mathbf{x})\frac{\partial u}{\partial n} = h(\mathbf{x},t) & \text{on } \Gamma_a \times [0,T]\,, \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{in } \Omega\,. \end{cases} \tag{1.2}$$

The coefficients $a(\mathbf{x})$ and $c(\mathbf{x})$ are the heat conductivity and radiative coefficient, satisfying that $\underline{a} \le a(\mathbf{x}) \le \bar{a}$ and $\underline{c} \le c(\mathbf{x}) \le \bar{c}$ in Ω, where $\underline{a}$, $\bar{a}$ and $\underline{c}$, $\bar{c}$ are positive constants. Functions f, g and h are the source strength, ambient temperature and heat flux respectively. Both coefficients $\gamma(\mathbf{x})$ in (1.1) and (1.2) represent the Robin coefficients, which will be the focus of our interest and is assumed to stay in the following feasible constraint set:

$$K := \Big\{\gamma \in L^2(\Gamma_i);\ 0 < \gamma_1 \le \gamma(\mathbf{x}) \le \gamma_2 \ a.e. \text{ on } \Gamma_i\Big\},$$

where γ_1 and γ_2 are two positive constants. For convenience, we often write the solutions of the systems (1.1) and (1.2) as $u(\gamma)$ to emphasize their dependence on the Robin coefficient γ.

We are now ready to formulate the inverse problems of our interest in this work.

Elliptic Inverse Robin Problem: recover the Robin coefficient $\gamma(\mathbf{x})$ in (1.1) on the inaccessible part Γ_i from the measurable data z of u on the accessible part Γ_a.

Parabolic inverse Robin problem: recover the Robin coefficient $\gamma(\mathbf{x})$ in (1.2) on the inaccessible part Γ_i from the measurable data z of u on the accessible part Γ_a over the time period $[0,T]$.

Inverse problems are generally ill-posed, namely at least one of the three criteria for well-posedness, *i.e.*, the existence, uniqueness and stability, is violated [2]. Like most inverse problems, the inverse Robin problems are usually ill-posed, since there exists always noise in the measurable data z^δ of the forward solution u. In view of the measurement noise, the data z^δ may not have any desired regularity, hence the existence and stability do not hold generally for inverse Robin problems. However, these inverse problems may be well-posed under some restrictive conditions on the given data and the constraint set of the feasible Robin coefficients; see Section 2.3. The inverse Robin problems have been widely studied in literatures; see [4, 10, 18–20] and the references therein. The Gauss-Newton method was applied in [10] to solve the least-squares formulation of the elliptic inverse Robin problem, but with no consideration of regularizations. An L^1-tracking functional approach was suggested for the elliptic inverse Robin problem in [4]. Effectiveness and justifications of least-squares formulations with regularizations were analysed in [18–20] for the Robin inverse problems, and some iterative methods were applied to solve the resulting nonlinear least-squares minimizations. However, we may observe a common feature of these existing methods, which solve directly the nonlinear optimizations resulting from least-squares formulations with regularisations, but these optimisation problems are highly non-convex as the forward solution $u(\gamma)$ is nonlinear with respect to γ, and strongly unstable at discrete level with fine mesh sizes and time step sizes due to the severe ill-posedness of the inverse problems and the fact that noise is always present in the observation data.

In order to alleviate the effects of these drawbacks, we shall apply the L-M iterative method [9, 13, 23, 26, 30] to solve the nonlinear optimizations resulting from least-squares formulations with regularisations for the concerned inverse Robin problems. With the L-M method, we need only to solve a convex optimization at each iteration. Furthermore, in combination with the surrogate functional technique, we will not require the solution of any optimisation problems in each iteration as the minimisers can be computed explicitly. Another important novelty of this work is its establishment of the quadratic rate of convergence of the L-M method for both the elliptic and parabolic inverse Robin problems. This appears to be the first time in literature to demonstrate the quadratic convergence of the L-M method for a highly nonlinear ill-posed inverse problem. Compared with general optimal control problems or general direct nonlinear optimisation systems, the analysis on the quadratic rate of convergence of the L-M method here is much more delicate and tricky, due to the severe ill-posedness, high nonlinearity and strong instability of the current inverse problems and the direct effect on the convergence from two crucial parameters involved, namely the regularization parameter and the noise level in the data.

The rest of the paper is organized as follows. In Section 2, we present the well-posedness of the forward solutions to the considered elliptic and parabolic systems and discuss the uniqueness and stability of the elliptic and parabolic inverse Robin problems. In Sections 3 and 4, we formulate the Tikhonov regularizations for the nonlinear elliptic and parabolic inverse Robin problem respectively and study some mathematical properties of the resulting nonlinear optimisations. In Sections 3.1 and 4.1, Fréchet derivatives of the forward solution of (1.1) and (1.2) and corresponding adjoint operators are derived respectively. In Sections 3.2 and 4.2, the L-M iterative methods are formulated and their quadratic convergences are established. The surrogate functional approach is applied in Sections 3.3 and 4.3 to solve the convex minimization at each L-M iteration for the nonlinear elliptic and parabolic inverse Robin problem respectively. Several numerical experiments are presented in Section 5 to illustrate the efficiency, accuracy and quadratic convergence of the proposed methods. Some concluding remarks are given in Section 6.

Throughout this work, C is often used for a generic positive constant. We shall use the symbol $\langle\cdot,\cdot\rangle$ for the general inner product, and write the norms of the spaces $H^m(\Omega)$, $L^2(\Omega)$, $H^{1/2}(\Gamma)$ and $L^2(\Gamma)$ (for some $\Gamma \subset \partial\Omega$) respectively as $\|\cdot\|_{m,\Omega}$, $\|\cdot\|_{\Omega}$, $\|\cdot\|_{1/2,\Gamma}$ and $\|\cdot\|_{\Gamma}$. For any $\gamma^* \in K$, we shall frequently use its neighborhood of γ^* with radius $b > 0$ on Γ_i:

$$N(\gamma^*, b) = \{\gamma \in K;\ \|\gamma - \gamma^*\|_{\Gamma_i} \leq b\}. \tag{1.3}$$

2. Well-posedness of the forward solutions and uniqueness of the inverse Robin problems

In this section, we shall present two preliminary lemmas for recalling the classical well-posedness of the forward solutions u to the elliptic and parabolic systems (1.1) and (1.2), then demonstrate and discuss the uniqueness and stability of the corresponding inverse Robin problems.

2.1. Well-posedness of the forward problems

We first introduce the following two well-posedness results, which can be found, *e.g.*, [12] (Thms. 2.4.1.3 and 2.4.2.6) for the elliptic system (1.1), and [24] (Thm. 6.2) for the parabolic system (1.2), for an open bounded and connected domain with C^2-boundary. The domain was assumed to be C^∞-smooth in [24], but it is not essential, and C^2-smoothness is sufficient with some natural modifications of the arguments.

Lemma 2.1. *Assume that $a(\mathbf{x}) \in C^1(\bar{\Omega})$ and $c(\mathbf{x}) \in L^\infty(\Omega)$, both with a positive lower bound, and $\gamma(\mathbf{x}) \in K$, $f(\mathbf{x}) \in L^2(\Omega)$, $g(\mathbf{x}) \in H^{\frac{1}{2}}(\Gamma_i)$ and $h(\mathbf{x}) \in H^{\frac{1}{2}}(\Gamma_a)$. Then there exists a unique solution $u \in H^2(\Omega)$ to the system (1.1) with the estimate*

$$\|u\|_{2,\Omega} \leq C(\|f\|_{\Omega} + \|g\|_{\frac{1}{2},\Gamma_i} + \|h\|_{\frac{1}{2},\Gamma_a}). \tag{2.1}$$

Lemma 2.2. *Assume that* $a(\mathbf{x}) \in C^1(\bar{\Omega})$ *with a positive lower bound,* $\gamma(\mathbf{x}) \in K$*,* $f(\mathbf{x},t) \in L^2(0,T;L^2(\Omega))$*,* $g(\mathbf{x},t) \in L^2(0,T;H^{\frac{1}{2}}(\Gamma_i)) \cap H^{\frac{1}{4}}(0,T;L^2(\Gamma_i))$*,* $h(\mathbf{x},t) \in L^2(0,T;H^{\frac{1}{2}}(\Gamma_a)) \cap H^{\frac{1}{4}}(0,T;L^2(\Gamma_a))$ *and* $u_0(\mathbf{x}) \in H^1(\Omega)$*. Then there exists a unique solution* $u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$ *to the system (1.2) with the estimate:*

$$\begin{aligned}\|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{H^1(0,T;L^2(\Omega))} \le C(&\|f\|_{L^2(0,T;L^2(\Omega))} + \|g\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_i))} + \|g\|_{H^{\frac{1}{4}}(0,T;L^2(\Gamma_i))} \\ &+\|h\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma_a))} + \|h\|_{H^{\frac{1}{4}}(0,T;L^2(\Gamma_a))} + \|u_0\|_{1,\Omega}).\end{aligned} \tag{2.2}$$

2.2. Uniqueness of the inverse Robin problems

We first study the uniqueness of the elliptic inverse Robin problem.

Theorem 2.3. *Let* γ_1 *and* γ_2 *be two solutions to the elliptic inverse Robin problem as stated in Section 1. Furthermore, we assume that* $\text{meas}(\{\mathbf{x} \in \Gamma_i;\, u(\mathbf{x}) = 0\}) = 0$ *where* u *is the solution to the forward system (1.1), then* $\gamma_1 = \gamma_2$ *almost everywhere on* Γ_i*.*

Proof. It is straightforward to verify using (1.1) that $u(\gamma_1) - u(\gamma_2)$ satisfies

$$\begin{cases} -\nabla \cdot (a(\mathbf{x})\nabla(u(\gamma_1) - u(\gamma_2))) + c(\mathbf{x})(u(\gamma_1) - u(\gamma_2)) = 0 & \text{in } \Omega\,, \\ a(\mathbf{x})\frac{\partial(u(\gamma_1)-u(\gamma_2))}{\partial n} = 0 & \text{on } \Gamma_a\,, \\ u(\gamma_1) - u(\gamma_2) = 0 & \text{on } \Gamma_a\,, \end{cases} \tag{2.3}$$

and on the boundary Γ_i,

$$a(\mathbf{x})\frac{\partial(u(\gamma_1) - u(\gamma_2))}{\partial n} + \gamma_1 u(\gamma_1) - \gamma_2 u(\gamma_2) = 0. \tag{2.4}$$

The unique continuation principle [16] implies that $u(\gamma_1) - u(\gamma_2) = 0$ in Ω. Hence, by the trace theorem and the weak form of the system (2.3), we have

$$\|u(\gamma_1) - u(\gamma_2)\|_{\partial\Omega} \le C\|u(\gamma_1) - u(\gamma_2)\|_{1,\Omega} = 0,$$

and for any $\varphi \in H^1(\Omega)$,

$$\int_{\Gamma_i} a(\mathbf{x})\frac{\partial(u(\gamma_1) - u(\gamma_2))}{\partial n}\varphi \mathrm{d}s = \int_\Omega a(\mathbf{x})\nabla(u(\gamma_1) - u(\gamma_2)) \cdot \nabla\varphi + c(\mathbf{x})(u(\gamma_1) - u(\gamma_2))\varphi \mathrm{d}\mathbf{x} = 0\,.$$

Therefore we immediately see that

$$a(\mathbf{x})\frac{\partial(u(\gamma_1) - u(\gamma_2))}{\partial n} = 0 \quad \text{and} \quad u(\gamma_1) = u(\gamma_2) \quad \text{on } \Gamma_i,$$

which, along with (2.4), leads to

$$u(\gamma_1)(\gamma_1 - \gamma_2) = 0 \quad \text{on} \quad \Gamma_i.$$

Now the assumption that $\text{meas}(\{\mathbf{x} \in \Gamma_i :\, u(\mathbf{x}) = 0\}) = 0$ implies $\gamma_1 = \gamma_2$ *a.e.* on Γ_i. □

Next, we establish the uniqueness of the parabolic inverse Robin problem.

