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INHOMOGENEOUS STEADY-STATE PROBLEM OF COMPLEX HEAT TRANSFER *

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Abstract. An inhomogeneous steady-state problem of radiative-conductive heat transfer in a three-dimensional domain is studied in the framework of the P_1 approximation of the nonlinear complex heat transfer model. The unique solvability of the problem is proved. The Lyapunov stability of solutions is shown.

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1. Introduction

The steady-state normalized P_1 approximation of the complex heat transfer model describing radiative and conductive contributions is considered in a bounded domain $\Omega \subset \mathbb{R}^3$. The model has the following form [1]:

$$-a\Delta\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = f, \tag{1.1}$$

$$-\alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = g. \tag{1.2}$$

Here, θ is the normalized temperature, φ the normalized radiation intensity averaged over all directions, f, g the volume densities of temperature and intensity sources, and κ_a the absorption coefficient. The constants a, b, and α are given by the formulas:

$$a = \frac{k}{\rho c_p}, \quad b = \frac{4\sigma n^2 T_{\max}^3}{\rho c_p}, \quad \alpha = \frac{1}{3\kappa - A\kappa_s},$$

where k is the thermal conductivity, c_p the specific heat capacity, ρ the density, σ the Stefan–Boltzmann constant, n the refractive index, $T_{\max}$ the maximum temperature in the unnormalized model, $\kappa := \kappa_s + \kappa_a$ the extinction coefficient (total attenuation factor), and κ_s the scattering coefficient. The coefficient $A \in [-1, 1]$ describes the anisotropy of the scattering.

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The following boundary conditions on $\Gamma = \partial\Omega$ are assumed:

$$a\partial\theta/\partial n + \beta(\theta - \theta_b)|_\Gamma = 0, \quad \alpha\partial\varphi/\partial n + \gamma(\varphi - |\theta_b|\theta_b^3)|_\Gamma = 0. \tag{1.3}$$

Here, the boundary functions, $\theta_b = \theta_b(x)$, $\beta = \beta(x)$, and the function $\gamma = \gamma(x)$, $x \in \Gamma$, describing reflecting properties of the boundary, are fixed.

The problems of complex heat transfer in scattering media with reflecting boundaries are of growing interest in connection with engineering applications (see *e.g.* [2–6]). A considerable number of works is devoted to theoretical analysis of complex heat transfer models. In [7], the solvability of a homogeneous non-stationary boundary-value problem of complex heat transfer (1.1)–(1.3) (the case $f = g = 0$) and stability of steady-state solutions are proved. Theoretical analysis of similar non-stationary models is conducted in [8–10]. In [9], an inhomogeneous boundary-value problem for the SP_3 approximation of a complex heat transfer model is studied. For the case of bounded temperature sources, the unique solvability of the problem is proved. In [10], the unique solvability of an inhomogeneous boundary-value problem for P_1 approximation is proved using softer constraints on the sources. Theoretical aspects of homogeneous steady-state boundary-value problems of complex heat transfer are studied in [11–18], in particular, under some physical constraints [11–15].

The study of complex heat transfer models with sources which are described by functionals or p-integrable functions has not only theoretical interest. The estimates of solutions can be applied, particularly in analysis of inverse problems for complex heat transfer models. The main results of the current work consist in the derivation of new a priori estimates for a solution of the boundary-value problem (1.1)–(1.3) and the proof of its unique solvability. Also, Lyapunov stability of steady-state solutions of a non-stationary complex heat transfer problem is proved. This is very important for the problem of the adequacy of the steady-state complex heat transfer model.

2. Problem formalization

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the notation L^p, $1 \leq p \leq \infty$, stand for the space of p-integrable functions, and H^s denote the Sobolev space W_2^s. Denote $H = L^2(\Omega)$ and $V = H^1(\Omega)$. Let V' be the adjoint space of V. Identifying H with the dual space H' yields the Gelfand triple $V \subset H = H' \subset V'$. Let $\|\cdot\|$, $\|\cdot\|_V$, and $\|\cdot\|_{V'}$ be the norms in the spaces H, V, and V', respectively. Let the value of a functional $f \in V'$ on an element $v \in V$ be denoted by (f, v). Notice that (f, v) is the inner product in H if f and v are elements of H.

