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ANALYSIS OF AN UNGAUGED $\boldsymbol{T},\phi$–ϕ FORMULATION OF THE EDDY CURRENT PROBLEM WITH CURRENTS AND VOLTAGE EXCITATIONS*

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Abstract. The objective of this work is the analysis of a time-harmonic eddy current problem with prescribed currents or voltage drops on the boundary of the conducting domain. We will focus on an ungauged formulation that splits the magnetic field into three terms: a vector potential $\boldsymbol{T}$, defined in the conducting domain, a scalar potential ϕ, supported in the whole domain, and a linear combination of source fields, only depending on the geometry. To compute the source field functions we make use of the analytical expression of the Biot−Savart law in the dielectric domain. The most important advantage of this methodology is that it eliminates the need of multivalued scalar potentials. Concerning the discretisation, edge finite elements will be employed for the approximation of both the source field and the vector potential, and standard Lagrange finite elements for the scalar potential. To perform the analysis, we will establish an equivalence between the $\boldsymbol{T},\phi$–ϕ formulation of the problem and a slight variation of a magnetic field formulation whose well-possedness has already been proved. This equivalence will also be the key to prove convergence results for the discrete scheme. Finally, we will present some numerical results that corroborate the analytical ones.

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1. Introduction

This work deals with the mathematical analysis of the so-called $\boldsymbol{T}, \phi-\phi$ formulation for solving time-harmonic eddy current problems defined in three-dimensional bounded domains containing both conducting and dielectric materials. This kind of problem often arises in electrical engineering in the numerical simulation of varied devices, such as electrical machines, metallurgical furnaces, non-destructive testing tools, *etc.*, (see [4]). We will focus on

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the case in which some of the conducting subdomains are not strictly contained in the computational one, with sources given either in terms of the current intensities crossing their intersections with the outer boundary or in terms of the potential drops between them. This particular case is referred in the literature in different ways, such as the eddy current problem with *electric ports*, with *non-local boundary conditions* or *coupled with electric circuits*. Thanks to its widespread applicability, this problem has been subjected to thorough study during the last decades, by using different unknowns and formulations. We refer the reader to Chapter 8 of [4], where we can find a quite comprehensive review of the most relevant formulations, along with the main results from a mathematical and numerical point of view. Additionally, we can cite [1,7], more recent publications that analyse other relevant formulations of the eddy current problem with electric ports.

In the present paper, we will focus on the well-known $\boldsymbol{T}, \phi - \phi$ formulation, which combines a vector potential $\boldsymbol{T}$, defined only in the conducting domain and discretised using edge elements, with a scalar potential ϕ supported in the whole domain and discretised by nodal elements. One great advantage of this methodology is the low computational effort needed for its solution because the only vector unknown, $\boldsymbol{T}$, has to be computed only in conductors, where there are generally far fewer degrees of freedom. Therefore, this kind of formulation is one of the most used in commercial software for the solution of three-dimensional eddy current problems (*e.g.*, Altair Flux® or ANSYS Maxwell®).

While the $\boldsymbol{T}, \phi - \phi$ formulation has been widely used by electrical engineers (see, for instance, [11, 12, 20, 23]), the existing literature related to its mathematical analysis in both the continuous and discrete cases is comparatively limited. In particular, the theoretical analysis usually covers a formulation with a gauge condition for the electrical vector potential and uses a nodal finite element for its approximation. In this framework, we refer the reader to Section 8.1.3 of [4], where a continuous formulation is studied, and to the papers [14, 18], which perform the analysis in the transient case. Also, a nodal ungauged transient formulation involving only volumic sources instead of boundary ones is analysed in [19] at a discrete level. However, the formulation implemented in commercial software is usually ungauged and based on edge finite elements and, to the best of the authors' knowledge, a rigorous analysis for this case with electric ports has not yet been performed. To attain this goal, we will rest upon the uniqueness of the magnetic field, even though its decomposition in vector and scalar potentials is not unique. In this way, we will establish an equivalence between the $\boldsymbol{T}, \phi - \phi$ formulation of the problem and a slight variation of the magnetic field formulation anaysed in [10]. This equivalence, proved at both continuous and discrete levels, will be the key to obtain the uniqueness of the magnetic field reconstructed from the scalar and vector potentials, and to obtain the convergence result for the discrete scheme.