Theorem 2.4. *Let γ_1 and γ_2 be two solutions to the parabolic inverse Robin problem as stated in Section 1. Moreover, we assume that* $\mathrm{meas}(\{\mathbf{x} \in \varGamma_i;\ u(\mathbf{x},t) = 0 \text{ for } t \in (0,T)\}) = 0$, *where u is the solution to the forward system (1.2), then $\gamma_1 = \gamma_2$ almost everywhere on $\varGamma_i$.*

Proof. It is straightforward to verify using (1.2) that $u(\gamma_1) - u(\gamma_2)$ satisfies

$$\begin{cases} \partial_t(u(\gamma_1) - u(\gamma_2)) - \nabla \cdot (a(\mathbf{x})\nabla(u(\gamma_1) - u(\gamma_2))) = 0 & \text{in } \varOmega \times [0,T]\,, \\ a(\mathbf{x})\frac{\partial(u(\gamma_1)-u(\gamma_2))}{\partial n} = 0 & \text{on } \varGamma_a \times [0,T]\,, \\ u(\gamma_1) - u(\gamma_2) = 0 & \text{on } \varGamma_a \times [0,T]\,, \\ (u(\gamma_1) - u(\gamma_2))(\mathbf{x},0) = 0 & \text{in } \varOmega\,, \end{cases} \tag{2.5}$$

and on the boundary $\varGamma_i \times [0,T]$,

$$a(\mathbf{x})\frac{\partial(u(\gamma_1) - u(\gamma_2))}{\partial n} + \gamma_1 u(\gamma_1) - \gamma_2 u(\gamma_2) = 0. \tag{2.6}$$

The unique continuation principle [16] implies that $u(\gamma_1) - u(\gamma_2) = 0$ in $\varOmega \times [0,T]$. Hence, by the trace theorem and the weak form of the system (2.5), we have

$$\|u(\gamma_1) - u(\gamma_2)\|_{\partial\varOmega} \le C\|u(\gamma_1) - u(\gamma_2)\|_{1,\varOmega} = 0,$$

and for any $\varphi \in L^2(0,T;H^1(\varOmega))$,

$$\begin{aligned} \int_0^T \int_{\varGamma_i} a(\mathbf{x})\frac{\partial(u(\gamma_1) - u(\gamma_2))}{\partial n}\varphi \mathrm{d}s\mathrm{d}t &= \int_0^T \int_{\varOmega} a(\mathbf{x})\nabla(u(\gamma_1) - u(\gamma_2)) \cdot \nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}t \\ &\quad + \int_0^T \int_{\varOmega} \partial_t(u(\gamma_1) - u(\gamma_2))\varphi \mathrm{d}\mathbf{x}\mathrm{d}t = 0\,. \end{aligned}$$

Therefore we immediately see that

$$a(\mathbf{x})\frac{\partial(u(\gamma_1) - u(\gamma_2))}{\partial n} = 0 \quad \text{and} \quad u(\gamma_1) = u(\gamma_2) \quad \text{on } \varGamma_i \times [0,T],$$

which, along with (2.6), yields sthat

$$u(\gamma_1)(\gamma_1 - \gamma_2) = 0 \quad \text{on } \varGamma_i \times [0,T].$$

Now the assumption $\mathrm{meas}(\{\mathbf{x} \in \varGamma_i;\ u(\mathbf{x},t) = 0 \text{ for } t \in (0,T)\}) = 0$ implies $\gamma_1 = \gamma_2$ *a.e.* on $\varGamma_i$. ∎

2.3. Stability of the inverse Robin problems

In the next section we shall derive and establish our main results of this work, namely, the quadratic convergence of the Levenberg-Marquardt method for solving the nonlinear optimisations resulting from the least-squares formulation of the elliptic and parabolic inverse Robin problems. For this purpose, the uniqueness of the inverse problems we demonstrated in the previous section is insufficient. Instead we shall need the following stability conditions:

$$\|u(\gamma) - u(\gamma^*)\|_{\varGamma_a} \ge c_1\|\gamma - \gamma^*\|_{\varGamma_i} \quad \forall\, \gamma \in N(\gamma^*, b) \tag{2.7}$$

for the elliptic inverse Robin problem, and

$$\int_0^T \|u(\gamma) - u(\gamma^*)\|^2_{\Gamma_a} \mathrm{d}t \geq \bar{c}_1 \|\gamma - \gamma^*\|^2_{\Gamma_i}, \quad \forall\, \gamma \in N(\gamma^*, b) \tag{2.8}$$

for the parabolic inverse Robin problem. Here c_1, $\bar{c}_1$ and b are positive constants, with $b \in (0, 1)$.

Conditions (2.7) and (2.8) are the so-called local Lipschitz stability in a neighborhood of the true Robin coefficient γ^* for the elliptic and parabolic inverse Robin problems respectively [6, 17]. A global Lipschitz stability was established in [28] (Thm. 2.4) for some physically very important cases, namely the Robin coefficients are piecewise constant. A monotone global Lipschitz stability was demonstrated in [5] (Thm. 3). When the feasible Robin coefficients are restricted on a line segment formed by two functions in the constraint set K, the local Lipschitz stability was also verified (see [5], Thm. 2).

Stability analysis is a very important topic for inverse problems, and there are numerous studies for Lipschitz stability of various inverse problems in the literature, for example, for reconstructing the conductivity [1, 11], the source strength [8] and the potential [3] in elliptic systems, for determining the coefficients in a non-stationary transport equation [22] and the radiative transport equation [25], and for the identification of the source strength [14] (Thm. 3.1), the conductivity [31] (Thm. 1.1) and the radiative coefficient [29] (Thm. 2.1) in parabolic systems.

The main results and their analyses in this work, namely the quadratic convergence of the L-M method for solving the nonlinear regularized optimizations associated with the inverse Robin problems, are very general in the sense that they may be extended to other inverse problems principally, as long as Lipschitz stabilities similar to the conditions (2.7) and (2.8) are valid for the concerned inverse problems. We emphasize that the exact formulations of the Lipschitz stabilities should be different from (2.7) and (2.8), depending on the individual inverse problems and the concrete forms of measurement data, and this may lead to many natural modifications of the detailed analyses in this work.

3. Elliptic inverse Robin problem and its L-M solution

3.1. Tikhonov regularization for elliptic inverse Robin problem

In this section we first formulate the Levenberg-Marquardt method for solving the nonlinear non-convex optimisation problems resulting from the least-squares formulation of the elliptic inverse Robin problem as stated in Section 1, incorporated with Tikhonov regularization to handle its ill-posedness due to the presence of the noise in the observation data [20]. We assume the noise level in the observation data z^δ of the true solution u to the elliptic system (1.1) is of order δ, namely

$$\|u(\gamma^*) - z^\delta\|_{\Gamma_a} \leq \delta\,, \tag{3.1}$$

where γ^* is the true Robin coefficient. The elliptic inverse Robin problem is frequently transformed into the following stabilized minimization system with Tikhonov regularization:

$$\min_{\gamma \in K} \mathcal{J}(\gamma) = \|u(\gamma) - z^\delta\|^2_{\Gamma_a} + \beta \|\gamma\|^2_{\Gamma_i}\,, \tag{3.2}$$

where β is the regularization parameter. The formulation (3.2) was shown to be stable in the sense that its minimizer depends continuously on the change of the noise in the data z^δ [20].

For the subsequent analysis on the convergence of the Levenberg-Marquardt method for solving the optimisation (3.2), we shall frequently need the Fréchet derivative of the forward solution $u(\gamma)$ of system (1.1). Let

$w := u'(\gamma)d$ be the Fréchet derivative at direction d, then it solves the following system:

$$\begin{cases} -\nabla\cdot(a(\mathbf{x})\nabla w)+c(\mathbf{x})w=0 & \text{in } \Omega\,, \\ a(\mathbf{x})\frac{\partial w}{\partial n}+\gamma\, w=-d\, u(\gamma) & \text{on } \Gamma_i\,, \\ a(\mathbf{x})\frac{\partial w}{\partial n}=0 & \text{on } \Gamma_a\,. \end{cases} \tag{3.3}$$

Let $u'(\gamma)^*$ be the adjoint operator of the Fréchet derivative $u'(\gamma)$, then it it easy to verify that $w^* := u'(\gamma)^* p \in H^1(\Omega)$ at a general direction p solves the following system

$$\begin{cases} -\nabla\cdot(a(\mathbf{x})\nabla w^*)+c(\mathbf{x})w^*=0 & \text{in } \Omega\,, \\ a(\mathbf{x})\frac{\partial w^*}{\partial n}+\gamma\, w^*=0 & \text{on } \Gamma_i\,, \\ a(\mathbf{x})\frac{\partial w^*}{\partial n}=-p\, u(\gamma) & \text{on } \Gamma_a\,. \end{cases} \tag{3.4}$$

The following lemma gives an important relation for our later study.

Lemma 3.1. *The following relation holds for any directions d and p:*

$$\langle w,\, u(\gamma)p\rangle_{\Gamma_a} = \langle u(\gamma)d,\, w^*\rangle_{\Gamma_i}. \tag{3.5}$$

Proof. For any $\varphi, \psi \in H^1(\Omega)$, we can readily derive the variational forms of systems (3.1) and (3.1):

$$\int_\Omega a(\mathbf{x})\nabla w\cdot\nabla\varphi \mathrm{d}\mathbf{x} + \int_\Omega c(\mathbf{x}) w\varphi \mathrm{d}\mathbf{x} = \int_{\Gamma_i} (-du(\gamma)-\gamma w)\varphi \mathrm{d}s, \tag{3.6}$$

$$\int_\Omega a(\mathbf{x})\nabla w^*\cdot\nabla\psi \mathrm{d}\mathbf{x} + \int_\Omega c(\mathbf{x}) w^*\psi \mathrm{d}\mathbf{x} = -\int_{\Gamma_i} \gamma w^*\psi \mathrm{d}s - \int_{\Gamma_a} pu(\gamma)\psi \mathrm{d}s. \tag{3.7}$$

Now (3.5) follows by taking $\varphi = w^*$ and $\psi = w$ respectively in (3.6) and (3.7). □

3.2. Levenberg-Marquardt method and its convergence

The nonlinearity of the forward solution $u(\gamma)$ of the system (1.1) makes the minimization (3.2) highly non-linear and non-convex with respect to the Robin coefficient γ, as well as strongly unstable at discrete level with the fine mesh size due to the ill-posedness of the inverse problem and the fact that noise is always present in the observation data. To alleviate these difficulties in numerical solutions, we shall apply the Levenberg-Marquardt method to solve (3.2). For a given $\bar\gamma \in K$, we apply the linearization

$$u(\gamma) \approx u(\bar\gamma) + u'(\bar\gamma)(\gamma-\bar\gamma),$$

then we may solve the minimization system (3.2) by the following L-M iteration, which is widely used for general nonlinear optimization problems [9, 30]:

$$J_k(\gamma^{k+1}) = \min_{\gamma\in K} J_k(\gamma) := \|u'(\gamma^k)(\gamma-\gamma^k) - (z^\delta - u(\gamma^k))\|^2_{\Gamma_a} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i}\,. \tag{3.8}$$

Before our study of the convergence of the iteration (3.8), we shall develop some auxiliary results.