Suppose that the problem data satisfy the following conditions:

(i) $\beta, \gamma \in L^\infty(\Gamma)$, $\beta \geq \beta_0 > 0$, $\gamma \geq \gamma_0 > 0$, $\beta_0, \gamma_0 = \text{const}$, $\theta_b \in L^{16/3}(\Gamma)$,
(ii) $f \in V'$, $g \in L^{6/5}(\Omega)$.

Let us introduce the functions $h_p(s) := |s|^p \operatorname{sign} s$, $p > 0$, $s \in \mathbb{R}$. Notice that $h_p'(s) = p|s|^{p-1}$ and $h_p(h_q(s)) = h_{pq}(s)$.

Assuming that θ, φ, v are arbitrary elements of V, define operators and functionals $A_1, A_2 \colon V \to V'$, $f_1, f_2 \in V'$ by the following relations:

$$(A_1\theta, v) = a(\nabla\theta, \nabla v) + \int_\Gamma \beta\theta v \mathrm{d}\Gamma, \quad (A_2\varphi, v) = \alpha(\nabla\varphi, \nabla v) + \int_\Gamma \gamma\varphi v \mathrm{d}\Gamma,$$
$$(f_1, v) = \int_\Gamma \beta\theta_b v \mathrm{d}\Gamma, \quad (g_1, v) = \int_\Gamma \gamma h_4(\theta_b) v \mathrm{d}\Gamma.$$

Notice that the bilinear forms $(A_1 u, v)$, $(A_2 u, v)$ can be considered as inner products of V. The corresponding norms are equivalent to the conventional norm of V. Therefore, the bounded inverse operators $A_1^{-1}, A_2^{-1} : V' \mapsto V$ are well defined.

Definition 2.1. A pair $\{\theta,\varphi\} \in V \times V$ is called a weak solution of the problem (1.1)–(1.3) if it satisfies the equations

$$A_1\theta + b\kappa_a(h_4(\theta) - \varphi) = f + f_1, \tag{2.1}$$

$$A_2\varphi + \kappa_a(\varphi - h_4(\theta)) = g + g_1. \tag{2.2}$$

3. Unique solvability

Let us reduce the boundary-value problem (2.1), (2.2) to an operator equation in the Hilbert space $\mathbf{V} = V \times V$ with inner product

$$((y,z)) = (A_1u_1, v_1) + (A_2u_2, v_2) \;\; \forall y = \{u_1,u_2\},\, z = \{v_1,v_2\} \in \mathbf{V}.$$

Define a non-linear operator $F : \mathbf{V} \mapsto \mathbf{V}$,

$$Fy = \{A_1^{-1}(f + f_1 - b\kappa_a(h_4(\theta) - \varphi)), A_2^{-1}(g + g_1 + \kappa_a(h_4(\theta) - \varphi))\} \;\; \forall y = \{\theta,\varphi\} \in \mathbf{V}.$$

The problem (2.1), (2.2) is reduced to finding the fixed point of the operator F,

$$y = Fy, \;\; y = \{\theta,\varphi\} \in \mathbf{V}. \tag{3.1}$$

Lemma 3.1. *The operator F is completely continuous.*

Proof. Let $y_1 = \{\theta_1,\varphi_1\}$, $y_2 = \{\theta_2,\varphi_2\}$, $z = \{z_1,z_2\}$ be arbitrary elements of the space $\mathbf{V}$, $\theta = \theta_1 - \theta_2$, $\varphi = \varphi_1 - \varphi_2$.

$$\begin{aligned}((Fy_1 - Fy_2, z)) &= -\kappa_a b(|\theta_1|\theta_1^3 - |\theta_2|\theta_2^3 - \varphi, z_1) + \kappa_a(|\theta_1|\theta_1^3 - |\theta_2|\theta_2^3 - \varphi, z_2) \\ &\le 2\kappa_a b\left(\|\theta_1\|^3_{L^6(\Omega)} + \|\theta_2\|^3_{L^6(\Omega)}\right)\|\theta\|_{L^4(\Omega)}\|z_1\|_{L^4(\Omega)} + \kappa_a b\|\varphi\|\|z_1\| + \kappa_a\|\varphi\|\|z_2\| \\ &\quad + 2\kappa_a\left(\|\theta_1\|^3_{L^6(\Omega)} + \|\theta_2\|^3_{L^6(\Omega)}\right)\|\theta\|_{L^4(\Omega)}\|z_2\|_{L^4(\Omega)}.\end{aligned}$$