Lemma 3.2. *There exist two positive constants L and c_2 such that it holds for all γ, $\bar{\gamma} \in K$ that*

$$\|u(\gamma) - u(\bar{\gamma})\|_{\Gamma_a} \leq L\|\gamma - \bar{\gamma}\|_{\Gamma_i}, \tag{3.9}$$
$$\|u'(\bar{\gamma})(\gamma - \bar{\gamma}) - (u(\gamma) - u(\bar{\gamma}))\|_{\Gamma_a} \leq c_2\|\gamma - \bar{\gamma}\|^2_{\Gamma_i}. \tag{3.10}$$

Proof. From the variational form of the system (1.1), we can easily find that

$$\int_\Omega a(\mathbf{x})\nabla(u(\gamma) - u(\bar{\gamma})) \cdot \nabla\varphi \mathrm{d}\mathbf{x} + \int_\Omega c(\mathbf{x})(u(\gamma) - u(\bar{\gamma}))\varphi \mathrm{d}\mathbf{x} + \int_{\Gamma_i} \bar{\gamma}(u(\gamma) - u(\bar{\gamma}))\varphi \mathrm{d}s$$
$$= -\int_{\Gamma_i} (\gamma - \bar{\gamma})u(\gamma)\varphi \mathrm{d}s. \tag{3.11}$$

Taking $\varphi = u(\gamma) - u(\bar{\gamma})$ and using the lower bounds of $a(\mathbf{x})$, $c(\mathbf{x})$, and $\bar{\gamma}$, we derive

$$\min\{\underline{a}, \underline{c}\}\|u(\gamma) - u(\bar{\gamma})\|^2_{1,\Omega} + \gamma_1\|u(\gamma) - u(\bar{\gamma})\|^2_{\Gamma_i} \leq |\int_{\Gamma_i} (\gamma - \bar{\gamma})u(\gamma)(u(\gamma) - u(\bar{\gamma}))\mathrm{d}s|$$
$$\leq \|u(\gamma)\|_{L^\infty(\Gamma_i)}\|\gamma - \bar{\gamma}\|_{\Gamma_i}\|u(\gamma) - u(\bar{\gamma})\|_{\Gamma_i}.$$

The Sobolev embedding theorem and estimate (2.1) imply that $\|u(\gamma)\|_{L^\infty(\Gamma_i)} \leq C\|u(\gamma)\|_{2,\Omega} \leq C$. Then it follows by the Cauchy-Schwarz inequality that

$$\min\{\underline{a}, \underline{c}\}\|u(\gamma) - u(\bar{\gamma})\|^2_{1,\Omega} \leq \frac{C^2}{2\gamma_1}\|\gamma - \bar{\gamma}\|^2_{\Gamma_i}.$$

Now estimate (3.9) follows directly from this inequality and the trace theorem. To verify the estimate (3.10), we first show

$$\|u'(\gamma)d\|_{\partial\Omega} \leq C\|d\|_{\Gamma_i}. \tag{3.12}$$

Indeed, choosing $\varphi = u'(\gamma)d$ in (3.6), we readily get

$$\int_\Omega a(\mathbf{x})|\nabla(u'(\gamma)d)|^2 \mathrm{d}\mathbf{x} + \int_\Omega c(\mathbf{x})|u'(\gamma)d|^2 \mathrm{d}\mathbf{x} + \int_{\Gamma_i} \gamma|u'(\gamma)d|^2 \mathrm{d}s = -\int_{\Gamma_i} du(\gamma)(u'(\gamma)d)\mathrm{d}s.$$

Then it follows by the Cauchy-Schwarz inequality that

$$\min\{\underline{a}, \underline{c}\}\|u'(\gamma)d\|^2_{1,\Omega} \leq \frac{C^2}{2\gamma_1}\|d\|^2_{\Gamma_i},$$

which, along with the trace theorem, gives (3.12) immediately.

Next, we prove the estimate (3.10). Taking $\gamma = \bar{\gamma}$ and $d = \gamma - \bar{\gamma}$ in (3.6), we have

$$\int_\Omega a(\mathbf{x})\nabla u'(\bar{\gamma})(\gamma - \bar{\gamma}) \cdot \nabla\varphi \mathrm{d}\mathbf{x} + \int_\Omega c(\mathbf{x})u'(\bar{\gamma})(\gamma - \bar{\gamma})\varphi \mathrm{d}\mathbf{x} + \int_{\Gamma_i} \bar{\gamma}u'(\bar{\gamma})(\gamma - \bar{\gamma})\varphi \mathrm{d}s$$
$$= -\int_{\Gamma_i} (\gamma - \bar{\gamma})u(\bar{\gamma})\varphi \mathrm{d}s \quad \forall\, \varphi \in H^1(\Omega). \tag{3.13}$$

Subtracting (3.13) from (3.11) yields

$$\int_\Omega a(\mathbf{x})\nabla(u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma)))\cdot\nabla\varphi \mathrm{d}\mathbf{x}+\int_{\Gamma_i}\bar\gamma(u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma)))\varphi \mathrm{d}s$$
$$+\int_\Omega c(\mathbf{x})(u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma)))\varphi \mathrm{d}\mathbf{x}=\int_{\Gamma_i}(\gamma-\bar\gamma)(u(\gamma)-u(\bar\gamma))\varphi \mathrm{d}s.$$

Then applying the trace theorem, Lagrange mean value theorem and inequality (3.12), we derive

$$\begin{aligned}&\|u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma))\|_{\Gamma_a}\le C\|u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma))\|_{1,\Omega}\le C\|(\gamma-\bar\gamma)(u(\gamma)-u(\bar\gamma))\|_{\Gamma_i}\\&\quad=C\|(\gamma-\bar\gamma)u'(\xi)(\gamma-\bar\gamma)\|_{\Gamma_i}\le c_2\|\gamma-\bar\gamma\|^2_{\Gamma_i},\end{aligned}$$

where ξ is some element in K between γ and $\bar\gamma$. □

Now we are ready to establish a quadratic rate on the convergence of the L-M method (3.8), under the stability condition (2.7). This condition is the frequently adopted basic condition to ensure the quadratic convergence of the L-M method for most direct nonlinear optimization problems [9, 30], so it is natural to bring it to the current nonlinear ill-posed inverse problems.

It is a well-known technical difficulty in a practical numerical realisation of any Tikhonov regularised optimisation system like the ones (3.2) and (3.8) to choose a reasonable and effective regularization parameter β or β_k. Another important novelty of this work is our suggestion of a very simple and easy implementable choice of the parameter β_k based on the following rule:

$$\beta_k=\|u(\gamma^k)-z^\delta\|_{\Gamma_a}\,. \tag{3.14}$$

And surprisingly, as we shall demonstrate below, this choice of the regularization parameter β_k ensures a quadratical convergence of the resulting L-M iteration (3.8).

Considering the presence of the noise (see (3.1)), it is reasonable for us to terminate the L-M iteration (3.8) when its minimizer γ^k is accurate enough in terms of the noise level. This is also consistent with the popular discrepancy principle. More specifically, we shall terminate the iteration if the following criterion is realised:

$$c_1\|\gamma^k-\gamma^*\|_{\Gamma_i}<2\delta\quad\text{or}\quad\|u(\gamma^k)-z^\delta\|_{\Gamma_a}<\delta\,. \tag{3.15}$$

As it is usually hard to achieve the knowledge of c_1 in practice, the first condition serves mainly as a theoretical alternative condition to ensure the quadratic convergence of the L-M iteration. Instead, the second condition is more convenient to realize for applications.

Lemma 3.3. *Under the conditions (2.7), (3.14) and (3.15), if $\gamma^k\in N(\gamma^*,b)$ then γ^{k+1} generated by the iteration (3.8) satisfies*

$$\|u'(\gamma^k)(\gamma^{k+1}-\gamma^k)-(z^\delta-u(\gamma^k))\|_{\Gamma_a}\le c_3(\|\gamma^k-\gamma^*\|^2_{\Gamma_i}+\delta), \tag{3.16}$$
$$\|\gamma^{k+1}-\gamma^k\|^2_{\Gamma_i}\le c_4(\|\gamma^k-\gamma^*\|^2_{\Gamma_i}+\delta), \tag{3.17}$$

where constants c_3 and c_4 are given explicitly by $c_3=\sqrt{\max\{2{c_2}^2+2L^2+1,\,3\}}$ and $c_4=\max\{\frac{4c_2^2}{c_1}+1,\,2\}$.

Proof. As γ^{k+1} is a minimizer in (3.8), we derive using the estimates (3.1), (3.9), (3.10), equality (3.14) and the Cauchy-Schwarz inequality

$$\|u'(\gamma^k)(\gamma^{k+1}-\gamma^k)-(z^\delta-u(\gamma^k))\|^2_{\Gamma_a}\le J_k(\gamma^{k+1})\le J_k(\gamma^*)$$

$$
\begin{aligned}
&= \|u'(\gamma^k)(\gamma^* - \gamma^k) - (u(\gamma^*) - u(\gamma^k)) + u(\gamma^*) - z^\delta\|_{\Gamma_a}^2 + \beta_k \|\gamma^* - \gamma^k\|_{\Gamma_i}^2 \\
&\le 2{c_2}^2 \|\gamma^k - \gamma^*\|_{\Gamma_i}^4 + 2\delta^2 + \|u(\gamma^k) - u(\gamma^*) + u(\gamma^*) - z^\delta\|_{\Gamma_a}^2 \|\gamma^* - \gamma^k\|_{\Gamma_i}^2 \\
&\le 2{c_2}^2 \|\gamma^k - \gamma^*\|_{\Gamma_i}^4 + 2\delta^2 + 2L^2 \|\gamma^* - \gamma^k\|_{\Gamma_i}^4 + 2\delta^2 \|\gamma^* - \gamma^k\|_{\Gamma_i}^2 \\
&\le (2{c_2}^2 + 2L^2 + 1)\|\gamma^k - \gamma^*\|_{\Gamma_i}^4 + (2 + \delta^2)\delta^2 \\
&\le \max\{2{c_2}^2 + 2L^2 + 1,\, 3\}(\|\gamma^k - \gamma^*\|_{\Gamma_i}^4 + \delta^2),
\end{aligned}
$$

which implies (3.16) immediately.

Again, using the minimizing property of γ^{k+1} in (3.8) and the estimates (3.1) and (3.10), we can deduce as follows:

$$
\begin{aligned}
\|\gamma^{k+1} - \gamma^k\|_{\Gamma_i}^2 &\le \frac{1}{\beta_k} J_k(\gamma^{k+1}) \le \frac{1}{\beta_k} J_k(\gamma^*) \\
&= \frac{1}{\beta_k} \|u'(\gamma^k)(\gamma^* - \gamma^k) - (z^\delta - u(\gamma^k))\|_{\Gamma_a}^2 + \|\gamma^* - \gamma^k\|_{\Gamma_i}^2 \\
&\le \frac{1}{\beta_k}(2{c_2}^2 \|\gamma^k - \gamma^*\|_{\Gamma_i}^4 + 2\delta^2) + \|\gamma^* - \gamma^k\|_{\Gamma_i}^2 \\
&= \frac{2{c_2}^2 \|\gamma^k - \gamma^*\|_{\Gamma_i}^4}{\|u(\gamma^k) - z^\delta\|_{\Gamma_a}} + \frac{2\delta^2}{\|u(\gamma^k) - z^\delta\|_{\Gamma_a}} + \|\gamma^* - \gamma^k\|_{\Gamma_i}^2 .
\end{aligned} \tag{3.18}
$$

As stated in (3.15), the iterative process (3.8) terminates if $c_1\|\gamma^k - \gamma^*\|_{\Gamma_i} < 2\delta$ or $\|u(\gamma^k) - z^\delta\|_{\Gamma_a} < \delta$. Otherwise we have $c_1\|\gamma^k - \gamma^*\|_{\Gamma_i} \ge 2\delta$ and $\|u(\gamma^k) - z^\delta\|_{\Gamma_a} \ge \delta$. Then we can easily see that $\frac{2\delta^2}{\|u(\gamma^k) - z^\delta\|_{\Gamma_a}} \le 2\delta$ and

$$
\begin{aligned}
\|u(\gamma^k) - z^\delta\|_{\Gamma_a} &\ge \|u(\gamma^k) - u(\gamma^*)\|_{\Gamma_a} - \|u(\gamma^*) - z^\delta\|_{\Gamma_a} \\
&\ge c_1\|\gamma^k - \gamma^*\|_{\Gamma_i} - \delta \ge c_1\|\gamma^k - \gamma^*\|_{\Gamma_i} - \frac{c_1}{2}\|\gamma^k - \gamma^*\|_{\Gamma_i} \\
&= \frac{c_1}{2}\|\gamma^k - \gamma^*\|_{\Gamma_i} \ge \frac{c_1}{2}\|\gamma^k - \gamma^*\|_{\Gamma_i}^2 ,
\end{aligned}
$$

where we have used the fact that $\|\gamma^k - \gamma^*\|_{\Gamma_i} \le b < 1$ in the last inequality. Now the desired result (3.17) follows readily from these two estimates and (3.18). ☐

Lemma 3.4. *Under the conditions (2.7), (3.14) and (3.15), let γ^k and γ^{k+1} be two consequent iterates generated by the iteration (3.8) such that both γ^k and γ^{k+1} lie in $N(\gamma^*, b)$, then*

$$
\|\gamma^{k+1} - \gamma^*\|_{\Gamma_i} \le c_5(\|\gamma^k - \gamma^*\|_{\Gamma_i}^2 + \delta), \tag{3.19}
$$

where constant c_5 is given explicitly by $c_5 = (c_3 + c_2c_4 + 1)/c_1$.

Proof. It follows from (2.7), (3.10), (3.16) and (3.17) that

$$
\begin{aligned}
c_1\|\gamma^{k+1} - \gamma^*\|_{\Gamma_i} &\le \|u(\gamma^{k+1}) - u(\gamma^*)\|_{\Gamma_a} \le \|u(\gamma^{k+1}) - z^\delta\|_{\Gamma_a} + \delta \\
&= \|u'(\gamma^k)(\gamma^{k+1} - \gamma^k) + u(\gamma^k) - z^\delta - \{u'(\gamma^k)(\gamma^{k+1} - \gamma^k) + u(\gamma^k) - u(\gamma^{k+1})\}\|_{\Gamma_a} + \delta \\
&\le \|u'(\gamma^k)(\gamma^{k+1} - \gamma^k) + u(\gamma^k) - z^\delta)\|_{\Gamma_a} + \|u'(\gamma^k)(\gamma^{k+1} - \gamma^k) + u(\gamma^k) - u(\gamma^{k+1})\|_{\Gamma_a} + \delta \\
&\le c_3(\|\gamma^k - \gamma^*\|_{\Gamma_i}^2 + \delta) + c_2\|\gamma^{k+1} - \gamma^k\|_{\Gamma_i}^2 + \delta \\
&\le c_3(\|\gamma^k - \gamma^*\|_{\Gamma_i}^2 + \delta) + c_2c_4(\|\gamma^k - \gamma^*\|_{\Gamma_i}^2 + \delta) + \delta \\
&\le (c_3 + c_2c_4 + 1)(\|\gamma^k - \gamma^*\|_{\Gamma_i}^2 + \delta),
\end{aligned}
$$

which implies the estimate (3.19). □

In order to establish the quadratic convergence of the L-M iteration, we now emphasize the dependence of all the constants $c_1, \cdots, c_5$ in our previous estimates on the radius b of the ball $N(\gamma^*, b)$. First, we know both constants c_2 and c_3 in (3.10) and (3.16) are independent of b. But constant c_1 in (2.7) depends on this radius b, so we will write $c_1(b)$ to emphasize this dependence. Similarly, we can write the constants c_4 and c_5 in the estimates (3.17) and (3.19) as $c_4(b)$ and $c_5(b)$.

We are now ready to establish our major convergence results in this work, the quadratic convergence and quadratic rate of convergence for the L-M iteration (3.8). For simplicity, we set

$$r(b,\delta) = \min\left\{ b,\, \frac{b - \sqrt{c_4\left(\frac{1}{3}\right)\delta}}{\sqrt{c_4\left(\frac{1}{3}\right)} + 1} \right\}, \quad \alpha = c_5\left(\frac{1}{3}\right)\left(\sqrt{c_4\left(\frac{1}{3}\right)} + 1\right), \quad \beta(x) = \alpha x + \sqrt{c_4\left(\frac{1}{3}\right)x}\,, \tag{3.20}$$

$$d_0 = \frac{1 + 2c_4\left(\frac{1}{3}\right) - 2\sqrt{c_4^2\left(\frac{1}{3}\right) + c_4\left(\frac{1}{3}\right)}}{4\alpha^2}, \quad d = \frac{1}{2}d_0\,. \tag{3.21}$$

Theorem 3.5. *For any $b \in \left[\frac{1-\sqrt{1-4\alpha\beta(d)}}{2\alpha}, \frac{1+\sqrt{1-4\alpha\beta(d)}}{2\alpha}\right]$, we assume the conditions (2.7), (3.14) and (3.15) are satisfied and the noise level δ is small such that*

$$\delta < \min\left\{ \frac{b^2}{c_4(\frac{1}{3})},\, d \right\}. \tag{3.22}$$

Then for any $\gamma^0 \in N(\gamma^*, r(b,\delta))$, the sequence $\{\gamma^k\}$ generated by (3.8) stays always in $N(\gamma^*, b)$ and satisfies

$$\|\gamma^{k+1} - \gamma^*\|_{\Gamma_i} \le c_5\left(\frac{1}{3}\right)(\|\gamma^k - \gamma^*\|^2_{\Gamma_i} + \delta).$$

Proof. From the results of Lemma 3.4, we only need to show that the sequence $\{\gamma^k\}$ generated by (3.8) stays always in $N(\gamma^*, b)$. This is proved below by the mathematical induction.

First, we derive several helpful estimates. We can readily see from (3.9) and assumption (2.7) that $c_1(b) \le L$. Then by the definitions of c_3, c_4, c_5 and α, we derive

$$c_5(b) > \frac{c_3}{c_1(b)} > \frac{\sqrt{2L^2}}{c_1(b)} \ge \sqrt{2}\,, \quad c_4(b) \ge 2\,, \quad \alpha \ge \sqrt{2}(2+1) > 3\,.$$

On the other hand, using the definitions of $r(b,\delta)$ and the fact that δ is smaller than the first fraction in the condition (3.22), we see $r(b,\delta) > 0$. Hence, by the choice b we know $0 < b < 1/3$, and $\gamma^0 \in N(\gamma^*, r(b,\delta)) \subset N(\gamma^*, 1/3)$. Then by the triangle inequality and estimate (3.17) with $b = 1/3$ we deduce

$$\begin{aligned}\|\gamma^1 - \gamma^*\|_{\Gamma_i} \le \|\gamma^1 - \gamma^0\|_{\Gamma_i} + \|\gamma^0 - \gamma^*\|_{\Gamma_i} &\le \left(\sqrt{c_4\left(\frac{1}{3}\right)} + 1\right)\|\gamma^0 - \gamma^*\|_{\Gamma_i} + \sqrt{c_4\left(\frac{1}{3}\right)\delta} \\ &\le \left(\sqrt{c_4\left(\frac{1}{3}\right)} + 1\right) r(b,\delta) + \sqrt{c_4\left(\frac{1}{3}\right)\delta} \le b,\end{aligned}$$

which implies $\gamma^1 \in N(\gamma^*, b) \subset N(\gamma^*, 1/3)$.

Now we show $\gamma^{l+1} \in N(\gamma^*, b)$ if $\gamma^k \in N(\gamma^*, b)$ for $k = 1, ..., l$. Indeed, we deduce from the triangle inequality, the estimate (3.17) for $\gamma^l \in N(\gamma^*, b)$ and the estimate (3.19) for $\gamma^l, \gamma^{l-1} \in N(\gamma^*, b)$ that

$$\begin{aligned}
\|\gamma^{l+1} - \gamma^*\|_{\Gamma_i} &\leq \|\gamma^{l+1} - \gamma^l\|_{\Gamma_i} + \|\gamma^l - \gamma^*\|_{\Gamma_i} \\
&\leq \left(\sqrt{c_4\left(\frac{1}{3}\right)} + 1\right)\|\gamma^l - \gamma^*\|_{\Gamma_i} + \sqrt{c_4\left(\frac{1}{3}\right)\delta} \\
&\leq \left(\sqrt{c_4\left(\frac{1}{3}\right)} + 1\right) c_5\left(\frac{1}{3}\right)(\|\gamma^{l-1} - \gamma^*\|^2_{\Gamma_i} + \delta) + \sqrt{c_4\left(\frac{1}{3}\right)\delta} \\
&< \left(\sqrt{c_4\left(\frac{1}{3}\right)} + 1\right) c_5\left(\frac{1}{3}\right)(\|\gamma^{l-1} - \gamma^*\|^2_{\Gamma_i} + d) + \sqrt{c_4\left(\frac{1}{3}\right)d} \\
&\leq \alpha b^2 + \beta(d) \leq b\,,
\end{aligned} \tag{3.23}$$

which implies that $\gamma^{l+1} \in N(\gamma^*, b)$, if it holds that $\alpha b^2 + \beta(d) \leq b$.

To see $\alpha b^2 + \beta(d) \leq b$, we need several helpful estimates. First, we use the definitions of β, d_0, d and α in (3.20) and (3.21) and direct computings to derive

$$\begin{aligned}
1 - 4\alpha\beta(d) &> 1 - 4\alpha\beta(d_0) = 1 - 4\alpha^2 d_0 - 4\alpha\sqrt{c_4\left(\frac{1}{3}\right)d_0} \\
&= 1 - \left\{1 + 2c_4\left(\frac{1}{3}\right) - 2\sqrt{c_4^2\left(\frac{1}{3}\right) + c_4\left(\frac{1}{3}\right)}\right\} - 2\sqrt{c_4\left(\frac{1}{3}\right)\left\{1 + 2c_4\left(\frac{1}{3}\right) - 2\sqrt{c_4^2\left(\frac{1}{3}\right) + c_4\left(\frac{1}{3}\right)}\right\}} \\
&= 2\left(\sqrt{c_4^2\left(\frac{1}{3}\right) + c_4\left(\frac{1}{3}\right)} - c_4\left(\frac{1}{3}\right)\right) - 2\left(\sqrt{c_4^2\left(\frac{1}{3}\right) + c_4\left(\frac{1}{3}\right)} - c_4\left(\frac{1}{3}\right)\right) = 0.
\end{aligned}$$

This, along with the fact that $\alpha > 3$, implies $1 + \sqrt{1 - 4\alpha\beta(d)} \leq 2 < (2\alpha)/3$. We are now ready to verify $\alpha b^2 + \beta(d) \leq b$. For this, we define a quadratic functional $f(b) = \alpha b^2 - b + \beta(d)$. As $1 - 4\alpha\beta(d) > 0$, it is easy to see $0 < b_1 = \frac{1-\sqrt{1-4\alpha\beta(d)}}{2\alpha} < 1/6$ and $b_1 < b_2 = \frac{1+\sqrt{1-4\alpha\beta(d)}}{2\alpha} < 1/3$, and b_1 and b_2 are two solutions of $f(b) = 0$. Clearly for any $b \in [b_1, b_2]$, we know $f(b) \leq 0$, namely, $\alpha b^2 + \beta(d) \leq b$. □

3.3. Surrogate functional technique

In each step of the L-M iteration we have to solve the minimization problem (3.8). Let us now derive its optimality system, *i.e.*, $J_k'(\gamma^{k+1})\xi = 0$ for any $\xi \in L^2(\Gamma_i)$. By direct computations, we have

$$\begin{aligned}
J_k'(\gamma)\xi &= 2\langle u'(\gamma^k)(\gamma - \gamma^k) - (z^\delta - u(\gamma^k)), u'(\gamma^k)(\xi)\rangle_{\Gamma_a} + 2\beta_k\langle \gamma - \gamma^k, \xi\rangle_{\Gamma_i} \\
&= 2\left\langle u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{u'(\gamma^k)(\gamma - \gamma^k) - (z^\delta - u(\gamma^k))}{u(\gamma^k)}\right)\right\}, \xi\right\rangle_{\Gamma_i} + 2\beta_k\langle \gamma - \gamma^k, \xi\rangle_{\Gamma_i},
\end{aligned}$$

where we have used the adjoint relation (3.5). This is equivalent to the following equation:

$$u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{u'(\gamma^k)(\gamma - \gamma^k)}{u(\gamma^k)}\right)\right\} + \beta_k(\gamma - \gamma^k) = u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k)}{u(\gamma^k)}\right)\right\}. \tag{3.24}$$

So we have to solve this rather complicated linear system (whose discretized system is highly ill-conditioned) to get the solution γ^{k+1} at each iteration of (3.8), *e.g.*, by some iterative method. This is still difficult and computationally very expensive.