Let $z = \{z_1,z_2\} := Fy_1 - Fy_2$. Taking into account the continuity of the embedding V in $L^s(\Omega)$, $1 \le s \le 6$, from the last inequality, it follows:

$$\|Fy_1 - Fy_2\|_{\mathbf{V}} \le C\left(\|\theta\|_{L^4(\Omega)} + \|\varphi\|\right). \tag{3.2}$$

Here, C depends only on κ_a, b, norms of embedding operators, and norms of θ_1 and θ_2 in the space V. From estimate (3.2), it follows the continuity of the operator $F : \mathbf{V} \mapsto \mathbf{V}$. Moreover, the compactness of embedding the space V into $L^4(\Omega)$, and into $L^2(\Omega)$ gives the compactness of F. □

Theorem 3.2. *If the conditions* (i), (ii) *are fulfilled, then the problem* (2.1), (2.2) *is solvable.*

Proof. To prove the existence of a fixed point of the completely continuous operator F, it is sufficient, based on the Leray−Schauder principle, to show the uniform boundedness (with respect to $\lambda \in (0,1]$) of the solutions of the operator equation

$$y = \lambda Fy, \;\; y = \{\theta,\varphi\} \in \mathbf{V}. \tag{3.3}$$

Equation (3.3) is equivalent to the following equalities:

$$a(\nabla\theta,\nabla v) + \int_\Gamma \beta\theta v \mathrm{d}\Gamma + \lambda b\kappa_a(h_4(\theta) - \varphi, v) = \lambda(f + f_1, v) \;\; \forall v \in V, \tag{3.4}$$

$$\alpha(\nabla\varphi,\nabla w) + \int_\Gamma \gamma\varphi w \mathrm{d}\Gamma - \lambda\kappa_a(h_4(\theta) - \varphi, w) = \lambda(g + g_1, w) \;\; \forall w \in V. \tag{3.5}$$

Denote $\varphi_1 = h_{1/4}(\varphi)$ and, for $\varepsilon > 0$, define

$$w_\varepsilon = \begin{cases} \varphi_1 - \varepsilon, & \varphi_1 > \varepsilon, \\ 0, & |\varphi_1| \le \varepsilon, \\ \varphi_1 + \varepsilon, & \varphi_1 < -\varepsilon. \end{cases}$$

Notice that if $\varphi \in V$, then $\varphi_1 \in L^{24}(\Omega)$, $\varphi_1|_\Gamma \in L^{16}(\Gamma)$, $w_\varepsilon \in V$, and

$$\nabla w_\varepsilon = \frac{1}{4} \begin{cases} |\varphi|^{-3/4} \nabla \varphi, & |\varphi_1| > \varepsilon, \\ 0, & \text{otherwise}. \end{cases}$$

It is important that

$$\int_\Gamma \gamma \varphi w_\varepsilon \mathrm{d}\Gamma - \lambda \kappa_a (h_4(\theta) - \varphi, w_\varepsilon) - \lambda (g + g_1, w_\varepsilon) = \int_\Gamma \gamma(\varphi - \lambda h_4(\theta_b))\varphi_1 \mathrm{d}\Gamma - \lambda \kappa_a (h_4(\theta) - \varphi, \varphi_1) - \lambda (g, \varphi_1) + c_\varepsilon.$$

Herewith $|c_\varepsilon| \le C\varepsilon$, where $C > 0$ does not depend on ε.

Set $v = \theta$ in (3.4), $w = bw_\varepsilon$ in (3.5), and add these equalities. Then, taking into account monotonicity of $(h_4(\theta) - \varphi)(\theta - h_{1/4}(\varphi)) \ge 0$, we obtain the inequality

$$a\|\nabla\theta\|^2 + \int_\Gamma \beta\theta^2 \mathrm{d}\Gamma + \frac{16}{25}\alpha b \int_{|\psi| > \varepsilon^{5/2}} |\nabla\psi|^2 \mathrm{d}x + b\int_\Gamma \gamma\psi^2 \mathrm{d}\Gamma \le \lambda(f + f_1, \theta) + \lambda b(g, \varphi_1) + \lambda b \int_\Gamma \gamma h_4(\theta_b)\varphi_1 \mathrm{d}\Gamma - bc_\varepsilon. \tag{3.6}$$