Next, we shall make use of the surrogate functional technique to greatly simplify the solution to the minimization (3.8), resulting in an explicit solution at each iteration. The resultant algorithm is computationally much less expensive. The surrogate functional technique was studied in [7] for solving a linear inverse operator equation of the form $Kh = f$. We now construct a surrogate functional $J_k^s(\gamma,\zeta)$ of $J_k(\gamma)$ in (3.8) for any $\zeta \in K$:

$$J_k^s(\gamma,\zeta) = J_k(\gamma) + A\|\gamma-\zeta\|^2_{\Gamma_i} - \|u'(\gamma^k)(\gamma-\zeta)\|^2_{\Gamma_a}, \tag{3.25}$$

where A can be any positive constant such that $\|u'(\gamma^k)d\|^2_{\Gamma_a} \le A\|d\|^2_{\Gamma_i}$ for all $d \in L^2(\Gamma_i)$. Next, we will simplify the expression (3.25). Using the adjoint relation (3.5), we can rewrite $J_k^s(\gamma,\zeta)$ as follows:

$$\begin{aligned}
J_k^s(\gamma,\zeta) &= \|u'(\gamma^k)(\gamma-\gamma^k) - (z^\delta - u(\gamma^k))\|^2_{\Gamma_a} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i} + A\|\gamma-\zeta\|^2_{\Gamma_i} - \|u'(\gamma^k)(\gamma-\zeta)\|^2_{\Gamma_a}\\
&= \|u'(\gamma^k)(\gamma)\|^2_{\Gamma_a} - 2\langle u'(\gamma^k)(\gamma),\, z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k)\rangle_{\Gamma_a} + \|z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k)\|^2_{\Gamma_a}\\
&\quad + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i} + A\|\gamma-\zeta\|^2_{\Gamma_i} - \|u'(\gamma^k)(\gamma)\|^2_{\Gamma_a} + 2\langle u'(\gamma^k)(\gamma),\, u'(\gamma^k)(\zeta)\rangle_{\Gamma_a} - \|u'(\gamma^k)(\zeta)\|^2_{\Gamma_a}\\
&= -2\left\langle \gamma,\, u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\right\rangle_{\Gamma_i} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i}\\
&\quad + A\|\gamma-\zeta\|^2_{\Gamma_i} + \|z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k)\|^2_{\Gamma_a} - \|u'(\gamma^k)(\zeta)\|^2_{\Gamma_a}\\
&= A\|\gamma - \zeta - \frac{1}{A}u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\|^2_{\Gamma_i} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i}\\
&\quad + \Big\{\|z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k)\|^2_{\Gamma_a} - \|u'(\gamma^k)(\zeta)\|^2_{\Gamma_a} + A\|\zeta\|^2_{\Gamma_i}\\
&\quad - A\|\zeta + \frac{1}{A}u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\|^2_{\Gamma_i}\Big\}.
\end{aligned} \tag{3.26}$$

We can see that the last term above is independent of γ, so does not affect the minimization. Hence we will drop that term in the functional $J_k^s(\gamma,\zeta)$ and obtain

$$\min_{\gamma\in K} J_k^s(\gamma,\zeta) = \min_{\gamma\in K} A\|\gamma-\zeta-\frac{1}{A}u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k)+u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\|^2_{\Gamma_i} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i}. \tag{3.27}$$

This is a simple quadratic minimization, and we can compute its minimizer exactly:

$$\operatorname*{argmin}_{\gamma\in K} J_k^s(\gamma,\zeta) = \frac{\beta_k}{A+\beta_k}\gamma^k + \frac{A}{A+\beta_k}\zeta + \frac{1}{A+\beta_k}u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}. \tag{3.28}$$

This motivates us with the following reconstruction algorithm for the Robin coefficient in (1.1), which is clearly much easier and computationally much less expensive than solving the minimization (3.8) directly.

Algorithm 3.6. Choose two tolerance parameters ϵ_1, $\epsilon_2 > 0$ and an initial value γ^0, and set $k := 0$.

1. Compute γ^{k+1}: set $\zeta_0 = \gamma^k$ and $n := 0$.
   (1.1). Compute

$$\zeta_{n+1} = \operatorname*{argmin}_{\gamma\in K} J_k^s(\gamma,\zeta_n) = \frac{1}{A+\beta_k}\left(\beta_k\gamma^k + A\zeta_n + u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k)+u'(\gamma^k)(\gamma^k-\zeta_n)}{u(\gamma^k)}\right)\right\}\right).$$

(1.2). If $\frac{\|\zeta_{n+1}-\zeta_n\|_{\Gamma_i}}{\|\zeta_n\|_{\Gamma_i}} \le \epsilon_1$, set $\gamma^{k+1} = \zeta_{n+1}$, go to Step 2; otherwise set $n := n+1$, go to Step (1.1).

2. If $\frac{\|\gamma^{k+1}-\gamma^k\|_{\Gamma_i}}{\|\gamma^k\|_{\Gamma_i}} \le \epsilon_2$, stop the iteration; otherwise set $k := k+1$, go to Step 1.

4. Parabolic inverse Robin problem and its L-M solution

4.1. Tikhonov regularization for the parabolic inverse Robin problem

We now follow what we did in Section 3 to convert our interested parabolic inverse Robin problem to a nonlinear optimization problem by a Tikhonov regularization, and then propose to solve the optimization by the L-M method. Let γ^* be the true Robin coefficient in the system (1.2), and the noise level in the observation data z^δ of the true solution u to the parabolic system (1.2) be of order δ, *i.e.*,

$$\int_0^T \|u(\gamma^*) - z^\delta\|^2_{\Gamma_a} \mathrm{d}t \le \delta^2. \tag{4.1}$$

Let β be a tregularization parameter, then we may transform the parabolic inverse Robin problem to the following minimization with Tikhonov regularization, which was shown to be a stable formulation [19]:

$$\min_{\gamma \in K} \mathcal{J}(\gamma) = \int_0^T \|u(\gamma) - z^\delta\|^2_{\Gamma_a} \mathrm{d}t + \beta \|\gamma\|^2_{\Gamma_i}\,. \tag{4.2}$$

For our later analysis, we need the Fréchet derivative of the forward solution $u(\gamma)$ of system (1.2). Let $w := u'(\gamma)d$ be the derivative at direction $d \in L^2(\Gamma_i)$, then $w \in L^2(0,T;H^1(\Omega))$ solves the system:

$$\begin{cases} \partial_t w - \nabla \cdot (a(\mathbf{x})\nabla w) = 0 & \text{in } \Omega \times (0,T)\,, \\ a(\mathbf{x})\frac{\partial w}{\partial n} + \gamma\, w = -d\, u(\gamma) & \text{on } \Gamma_i \times (0,T)\,, \\ a(\mathbf{x})\frac{\partial w}{\partial n} = 0 & \text{on } \Gamma_a \times (0,T) \\ w(\mathbf{x},0) = 0 & \text{in } \Omega\,, \end{cases} \tag{4.3}$$

For any direction $p \in L^2(0,T;L^2(\Gamma_a))$, we define the adjoint $w^* := u'(\gamma)^* p$, then $w^* \in L^2(0,T;H^1(\Omega))$ solves the following parabolic system:

$$\begin{cases} -\partial_t w^* - \nabla \cdot (a(\mathbf{x})\nabla w^*) = 0 & \text{in } \Omega \times [0,T]\,, \\ a(\mathbf{x})\frac{\partial w^*}{\partial n} + \gamma\, w^* = 0 & \text{on } \Gamma_i \times [0,T]\,, \\ a(\mathbf{x})\frac{\partial w^*}{\partial n} = -p\, u(\gamma) & \text{on } \Gamma_a \times [0,T]\,, \\ w^*(\mathbf{x},T) = 0 & \text{in } \Omega\,. \end{cases} \tag{4.4}$$

We shall need the following useful relation between $u'(\gamma)$ and its adjoint $u'(\gamma)^*$.

Lemma 4.1. *It holds for any directions $d \in L^2(\Gamma_i)$ and $p \in L^2(0,T;L^2(\Gamma_a))$ that*

$$\int_0^T \langle u(\gamma)w,\, p\rangle_{\Gamma_a} dt = \langle d,\, \int_0^T u(\gamma) w^*\, dt\rangle_{\Gamma_i}. \tag{4.5}$$

Proof. For any $\varphi, \psi \in L^2(0,T;H^1(\Omega))$, it is easy to derive the variational forms of (4.1) and (4.4):

$$\int_0^T \int_\Omega \partial_t w \varphi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_\Omega a(\mathbf{x})\nabla w \cdot \nabla \varphi \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_{\Gamma_i} (-du(\gamma) - \gamma w)\varphi \mathrm{d}s\mathrm{d}t, \tag{4.6}$$

$$-\int_0^T\int_\Omega \partial_t w^*\psi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Omega a(\mathbf{x})\nabla w^*\cdot\nabla\psi \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$= -\int_0^T\int_{\Gamma_i} \gamma w^*\psi \mathrm{d}s\mathrm{d}t - \int_0^T\int_{\Gamma_a} pu(\gamma)\psi \mathrm{d}s\mathrm{d}t. \tag{4.7}$$

By integrating by parts with respect to t in (4.7), we derive

$$-\int_\Omega w^*(\mathbf{x},T)\psi(\mathbf{x},T)\mathrm{d}\mathbf{x} + \int_\Omega w^*(\mathbf{x},0)\psi(\mathbf{x},0)\mathrm{d}\mathbf{x} + \int_0^T\int_\Omega \partial_t\psi w^* \mathrm{d}\mathbf{x}\mathrm{d}t$$
$$+\int_0^T\int_\Omega a(\mathbf{x})\nabla w^*\cdot\nabla\psi \mathrm{d}\mathbf{x}\mathrm{d}t = -\int_0^T\int_{\Gamma_i}\gamma w^*\psi \mathrm{d}s\mathrm{d}t - \int_0^T\int_{\Gamma_a} pu(\gamma)\psi \mathrm{d}s\mathrm{d}t. \tag{4.8}$$

Taking $\varphi = w^*$ in (4.6), $\psi = w$ in (4.8) and noting that $w^*(\mathbf{x},T) = w(\mathbf{x},0) = 0$, we can easily find that

$$\int_0^T \langle u(\gamma)w,\, p\rangle_{\Gamma_a}\mathrm{d}t = \int_0^T \langle d,\, u(\gamma)w^*\rangle_{\Gamma_i}\mathrm{d}t = \langle d,\, \int_0^T u(\gamma)w^*\mathrm{d}t\rangle_{\Gamma_i}.$$

□

4.2. Levenberg-Marquardt method and its convergence

The minimization (4.2) is highly nonlinear and non-convex due to the nonlinearity of the parabolic forward solution $u(\gamma)$ to system (1.2) with respect to the Robin coefficient γ. Motivated by the same reasons as we pointed out earlier for the elliptic case, we apply the L-M iteration method to alleviate the nonlinearity and non-convexity of the minimization (4.2):

$$J_k(\gamma^{k+1}) = \min_{\gamma\in K} J_k(\gamma) =: \int_0^T \|u'(\gamma^k)(\gamma-\gamma^k) - (z^\delta - u(\gamma^k))\|^2_{\Gamma_a}\mathrm{d}t + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i}\,. \tag{4.9}$$

In the rest of this section, we establish our main result, namely, the quadratic convergence of the L-M iteration (4.9), under the basic condition (2.8). The same as we did in Section 3.2, we suggest a very simple and easy implementable strategy at each L-M iteration to choose the regularization parameter β_k adaptively:

$$\beta_k = \|u(\gamma^k) - z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}, \tag{4.10}$$

and terminate the iteration based on the following criterion:

$$\sqrt{\bar c_1}\|\gamma^k - \gamma^*\|_{\Gamma_i} < 2\delta \quad \text{or} \quad \|u(\gamma^k) - z^\delta\|_{L^2(0,T;L^2(\Gamma_a))} < \delta\,. \tag{4.11}$$

We now present several auxiliary results for our later analysis of the convergence of the iteration (4.9).