Here, $\psi = h_{5/8}(\varphi)$, $\varphi_1 = h_{2/5}(\psi)$. Let us show that from estimate (3.6) follows $\psi \in V$. Denote

$$\psi_\varepsilon = \begin{cases} \psi - \varepsilon^{5/2}, & \psi > \varepsilon^{5/2}, \\ 0, & |\psi| \le \varepsilon^{5/2}, \\ \psi + \varepsilon^{5/2}, & \psi < -\varepsilon^{5/2}. \end{cases}$$

Since $|\psi_\varepsilon - \psi| \le \varepsilon^{5/2}$, then $\psi_\varepsilon \to \psi$ in $L^2(\Omega)$ as $\varepsilon \to +0$. As follows from (3.6), the sequence ψ_ε is bounded in the space V and $\nabla\psi_\varepsilon \to \nabla\psi$ in $L^2(\Omega)$ weakly. Moreover, $\|\nabla\psi\| \le \liminf \|\nabla\psi_\varepsilon\|$.

Therefore, passing to the limit in inequality (3.6) as $\varepsilon \to +0$, we obtain

$$k_1\|\theta\|_V^2 + k_2\|\psi\|_V^2 \le |(f + f_1, \theta)| + b|(g, h_{2/5}(\psi))| + b\int_\Gamma \gamma |h_4(\theta_b) h_{2/5}(\psi)| \mathrm{d}\Gamma. \tag{3.7}$$

Here, $k_1 = \min\{a, \beta_0\}$, $k_2 = b\min\{\frac{16}{25}\alpha, \gamma_0\}$. The norm in the space V is defined by the following equality:

$$\|v\|_V^2 = \|\nabla v\|^2 + \int_\Gamma v^2 \mathrm{d}\Gamma.$$

Using Hölder and Young inequalities with parameter $\delta > 0$, we estimate the terms in the right-hand side of (3.7):

$$|(f + f_1, \theta)| \le \frac{\delta}{2}\|\theta\|_V^2 + \frac{1}{2\delta}\|f + f_1\|_{V'}^2,$$

$$|(g, h_{2/5}(\psi))| \le \frac{\delta^5}{5}\|\psi\|_{L^6(\Omega)}^2 + \frac{4}{5\delta^{5/4}}\|g\|_{L^{15/14}(\Omega)}^{5/4},$$

$$\int_\Gamma \gamma |h_4(\theta_b) h_{2/5}(\psi)| \mathrm{d}\Gamma \le \|\gamma\|_{L^\infty(\Gamma)} \left(\frac{\delta^5}{5}\|\psi\|_{L^4(\Gamma)}^2 + \frac{4}{5\delta^{5/4}}\|\theta_b\|_{L^{40/9}(\Gamma)}^5 \right).$$

Taking into account the continuity of the embedding of V into $L^6(\Omega)$, the continuity of the trace operator from V into $L^4(\Gamma)$, and choosing a sufficiently small δ, we obtain from (3.7) the uniform estimate with respect to $\lambda \in (0,1]$:

$$\|\theta\|_V^2 + \|\psi\|_V^2 \le K_1 \left(\|f + f_1\|_{V'}^2 + \|g\|_{L^{15/14}(\Omega)}^{5/4} + \|\theta_b\|_{L^{40/9}(\Gamma)}^5 \right). \tag{3.8}$$

Here, K_1 depends only on a, α, b, β_0, γ_0, $\|\gamma\|_{L^\infty(\Gamma)}$, and the domain Ω.

The estimate of $\|\theta\|_V$ allows to obtain the uniform estimate of $\|\varphi\|_V$ with respect to λ. By setting $w = \varphi$ in (3.5), and denoting $k_3 = \min\{\alpha, \gamma_0\}$, we obtain the inequality

$$k_3\|\varphi\|_V^2 + \kappa_a\|\varphi\|^2 \le |(g,\varphi)| + \kappa_a|(h_4(\theta),\varphi)| + \int_\Gamma \gamma|h_4(\theta_b)\varphi| \mathrm{d}\Gamma.$$

Using Hölder and Young inequalities with parameter $\delta > 0$, we estimate the terms in the right-hand side:

$$|(g,\varphi)| \le \frac{\delta}{2}\|\varphi\|_{L^6(\Omega)}^2 + \frac{1}{2\delta}\|g\|_{L^{6/5}(\Omega)}^2,$$

$$|(h_4(\theta),\varphi)| \le \frac{\delta}{2}\|\varphi\|_{L^3(\Omega)}^2 + \frac{1}{2\delta}\|\theta\|_{L^6(\Omega)}^8,$$

$$\int_\Gamma \gamma|h_4(\theta_b)\varphi|\mathrm{d}\Gamma \le \|\gamma\|_{L^\infty(\Gamma)} \left(\frac{\delta}{2}\|\varphi\|_{L^4(\Gamma)}^2 + \frac{1}{2\delta}\|\theta_b\|_{L^{16/3}(\Gamma)}^8 \right).$$

Taking into account the continuity of the embedding of V into $L^6(\Omega)$, the continuity of the trace operator from V into $L^4(\Gamma)$, and a sufficiently small δ, we obtain the uniform estimate of $\|\varphi\|_V$ with respect to $\lambda \in (0,1]$:

$$\|\varphi\|_V^2 \le K_2 \left(\|g\|_{L^{6/5}(\Omega)}^2 + \|\theta_b\|_{L^{16/3}(\Gamma)}^8 + \|\theta\|_V^8 \right). \tag{3.9}$$

Here, K_2 depends only on α, γ_0, κ_a $\|\gamma\|_{L^\infty(\Gamma)}$, and the domain Ω. The estimates (3.8) and (3.9) give the boundedness (uniform with respect to $\lambda \in (0,1]$) of the set of solutions of the operator equation (3.3). This proves the theorem. □

Theorem 3.3. *The problem* (2.1), (2.2) *has a unique solution.*

Proof. Let $\{\theta_1,\varphi_1\}$, $\{\theta_2,\varphi_2\} \in \mathbf{V}$ be the solutions of the problem (2.1), (2.2). Let $\theta = \theta_1 - \theta_2$, $\varphi = \varphi_1 - \varphi_2$, and $w = h_4(\theta_1) - h_4(\theta_2)$. Then we obtain

$$A_1\theta + b\kappa_a(w - \varphi) = 0, \quad A_2\varphi + \kappa_a(\varphi - w) = 0. \tag{3.10}$$

Let us consider a regularization of sign function: $r_\delta(s) = s/|s|$, if $|s| \ge \delta$, and $r_\delta(s) = s/\delta$, if $|s| < \delta$. Multiplying the first equation in (3.10), in the sense of the inner product of H, by $r_\delta(\theta)$, the second one by $br_\delta(\varphi)$, and adding these equalities, we obtain

$$a(\nabla\theta, r'_\delta(\theta)\nabla\theta) + \int_\Gamma \beta\theta r_\delta(\theta)\mathrm{d}\Gamma + \alpha b(\nabla\varphi, r'_\delta(\varphi)\nabla\varphi) + b\int_\Gamma \gamma\varphi r_\delta(\varphi)\mathrm{d}\Gamma + b\kappa_a(w - \varphi, r_\delta(\theta) - r_\delta(\varphi)) = 0. \tag{3.11}$$

Note that $r'_\delta(s) \ge 0$, $s \in \mathbb{R}$. Moreover, the values of the functions θ and w have the same sign. Therefore, from (3.11), it follows:

$$\int_\Gamma \beta\theta r_\delta(\theta)\mathrm{d}\Gamma + b\int_\Gamma \gamma\varphi r_\delta(\varphi)\mathrm{d}\Gamma + b\kappa_a \int_{w,\varphi\ne 0} (w - \varphi)(r_\delta(\theta) - r_\delta(\varphi))\mathrm{d}x \le 0.$$

In the limit as $\delta \to +0$, we obtain

$$\int_\Gamma \beta|\theta|\mathrm{d}\Gamma + b\int_\Gamma \gamma|\varphi|\mathrm{d}\Gamma + b\kappa_a \int_{w,\varphi\ne 0} (w - \varphi)(\mathrm{sign}\theta - \mathrm{sign}\varphi)\mathrm{d}x \le 0.$$

Therefore, $\theta|_\Gamma = \varphi|_\Gamma = 0$. Further, from (3.10), it follows that

$$A_1\theta + bA_2\varphi = 0. \tag{3.12}$$

Multiplying (3.12), in the sense of the inner product of H, by $a\theta + \alpha b\varphi$, and taking into account zero boundary values of θ and φ, we obtain $\|\nabla(a\theta + \alpha b\varphi)\|^2 = 0$. Therefore, $a\theta + \alpha b\varphi = 0$. As a result, from the first equation of (3.10), it follows:

$$a(\nabla\theta, \nabla v) + b\kappa_a \left(w + \frac{a}{\alpha b}\theta, v\right) = 0 \quad \forall v \in V. \tag{3.13}$$