Lemma 4.2. *There exist two positive constants $\bar L$ and $\bar c_2$ such that it holds for any $\gamma,\,\bar\gamma\in K$ that*

$$\int_0^T \|u(\gamma) - u(\bar\gamma)\|^2_{\Gamma_a}dt \le \bar L\|\gamma-\bar\gamma\|^2_{\Gamma_i}, \tag{4.12}$$

$$\int_0^T \|u'(\bar\gamma)(\gamma-\bar\gamma) - (u(\gamma)-u(\bar\gamma))\|^2_{\Gamma_a}dt \le \bar c_2\|\gamma-\bar\gamma\|^4_{\Gamma_i}\,. \tag{4.13}$$

Proof. From the variational form of the system (1.2), we can easily see for any $\varphi \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}&\int_\Omega \partial_t(u(\gamma)-u(\bar\gamma))\varphi \mathrm{d}\mathbf{x} + \int_\Omega a(\mathbf{x})\nabla(u(\gamma)-u(\bar\gamma))\cdot\nabla\varphi \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{x}\\ &+\int_{\Gamma_i}\bar\gamma(u(\gamma)-u(\bar\gamma))\varphi \mathrm{d}s = -\int_{\Gamma_i}(\gamma-\bar\gamma)u(\gamma)\varphi \mathrm{d}s.\end{aligned}\tag{4.14}$$

Taking $\varphi = u(\gamma)-u(\bar\gamma)$ in (4.14) and integrating by parts with respect to t over $[0,\tau]$ for $\tau\in[0,T]$, then using the Cauchy-Schwarz inequality, Sobolev embedding theorem and estimate (2.2), we derive

$$\begin{aligned}&\frac{1}{2}\|u(\gamma)(\mathbf{x},\tau)-u(\bar\gamma)(\mathbf{x},\tau)\|_\Omega^2 + \underline{a}\int_0^\tau \|\nabla u(\gamma)-\nabla u(\bar\gamma)\|_\Omega^2 \mathrm{d}t + \gamma_1\int_0^\tau \|u(\gamma)-u(\bar\gamma)\|_{\Gamma_i}^2\mathrm{d}t\\ &\le |\int_0^\tau\int_{\Gamma_i}(\gamma-\bar\gamma)u(\gamma)(u(\gamma)-u(\bar\gamma))\mathrm{d}s\mathrm{d}t|\\ &\le \|u(\gamma)\|_{L^2(0,T;L^\infty(\Gamma_i))}\sqrt{\tau}\|\gamma-\bar\gamma\|_{\Gamma_i}\|u(\gamma)-u(\bar\gamma)\|_{L^2(0,\tau;L^2(\Gamma_i))}\\ &\le C\|u(\gamma)\|_{L^2(0,T;H^2(\Omega))}\sqrt{\tau}\|\gamma-\bar\gamma\|_{\Gamma_i}\|u(\gamma)-u(\bar\gamma)\|_{L^2(0,\tau;L^2(\Gamma_i))}\\ &\le C\|\gamma-\bar\gamma\|_{\Gamma_i}\|u(\gamma)-u(\bar\gamma)\|_{L^2(0,\tau;L^2(\Gamma_i))}.\end{aligned}$$

Now a direct application of the Young's inequality gives

$$\|u(\gamma)(\mathbf{x},\tau)-u(\bar\gamma)(\mathbf{x},\tau)\|_\Omega^2 + \int_0^\tau \|\nabla u(\gamma)-\nabla u(\bar\gamma)\|_\Omega^2\mathrm{d}t \le C\|\gamma-\bar\gamma\|_{\Gamma_i}^2,$$

from which and the trace theorem, we obtain

$$\int_0^T \|u(\gamma)-u(\bar\gamma)\|_{\Gamma_a}^2\mathrm{d}t \le C\int_0^T \|u(\gamma)-u(\bar\gamma)\|_{1,\Omega}^2\mathrm{d}t \le \bar{L}\|\gamma-\bar\gamma\|_{\Gamma_i}^2.$$

To verify the estimate (4.13), we first have by taking $\varphi = u'(\gamma)d$ in (4.6) and then following the same technique as we did for (4.12) that

$$\int_0^T \|u'(\gamma)d\|_{\partial\Omega}^2\mathrm{d}t \le C\|\gamma-\bar\gamma\|_{\Gamma_i}^2\,.\tag{4.15}$$

Next, we take $\gamma=\bar\gamma$ and $d=\gamma-\bar\gamma$ in (4.6) to deduce

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t(u'(\bar\gamma)(\gamma-\bar\gamma))\varphi\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Omega a(\mathbf{x})\nabla(u'(\bar\gamma)(\gamma-\bar\gamma))\cdot\nabla\varphi\mathrm{d}\mathbf{x}\mathrm{d}t\\ &\qquad = \int_0^T\int_{\Gamma_i}(-(\gamma-\bar\gamma)u(\bar\gamma)-\bar\gamma(u'(\bar\gamma)(\gamma-\bar\gamma)))\varphi\mathrm{d}s\mathrm{d}t.\end{aligned}\tag{4.16}$$

Subtracting (4.16) from (4.14) gives

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t(u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma)))\varphi\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_\Omega a(\mathbf{x})\nabla(u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma)))\cdot\nabla\varphi\mathrm{d}\mathbf{x}\mathrm{d}t\\ &+\int_0^T\int_{\Gamma_i}\bar\gamma(u'(\bar\gamma)(\gamma-\bar\gamma)-(u(\gamma)-u(\bar\gamma)))\varphi\mathrm{d}s\mathrm{d}t = \int_0^T\int_{\Gamma_i}(\gamma-\bar\gamma)(u(\gamma)-u(\bar\gamma))\varphi\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Now applying the trace theorem, Lagrange mean value theorem and estimate (4.15), we can derive

$$
\begin{aligned}
&\int_0^T \|u'(\bar{\gamma})(\gamma-\bar{\gamma})-(u(\gamma)-u(\bar{\gamma}))\|_{\Gamma_a}^2 \mathrm{d}t \\
&\qquad \le C\int_0^T \|u'(\bar{\gamma})(\gamma-\bar{\gamma})-(u(\gamma)-u(\bar{\gamma}))\|_{1,\Omega}^2 \mathrm{d}t \\
&\qquad \le C\int_0^T \|(\gamma-\bar{\gamma})(u(\gamma)-u(\bar{\gamma}))\|_{\Gamma_i}^2 \mathrm{d}t \\
&\qquad = C\int_0^T \|(\gamma-\bar{\gamma})u'(\xi)(\gamma-\bar{\gamma})\|_{\Gamma_i}^2 \mathrm{d}t \le \bar{c}_2\|\gamma-\bar{\gamma}\|_{\Gamma_i}^4,
\end{aligned}
$$

where ξ is some element in K between γ and $\bar{\gamma}$. □

Lemma 4.3. *Under the conditions (2.8), (4.10) and (4.11), if $\gamma^k \in N(\gamma^*, b)$, then γ^{k+1} generated by the iteration (4.9) satisfies*

$$
\int_0^T \|u'(\gamma^k)(\gamma^{k+1}-\gamma^k)-(z^\delta-u(\gamma^k))\|_{\Gamma_a}^2 dt \le \bar{c}_3(\|\gamma^k-\gamma^*\|_{\Gamma_i}^4+\delta^2), \tag{4.17}
$$

$$
\|\gamma^{k+1}-\gamma^k\|_{\Gamma_i}^2 \le \bar{c}_4(\|\gamma^k-\gamma^*\|_{\Gamma_i}^2+\delta), \tag{4.18}
$$

where $\bar{c}_3$ and $\bar{c}_4$ are two constants given explicitly by $\bar{c}_3 = \max\{2\bar{c}_2+2\bar{L}+1,\, 3\}$ and $\bar{c}_4 = \max\{\frac{4\bar{c}_2}{\sqrt{\bar{c}_1}}+1,\, 2\}$.

Proof. As γ^{k+1} is a minimizer in (4.9), then by using the estimates (4.1), (4.12)-(4.13), the equality (4.10) and Cauchy-Schwarz inequality, we derive

$$
\begin{aligned}
&\int_0^T \|u'(\gamma^k)(\gamma^{k+1}-\gamma^k)-(z^\delta-u(\gamma^k))\|_{\Gamma_a}^2 \mathrm{d}t \le J_k(\gamma^{k+1}) \le J_k(\gamma^*) \\
&= \int_0^T \|u'(\gamma^k)(\gamma^*-\gamma^k)-(u(\gamma^*)-u(\gamma^k))+u(\gamma^*)-z^\delta\|_{\Gamma_a}^2 \mathrm{d}t + \beta_k\|\gamma^*-\gamma^k\|_{\Gamma_i}^2 \\
&\le 2\bar{c}_2\|\gamma^k-\gamma^*\|_{\Gamma_i}^4 + 2\delta^2 + \int_0^T \|u(\gamma^k)-u(\gamma^*)+u(\gamma^*)-z^\delta\|_{\Gamma_a}^2 \mathrm{d}t\|\gamma^*-\gamma^k\|_{\Gamma_i}^2 \\
&\le 2\bar{c}_2\|\gamma^k-\gamma^*\|_{\Gamma_i}^4 + 2\delta^2 + 2\bar{L}\|\gamma^*-\gamma^k\|_{\Gamma_i}^4 + 2\delta^2\|\gamma^*-\gamma^k\|_{\Gamma_i}^2 \\
&\le (2\bar{c}_2+2\bar{L}+1)\|\gamma^k-\gamma^*\|_{\Gamma_i}^4 + (2+\delta^2)\delta^2 \\
&\le \max\{2\bar{c}_2+2\bar{L}+1,\, 3\}(\|\gamma^k-\gamma^*\|_{\Gamma_i}^4+\delta^2) \\
&\equiv \bar{c}_3(\|\gamma^k-\gamma^*\|_{\Gamma_i}^4+\delta^2).
\end{aligned}
$$

Using the minimizing property of γ^{k+1} in (4.9), the estimates (4.1) and (4.13), we can derive that

$$
\begin{aligned}
&\|\gamma^{k+1}-\gamma^k\|_{\Gamma_i}^2 \le \frac{1}{\beta_k}J_k(\gamma^{k+1}) \le \frac{1}{\beta_k}J_k(\gamma^*) \\
&= \frac{1}{\beta_k}\int_0^T \|u'(\gamma^k)(\gamma^*-\gamma^k)-(z^\delta-u(\gamma^k))\|_{\Gamma_a}^2 \mathrm{d}t + \|\gamma^*-\gamma^k\|_{\Gamma_i}^2 \\
&\le \frac{1}{\beta_k}(2\bar{c}_2\|\gamma^k-\gamma^*\|_{\Gamma_i}^4+2\delta^2) + \|\gamma^*-\gamma^k\|_{\Gamma_i}^2 \\
&= \frac{2\bar{c}_2\|\gamma^k-\gamma^*\|_{\Gamma_i}^4}{\|u(\gamma^k)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}} + \frac{2\delta^2}{\|u(\gamma^k)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}} + \|\gamma^*-\gamma^k\|_{\Gamma_i}^2.
\end{aligned} \tag{4.19}
$$