Setting $v = \theta$ in (3.13), we obtain $\theta = 0$. Therefore, also $\varphi = 0$. □

4. Lyapunov stability

In this section, the Lyapunov stability of steady-state solutions of a non-stationary complex heat transfer problem [7,10,19] is studied. This is very important for the problem of the adequacy of the steady-state complex heat transfer model. To formulate the problem of stability, we consider the following non-stationary system with initial and boundary conditions [10]:

$$\partial\theta/\partial t - a\Delta\theta + b\kappa_a(h_4(\theta) - \varphi) = f, \tag{4.1}$$

$$\mu\partial\varphi/\partial t - \alpha\Delta\varphi + \kappa_a(\varphi - h_4(\theta)) = g, \quad x \in \Omega, \quad t \in (0, +\infty), \tag{4.2}$$

$$a\partial\theta/\partial n + \beta(\theta - \theta_b)|_\Gamma = 0, \quad \alpha\partial\varphi/\partial n + \gamma(\varphi - h_4(\theta_b))|_\Gamma = 0, \tag{4.3}$$

$$\theta|_{t=0} = \theta_0, \quad \varphi|_{t=0} = \varphi_0. \tag{4.4}$$

Here, $\mu = 1/c$, where c is the speed of light in the medium. The functions f, g, and θ_b do not depend on time.

The unique solvability of the problem (4.1)–(4.4) is proved in [10] for any finite interval of time. Let $W = \{y \in L^2(0, T; V) \colon y' \in L^2(0, T; V')\}$. Hereinafter, $y' = \mathrm{d}y/\mathrm{d}t$. Suppose that the following conditions hold:

(j) $\beta, \gamma \in L^\infty(\Gamma)$, $\beta \geq \beta_0 > 0$, $\gamma \geq \gamma_0 > 0$, $\beta_0, \gamma_0 = \mathrm{const}$, $\theta_b \in L^8(\Gamma)$,
(jj) $f \in L^{15/11}(\Omega)$, $g \in L^{6/5}(\Omega)$,
(jjj) $\theta_0 \in L^5(\Omega)$, $\varphi_0 \in L^2(\Omega)$.

Theorem 4.1. *Let the conditions* (j)–(jjj) *hold. Then for any $T > 0$ there exists a unique pair $\{\theta, \varphi\} \in W \times W$ such that*

$$\theta' + A_1\theta + b\kappa_a(h_4(\theta) - \varphi) = f + f_1, \tag{4.5}$$

$$\mu\varphi' + A_2\varphi + \kappa_a(\varphi - h_4(\theta)) = g + g_1, \tag{4.6}$$

$$\theta|_{t=0} = \theta_0, \quad \varphi|_{t=0} = \varphi_0. \tag{4.7}$$

Moreover, $h_4(\theta) \in L^2(Q)$.

Let $\{\theta_s, \varphi_s\} \in \mathbf{V}$ be a stationary state of system (4.5), (4.6). Notice that this state is a solution of the system of the operator equations (2.1), (2.2). To perform the stability analysis, let us consider a pair $\{\theta, \varphi\}$ which is a solution of the problem (4.5)–(4.7) over the interval $(0, +\infty)$. Let $\zeta = \theta - \theta_s$ and $\xi = \varphi - \varphi_s$. Then

$$\zeta' + A_1\zeta + b\kappa_a(q(\zeta, x) - \xi) = 0, \quad \mu\xi' + A_2\xi + \kappa_a(\xi - q(\zeta, x)) = 0, \tag{4.8}$$

$$\zeta|_{t=0} = \zeta_0 = \theta_0 - \theta_s, \quad \xi|_{t=0} = \xi_0 = \varphi_0 - \varphi_s. \tag{4.9}$$

Here, $q(\zeta, x) = h_4(\theta_s(x) + \zeta) - h_4(\theta_s(x))$.

For any $\varepsilon > 0$, let us define the function: $r_\varepsilon(s) = s/|s|$, if $|s| \geq \varepsilon$, and $r_\varepsilon(s) = s/\varepsilon$, if $|s| < \varepsilon$. Also, we introduce the function

$$z_\varepsilon(s) = \begin{cases} -\varepsilon/2 - s, & s < -\varepsilon, \\ s^2/2\varepsilon, & |s| \leq \varepsilon, \\ -\varepsilon/2 + s, & s > \varepsilon. \end{cases}$$

Notice that $z'_\varepsilon(s) = r_\varepsilon(s)$.

Lemma 4.2. *Let $y \in W$, $\varepsilon > 0$. Then $r_\varepsilon(y) \in L^2(0,T;V)$,*

$$\int_0^t (y'(\tau), r_\varepsilon(y(\tau)))\mathrm{d}\tau = \int_\Omega z_\varepsilon(y(t))\mathrm{d}x - \int_\Omega z_\varepsilon(y(0))\mathrm{d}x, \quad t \in [0,T].$$

Proof. Notice that the space $H^1(Q)$ is dense in W ([20], p. 423). Let $y_j \in H^1(Q)$, $y_j \to y$ in W. Let us consider the following integral:

$$\int_0^t (y'_j, r_\varepsilon(y_j))\mathrm{d}\tau = \int_\Omega \int_0^t (z_\varepsilon(y_j))'\mathrm{d}\tau\mathrm{d}x = \int_\Omega z_\varepsilon(y_j(t))\mathrm{d}x - \int_\Omega z_\varepsilon(y_j(0))\mathrm{d}x. \tag{4.10}$$

Notice that $r_\varepsilon(y_j) \to r_\varepsilon(y)$ in $L^2(0,T;H)$ because

$$|r_\varepsilon(y_j) - r_\varepsilon(y)| \leq \frac{1}{\varepsilon}|y_j - y|, \quad \int_0^T \|r_\varepsilon(y_j) - r_\varepsilon(y)\|^2\mathrm{d}t \leq \frac{1}{\varepsilon^2}\int_0^T \|y_j - y\|^2\mathrm{d}t \to 0.$$

The sequence $r_\varepsilon(y_j)$ is bounded in $L^2(0,T;V)$ because $\nabla(r_\varepsilon(y_j)) = r'_\varepsilon(y_j)\nabla y_j$, and

$$\int_0^T \|\nabla(r_\varepsilon(y_j))\|^2\mathrm{d}t = \frac{1}{\varepsilon^2}\int_0^T \int\limits_{|y_j|<\varepsilon} |\nabla y_j|^2\mathrm{d}x\mathrm{d}t \leq \frac{1}{\varepsilon^2}\int_0^T \|\nabla y_j\|^2\mathrm{d}t.$$

Similarly, it can be proved that $r_\varepsilon(y) \in L^2(0,T;V)$. Therefore, $r_\varepsilon(y_j) \rightharpoonup r_\varepsilon(y)$ weakly in $L^2(0,T;V)$. Thus,

$$\int_0^t (y'_j, r_\varepsilon(y_j))\mathrm{d}\tau \to \int_0^t (y', r_\varepsilon(y))\mathrm{d}\tau.$$

Notice that the embedding $W \subset C([0,T];H)$ is continuous. Therefore, the trace $y|_{t=t_0}$ is valid, and besides $y_j(t) \to y(t)$, $y_j(0) \to y(0)$ in H. Thus, using the estimate $|z_\varepsilon(y_j) - z_\varepsilon(y)| \leq |y_j - y|$, we obtain that $z_\varepsilon(y_j(t)) \to z_\varepsilon(y(t))$ and $z_\varepsilon(y_j(0)) \to z_\varepsilon(y(0))$ in H.

Passing to the limit in (4.10), we obtain the statement of the lemma. □

In the following theorem, the Lyapunov stability of steady-state solutions of the problem (4.1)–(4.4) is proved.

Theorem 4.3. *Let the conditions* (j)–(jjj) *hold. Then the following estimate of the stability is true:*

$$\int_\Omega |\theta(t) - \theta_s|\mathrm{d}x + b\mu\int_\Omega |\varphi(t) - \varphi_s|\mathrm{d}x \leq \int_\Omega |\theta_0 - \theta_s|\mathrm{d}x + b\mu\int_\Omega |\varphi_0 - \varphi_s|\mathrm{d}x, \quad t > 0.$$