We know the iterative process (4.9) terminates if $\sqrt{\bar{c}_1}\|\gamma^k-\gamma^*\|_{\Gamma_i}<2\delta$ or $\|u(\gamma^k)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}<\delta$ by the criterion (4.11). Otherwise we have $\sqrt{\bar{c}_1}\|\gamma^k-\gamma^*\|_{\Gamma_i}\ge 2\delta$ and $\|u(\gamma^k)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}\ge\delta$, which yields that $2\delta^2/\|u(\gamma^k)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}\le 2\delta$ and

$$\begin{aligned}\|u(\gamma^k)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}&\ge\|u(\gamma^k)-u(\gamma^*)\|_{L^2(0,T;L^2(\Gamma_a))}-\|u(\gamma^*)-z^\delta\|_{L^2(0,T;L^2(\Gamma_a))}\\&\ge\sqrt{\bar{c}_1}\|\gamma^k-\gamma^*\|_{\Gamma_i}-\delta\ge\sqrt{\bar{c}_1}\|\gamma^k-\gamma^*\|_{\Gamma_i}-\frac{\sqrt{\bar{c}_1}}{2}\|\gamma^k-\gamma^*\|_{\Gamma_i}=\frac{\sqrt{\bar{c}_1}}{2}\|\gamma^k-\gamma^*\|_{\Gamma_i}\ge\frac{\sqrt{\bar{c}_1}}{2}\|\gamma^k-\gamma^*\|^2_{\Gamma_i}.\end{aligned}$$

Now the desired estimate (4.18) follows directly from (4.19) and the above two estimates. □

Lemma 4.4. *Under the conditions (2.8), (4.10) and (4.11), let γ^k and γ^{k+1} be two consequent iterates generated by the iteration (4.9) and satisfy that γ^k, $\gamma^{k+1}\in N(\gamma^*,b)$, then*

$$\|\gamma^{k+1}-\gamma^*\|_{\Gamma_i}\le\bar{c}_5(\|\gamma^k-\gamma^*\|^2_{\Gamma_i}+\delta),\tag{4.20}$$

where $\bar{c}_5$ is a constant given explicitly by $\bar{c}_5=\sqrt{(2\bar{c}_3+4\bar{c}_2\bar{c}_4^2+1)/\bar{c}_1}$.

Proof. Similarly to the proof of Lemma 3.4, it is direct to derive (4.20) as follows:

$$\begin{aligned}\bar{c}_1\|\gamma^{k+1}-\gamma^*\|^2_{\Gamma_i}&\le\int_0^T\|u(\gamma^{k+1})-u(\gamma^*)\|^2_{\Gamma_a}\mathrm{d}t\le\int_0^T\|u(\gamma^{k+1})-z^\delta\|^2_{\Gamma_a}\mathrm{d}t+\delta^2\\&\le 2\int_0^T\|u'(\gamma^k)(\gamma^{k+1}-\gamma^k)+u(\gamma^k)-(z^\delta)\|^2_{\Gamma_a}\mathrm{d}t+2\bar{c}_2\|\gamma^{k+1}-\gamma^k\|^4_{\Gamma_i}+\delta^2\\&\le(2\bar{c}_3+4\bar{c}_2\bar{c}_4^2+1)(\|\gamma^k-\gamma^*\|^4_{\Gamma_i}+\delta^2)\,.\end{aligned}$$

□

The same as we did in the previous section for the elliptic Robin problem, we make it clear now how all the constants $\bar{c}_1,\ldots,\bar{c}_5$ in our previous estimates depend on the size b of the neighborhood $N(\gamma^*;b)$. We can easily check that constants $\bar{c}_2$ and $\bar{c}_3$ in (4.13) and (4.17) are independent of the radius b, but the constants $\bar{c}_1$, $\bar{c}_4$ and $\bar{c}_5$ in (2.8) (4.18) and (4.20) respectively depend on b. So we write these three constants as $\bar{c}_1(b)$, $\bar{c}_4(b)$ and $\bar{c}_5(b)$ to emphasize their dependence on b. In addition, we introduce

$$\bar{r}(b,\delta)=\min\left\{b,\,\frac{b-\sqrt{\bar{c}_4\left(\frac13\right)\delta}}{\sqrt{\bar{c}_4\left(\frac13\right)}+1}\right\},\quad\bar{\alpha}=\left(\sqrt{\bar{c}_4\left(\frac13\right)}+1\right)\bar{c}_5\left(\frac13\right),\quad\bar{\beta}(x)=\bar{\alpha}x+\sqrt{\bar{c}_4\left(\frac13\right)x}\,,$$

$$\bar{d}_0=\frac{1+2\bar{c}_4\left(\frac13\right)-2\sqrt{\bar{c}_4^2\left(\frac13\right)+\bar{c}_4\left(\frac13\right)}}{4\bar{\alpha}^2},\quad\bar{d}=\frac12\bar{d}_0\,.$$

Following the same arguments as we did for Theorem 3.5, we can derive our main result, namely, the quadratic convergence of the L-M iteration (4.9).

Theorem 4.5. *For any $b\in\left[\frac{1-\sqrt{1-4\bar{\alpha}\bar{\beta}(\bar{d})}}{2\bar{\alpha}},\,\frac{1+\sqrt{1-4\bar{\alpha}\bar{\beta}(\bar{d})}}{2\bar{\alpha}}\right]$, we assume the noise level is small such that*

$$\delta<\min\left\{\frac{b^2}{\bar{c}_4\left(\frac13\right)},\,\bar{d}\right\}\,,\tag{4.21}$$

and the conditions (2.8), (4.10) *and* (4.11) *are satisfied, then for any* $\gamma^0 \in N(\gamma^*, \bar{r}(b,\delta))$, *the sequence* $\{\gamma^k\}$ *generated by* (4.9) *stay always in* $N(\gamma^*, b)$ *and satisfies*

$$\|\gamma^{k+1} - \gamma^*\|_{\Gamma_i} \le \bar{c}_5 \left(\frac{1}{3}\right) (\|\gamma^k - \gamma^*\|^2_{\Gamma_i} + \delta).$$

4.3. Surrogate functional method

Based on the same motivation as we did in Section 3.3 for the elliptic Robin inverse problem, we apply the surrogate functional method to essentially simplify the solution to the minimization (4.9) involved in each step of the L-M iteration, resulting in explicit solutions at each iteration. For this purpose, we construct an auxiliary surrogate functional $J_k^s(\gamma, \zeta)$ of $J_k(\gamma)$ of the form for a given $\zeta \in K$:

$$J_k^s(\gamma,\zeta) = J_k(\gamma) + A\|\gamma - \zeta\|^2_{\Gamma_i} - \int_0^T \|u'(\gamma^k)(\gamma-\zeta)\|^2_{\Gamma_a} \mathrm{d}t, \tag{4.22}$$

where A is any positive constant such that $\int_0^T \|u'(\gamma^k)d\|^2_{\Gamma_a}\mathrm{d}t \le A\|d\|^2_{\Gamma_i}$ for $d \in L^2(\Gamma_i)$. Now we can convert the functional $J_k^s(\gamma,\zeta)$ in (4.22) into a more explicit representation by using the relation (4.5):

$$\begin{aligned}
J_k^s(\gamma,\zeta) &= \int_0^T \|u'(\gamma^k)(\gamma-\gamma^k) - (z^\delta - u(\gamma^k))\|^2_{\Gamma_a}\mathrm{d}t + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i} + A\|\gamma-\zeta\|^2_{\Gamma_i} \\
&\quad - \int_0^T \|u'(\gamma^k)(\gamma-\zeta)\|^2_{\Gamma_a}\mathrm{d}t \\
&= -2\left\langle \gamma, \int_0^T u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\mathrm{d}t\right\rangle_{\Gamma_i} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i} \\
&\quad + A\|\gamma-\zeta\|^2_{\Gamma_i} + \int_0^T \|z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k)\|^2_{\Gamma_a}\mathrm{d}t - \int_0^T \|u'(\gamma^k)(\zeta)\|^2_{\Gamma_a}\mathrm{d}t \\
&= A\left\|\gamma - \zeta - \frac{1}{A}\int_0^T u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\mathrm{d}t\right\|^2_{\Gamma_i} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i} \\
&\quad + \left\{\int_0^T \|z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k)\|^2_{\Gamma_a}\mathrm{d}t - \int_0^T \|u'(\gamma^k)(\zeta)\|^2_{\Gamma_a}\mathrm{d}t + A\|\zeta\|^2_{\Gamma_i}\right. \\
&\quad \left. - A\left\|\zeta + \frac{1}{A}\int_0^T u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\mathrm{d}t\right\|^2_{\Gamma_i}\right\}.
\end{aligned} \tag{4.23}$$

We note the last term in (4.23) is a constant, so we drop the term in the functional $J_k^s(\gamma,\zeta)$ to get

$$\min_{\gamma\in K} J_k^s(\gamma,\zeta) = \min_{\gamma\in K} A\left\|\gamma-\zeta-\frac{1}{A}\int_0^T u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta-u(\gamma^k)+u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\mathrm{d}t\right\|^2_{\Gamma_i} + \beta_k\|\gamma-\gamma^k\|^2_{\Gamma_i}.$$

Then it is easy to find the exact minimizer to this quadratic minimization:

$$\operatorname*{argmin}_{\gamma\in K} J_k^s(\gamma,\zeta) = \frac{1}{A+\beta_k}\left(\beta_k\gamma^k + A\zeta + \int_0^T u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k-\zeta)}{u(\gamma^k)}\right)\right\}\mathrm{d}t\right). \tag{4.24}$$

This motivates us with the following reconstruction algorithm that is computationally much easy and less expensive than the minimization (4.9).

Algorithm 4.6. Choose two tolerance parameters ϵ_1, $\epsilon_2 > 0$ and an initial value γ^0, set $k := 0$.

1. Compute γ^{k+1}: set $\zeta_0 = \gamma^k$ and $n := 0$.
   (1.1). Compute

$$\zeta_{n+1} = \underset{\gamma \in K}{\operatorname{argmin}} J_k^s(\gamma, \zeta_n)$$
$$= \frac{1}{A+\beta_k}\left(\beta_k \gamma^k + A\zeta_n + \int_0^T u(\gamma^k)\left\{u'(\gamma^k)^*\left(\frac{z^\delta - u(\gamma^k) + u'(\gamma^k)(\gamma^k - \zeta_n)}{u(\gamma^k)}\right)\right\}\mathrm{d}t\right).$$

   (1.2). If $\frac{\|\zeta_{n+1}-\zeta_n\|_{\Gamma_i}}{\|\zeta_n\|_{\Gamma_i}} \le \epsilon_1$, set $\gamma^{k+1} = \zeta_{n+1}$, go to Step 2; otherwise set $n := n+1$, go to Step (1.1).
2. If $\frac{\|\gamma^{k+1}-\gamma^k\|_{\Gamma_i}}{\|\gamma^k\|_{\Gamma_i}} \le \epsilon_2$, stop the iteration; otherwise set $k := k+1$, go to Step 1.