Proof. Notice that the statement $h_4(\theta) \in L^2(Q)$ implies $q(\zeta, x) \in L^1(Q)$. Multiplying the first equation in (4.8), in the sense of the inner product of H, by $r_\varepsilon(\zeta)$, multiplying the second equation in (4.8) by $br_\varepsilon(\xi)$, integrating them with respect to t, and adding the obtained equalities, we obtain:

$$\int_0^t \Big((\zeta', r_\varepsilon(\zeta)) + a(\nabla\zeta, r'_\varepsilon(\zeta)\nabla\zeta) + \int_\Gamma \beta\zeta r_\varepsilon(\zeta)\mathrm{d}\Gamma + b\mu(\xi', r_\varepsilon(\xi))$$
$$+b\alpha(\nabla\xi, r'_\varepsilon(\xi)\nabla\xi) + b\int_\Gamma \gamma\xi r_\varepsilon(\xi)\mathrm{d}\Gamma + b\kappa_a\big(q(\zeta,x) - \xi, r_\varepsilon(\zeta) - r_\varepsilon(\xi)\big)\Big)\mathrm{d}\tau = 0. \tag{4.11}$$

Notice that the second, the third, the firth, and the sixth terms in (4.11) are nonnegative. Moreover, $(q(\zeta,x)-\xi)(r_\varepsilon(\zeta)-r_\varepsilon(\xi))\geq 0$, if $\zeta=0$ or $\xi=0$. Therefore,

$$\int_0^t \left((\zeta', r_\varepsilon(\zeta)) + b\mu(\xi', r_\varepsilon(\xi)) + b\kappa_a \int\limits_{\zeta,\xi\neq 0} (q(\zeta,x)-\xi)(r_\varepsilon(\zeta)-r_\varepsilon(\xi))\mathrm{d}x \right) \mathrm{d}\tau \leq 0. \tag{4.12}$$

Let us pass to the limit in (4.12) as $\varepsilon\to 0$. Consider the first term. By lemma 2

$$\int_0^t (\zeta'(\tau), r_\varepsilon(\zeta(\tau)))\mathrm{d}\tau = \int_\Omega z_\varepsilon(\zeta(t))\mathrm{d}x - \int_\Omega z_\varepsilon(\zeta(0))\mathrm{d}x.$$

Since

$$|z_\varepsilon(\zeta(t)) - |\zeta(t)|| \leq \varepsilon/2 \quad \forall t,$$

then

$$\int_\Omega z_\varepsilon(\zeta(t))\mathrm{d}x \to \int_\Omega |\zeta(t)|\mathrm{d}x, \quad \int_\Omega z_\varepsilon(\zeta(0))\mathrm{d}x \to \int_\Omega |\zeta(0)|\mathrm{d}x.$$

Therefore,

$$\int_0^t (\zeta'(\tau), r_\varepsilon(\zeta(\tau)))\mathrm{d}\tau \to \int_\Omega |\zeta(t)|\mathrm{d}x - \int_\Omega |\zeta(0)|\mathrm{d}x,$$

and similarly

$$\int_0^t (\xi'(\tau), r_\varepsilon(\xi(\tau)))\mathrm{d}\tau \to \int_\Omega |\xi(t)|\mathrm{d}x - \int_\Omega |\xi(0)|\mathrm{d}x.$$

Taking into account that $r_\varepsilon(\zeta)\to \mathrm{sign}\zeta$, $r_\varepsilon(\xi)\to\mathrm{sign}\xi$ a.e. in Ω, and applying the Lebesgue theorem, we obtain

$$\int_0^t \int_{\zeta,\xi\neq 0} (q(\zeta,x)-\xi)(r_\varepsilon(\zeta)-r_\varepsilon(\xi))\mathrm{d}x\mathrm{d}\tau \to \int_0^t \int_{\zeta,\xi\neq 0} (q(\zeta,x)-\xi)(\mathrm{sign}\zeta - \mathrm{sign}\xi)\mathrm{d}x\mathrm{d}\tau \geq 0.$$

Thus, passing to the limit in (4.12), and dropping the nonnegative terms, we obtain the statement of the theorem. □

5. Conclusion

The conducted study allows us to consider the cases of singularity sources. For example, setting the right-hand side in equation (1.1) from the space V',

$$(f,v) = \int\limits_S \widetilde{f} v \mathrm{d}\Gamma, \quad v\in V, \quad \widetilde{f}\in L^2(S),$$

where S is a surface in the domain Ω, we define a jump of the heat flow through the surface S. This allows us to consider the cases of practical interest with surface temperature sources.

Also, this article significantly generalizes the previous results obtained in [17], where the unique solvability of the homogeneous steady-state boundary-value problem was proved in the class $L^\infty(\Omega)$.

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