5. Numerical experiments

In this section, we present several numerical examples to check the efficiency and accuracy of Algorithms 3.6 and 4.6 for recovering the Robin coefficients in the elliptic and parabolic systems (1.1) and (1.2) respectively. We choose the domain $\Omega = (0,1) \times (0,2)$ and triangulate it into $N \times M$ small squares of equal size and further divide each square through its diagonal into two triangles. This results in a finite element triangulation of domain Ω. All the elliptic problems involved in Algorithms 3.6 are solved by the continuous linear finite element method, while all the parabolic problems in Algorithm 4.6 are solved by the continuous linear finite element method in space and the backward difference scheme in time.

The parameters involved in Algorithms 3.6 and 4.6 are chosen as follows. The initial guesses are set to be identically equal to some constants, which as we see are rather poor initial guesses for all the test problems. The noisy data z^δ is obtained by adding some uniform random noise to the exact data, *i.e.*, $z^\delta = u + \delta R u$, where R is a uniform random function varying in the range $[-1,1]$. Let γ^* and γ^k be the exact parameter and its numerical reconstruction by Algorithms 3.6 and 4.6 respectively at the kth L-M iteration. We shall compute the absolute and relative errors:

$$e_k = \|\gamma^k - \gamma^*\|_{\Gamma_i} \quad \text{and} \quad \bar{e}_k = \frac{\|\gamma^k - \gamma^*\|_{\Gamma_i}}{\|\gamma^*\|_{\Gamma_i}}.$$

We choose two examples for the Robin coefficient reconstructions on the partial boundary $\Gamma_i = \{(x,y);\ x = 1,\ 0 \le y \le 2\}$ in the elliptic system (1.1), where we take $a(\mathbf{x}) = c(\mathbf{x}) = 1$ in Ω, the ambient temperature $g = 2 + (\cos(\pi y)+1)\gamma(\mathbf{x})$ on Γ_i, the heat flux $h = 0$ on Γ_a, the source strength $f = (\pi^2+1)\cos(\pi y) + x^2 - 2$ and the exact forward solution $u = x^2 + \cos(\pi y)$ in Ω. We set the mesh $N = 16$ and $M = 32$, two tolerance parameters $\epsilon_1 = 2 \times 10^{-3}$, $\epsilon_2 = 0.01$, the constant $A = 1$ and the initial guess $\gamma^0 = 2$.

Example 5.1. In this example, we fix the noise level $\delta = 2\%$ and select two different exact Robin coefficients to verify the efficiency and accuracy of the proposed Algorithm 3.6:
(1) $\gamma = 3 - \sin(\frac{\pi}{2}y)$ on Γ_i;
(2) $\gamma = (y-1)^2 + 2$ on $\{(x,y) \in \Gamma_i;\ 0 \le y \le 1\}$ and $\gamma = -(y-1)^2 + 2$ on $\{(x,y) \in \Gamma_i;\ 1 \le y \le 2\}$.

Example 5.2. In this example, we take the exact Robin coefficient $\gamma = -(y-1)^2 + 2$ on Γ_i. In order to check the quadratic convergence of Algorithm 3.6, we turn off the noise, *i.e.*, the noise level $\delta = 0$.

Figure 1 presents the exact and reconstructed Robin coefficients, the L-M iteration number k and the relative error $\bar{e}_k$ for Example 5.1, while Table 1 lists the absolute error e_k and the ratio e_k/e_{k-1}^2 for Example 5.2. We can see from Figure 1 that the numerical reconstructed Robin coefficients appear to be quite satisfactory, even with very rough initial guesses (the same constant everywhere) in the presence of a 2% noise in the data. Moreover,

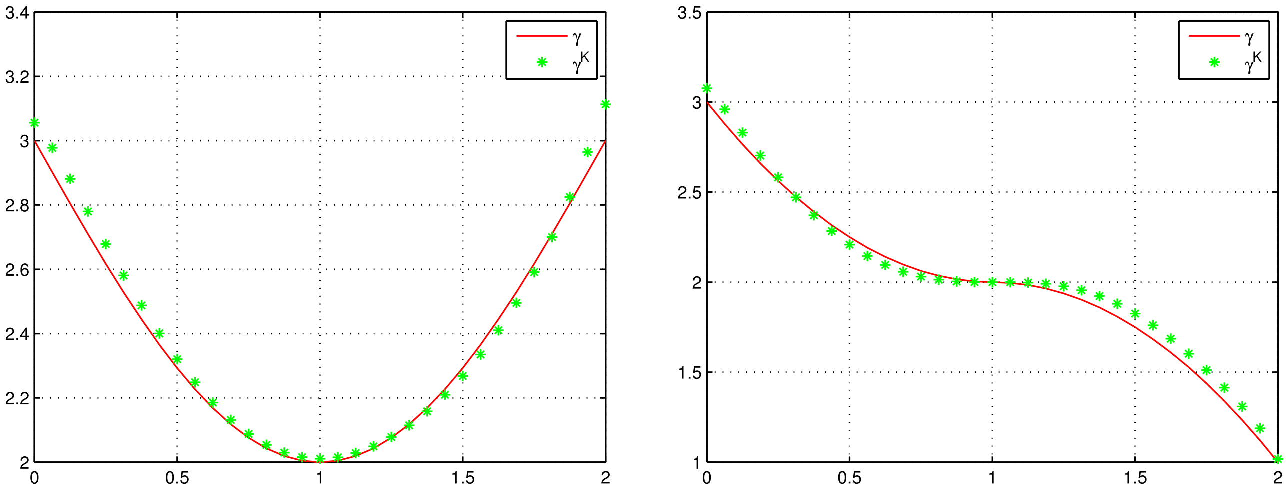

FIGURE 1. Exact and reconstructed Robin coefficients for Example 5.1: (1) $k = 4$, $\bar{e}_k = 0.0168$; (2) $k = 5$, $\bar{e}_k = 0.0252$.

TABLE 1. Numerical results for Example 5.2.

k	e_k	e_k/e_{k-1}^2
0	0.6325	–
1	0.3567	0.8915
2	0.1343	1.0555
3	0.0139	0.7695

Table 1 shows that e_k and e_{k-1}^2 are of the same order approximately, which indeed confirms the quadratic convergence of our proposed Algorithm 3.6.

Next, we demonstrate two numerical examples of reconstructing the Robin coefficient $\gamma(\mathbf{x})$ on the partial boundary $\Gamma_i = \{(x, y);\ x = 1,\ 0 \le y \le 2\}$ in the parabolic system (1.2) with $a(\mathbf{x}) = 1$ and $T = 2$. We take the ambient temperature $g = (2 + (\cos(\pi y) + 1)\gamma(\mathbf{x}))t$ on $\Gamma_i \times [0, T]$, the heat flux $h = 0$ on $\Gamma_a \times [0, T]$, the source strength $f = \cos(\pi y) + x^2 + (\pi^2 \cos(\pi y) - 2)t$ and the exact forward solution $u = (x^2 + \cos(\pi y))t$ in $\Omega \times [0, T]$. We set the mesh $N = 16$ and $M = 32$, two tolerance parameters $\epsilon_1 = 2 \times 10^{-3}$, $\epsilon_2 = 0.01$, the constant $A = 1$ and the initial guess $\gamma^0 = 2$.

Example 5.3. We take the noise level $\delta = 2\%$ and select two different exact Robin coefficients:
(1) $\gamma = -\frac{1}{2}(y-1)^2 + 2$ on Γ_i; (2) $\gamma = \frac{1}{2}(\sin(\frac{\pi}{2}y) + y^{\frac{1}{4}}) + 1$ on Γ_i
to verify the efficiency and accuracy of the proposed Algorithm 4.6.

Example 5.4. We take the exact Robin coefficient $\gamma = -(y-1)^2 + 2$ on Γ_i. In order to check the quadratic convergence of Algorithm 4.6, we turn off the noise, *i.e.*, the noise level $\delta = 0$.

Figure 2 presents the exact and reconstructed Robin coefficients, the L-M iteration number k and the relative error $\bar{e}_k$ for Example 5.3. Table 2 lists the absolute error e_k and the ratio e_k/e_{k-1}^2 for Example 5.4. As we can observe from Figure 2, the numerical recovered Robin coefficients are quite satisfactory, even with some very rough initial guesses (just a constant in the whole Γ_i) in the presence of a 2% noise in the data. And we also notice that Algorithm 4.6 converges quite fast, with about 5 to 6 iterations. More importantly, we can see from Table 2 that e_k and e_{k-1}^2 are about the same order, confirming a quadratic convergence of Algorithm 4.6.

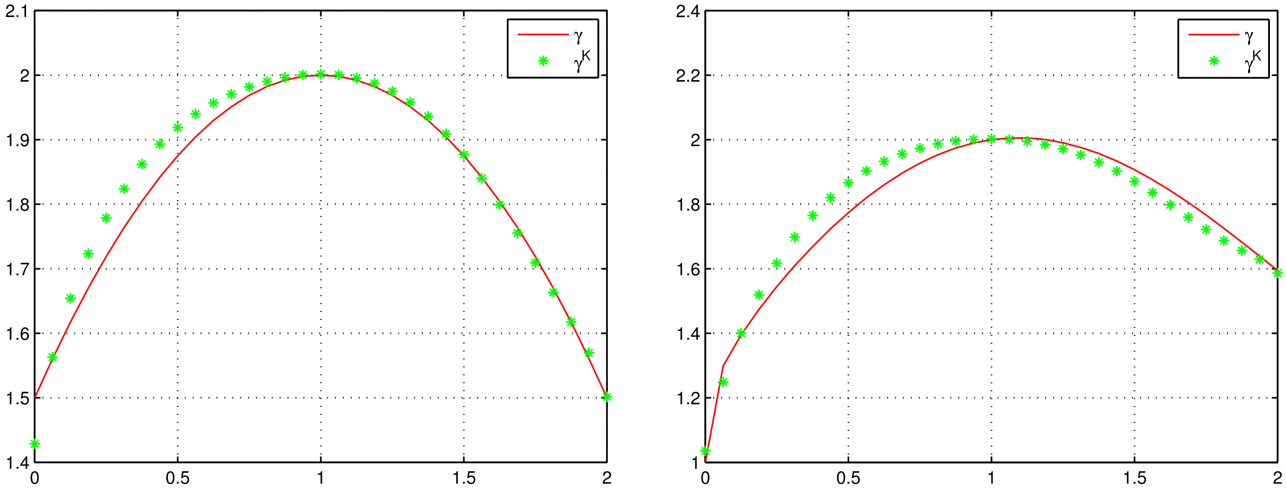

FIGURE 2. Exact and reconstructed Robin coefficients for Example 5.3: (1) $k = 5$, $\bar{e}_k = 0.0158$; (2) $k = 6$, $\bar{e}_k = 0.0275$.

TABLE 2. Numerical results for Example 5.4.

k	e_k	e_k/e_{k-1}^2
0	0.6325	–
1	0.4404	1.1007
2	0.2615	1.3479
3	0.1237	1.8082
4	0.0501	3.2745

6. Concluding remarks

We have justified in this work the uniqueness of the elliptic and parabolic Robin inverse problems. Then the L-M iterative method is formulated to solve the nonlinear Tikhonov regularized optimizations, which transform the original highly nonlinear and nonconvex minimizations into convex minimizations. We have established the quadratic convergence and the quadratic rate of convergence for the L-M iterations for the highly ill-posed nonlinear elliptic and parabolic Robin inverse problems. This appears to be the first time in literature to achieve the quadratic convergence and the quadratic rate of convergence for the L-M iterations rigorously for a highly nonlinear and ill-posed inverse problem, in combination with a simple and easily implementable choice rule of regularization parameters. The surrogate functional techniques have been applied to solve the convex minimizations at each L-M iteration, which lead to explicit expressions of the minimizers for both the elliptic and parabolic cases, resulting in two computationally very efficient solvers for the highly ill-posed nonlinear inverse problems. Numerical experiments have demonstrated the computational efficiency of the methods and their robustness against the noise in the observation data.

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