Concerning the discretization of the problem, "edge" finite elements will be employed for the approximation of the vector potential and standard Lagrange finite elements for that of the scalar potential. A drawback of this formulation is that it requires the computation of a source field in the dielectric domain, the so-called "impressed vector potential", which is not trivial if the dielectric domain is not simply connected. Based on the ideas introduced by Bíró and Preis in [12], we will compute this field by using the Biot−Savart law, what eliminates the necessity of using multivalued scalar potentials, even in the case of homologically non-trivial topologies. From the point of view of the mathematical analysis, this approach guarantees the convergence of the numerical method when sources are provided in terms of the currents crossing some parts of the boundary, but this is not the case if the potential drops are given. To overcome this theoretical difficulty, we also include in the paper the procedure introduced in [2] for constructing the impressed vector potential by computing the so-called loop fields, which would be suitable to prove the convergence in all cases; see [3], where this idea is also exploited in the implementation of a magnetic field/scalar potential formulation.

The outline of the paper is as follows: in Section 2 we present the eddy current model and recall a formulation to solve it presented in [10]; in Section 3, we derive the proposed $\boldsymbol{T}, \phi - \phi$ formulation for the eddy current problem; in Sections 4 and 5, we perform the mathematical analysis of this formulation in the continuous and discrete cases, respectively, through its equivalence with the one studied in [10]; in Section 6, we introduce a numerical procedure to compute the impressed vector potential and, with this end, we derive an expression to evaluate the Biot−Savart field corresponding to a polygonal filament carrying a unit current intensity; finally, in Section 7, some numerical results are reported.

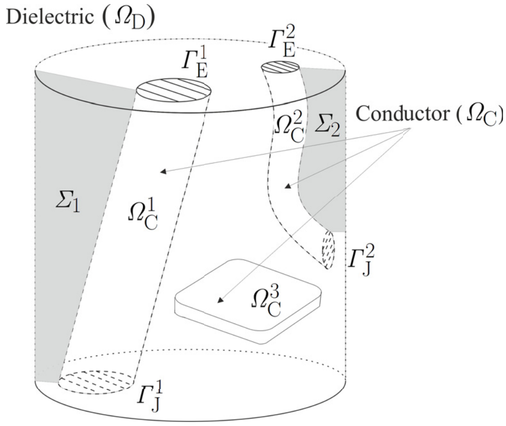


FIGURE 1. Sketch of the domain ($N = 2$, $M = 3$).

2. EDDY CURRENT MODEL WITH SOURCES AS BOUNDARY DATA

Eddy currents in linear, homogeneous and isotropic media are usually modeled by the low-frequency harmonic Maxwell equations,

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}, \tag{2.1}$$

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0}, \tag{2.2}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0, \tag{2.3}$$

along with Ohm's law

$$\boldsymbol{J} = \sigma\boldsymbol{E}, \tag{2.4}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{J}$ the current density, $\omega \neq 0$ the angular frequency (with $\omega = 2\pi f$, f being the current frequency), μ the magnetic permeability and σ the electric conductivity. Note that the latter is non-null only in conducting media.

Although equations (2.1)–(2.4) concern the whole space, for computational purposes we restrict them to a simply connected three-dimensional bounded domain Ω with a connected boundary. This domain Ω consists of two parts, $\Omega_{\rm C}$ and $\Omega_{\rm D}$, occupied by conductors and dielectrics, respectively (see Fig. 1), and we assume that $\Omega_{\rm D}$ is connected. Domains Ω, $\Omega_{\rm C}$ and $\Omega_{\rm D}$ are assumed to have Lipschitz-continuous boundaries. We denote by $\Gamma_{\rm C}$, $\Gamma_{\rm D}$ and $\Gamma_{\rm I}$ the open Lipschitz surfaces such that $\overline{\Gamma}_{\rm C} := \partial\Omega_{\rm C} \cap \partial\Omega$ is the outer boundary of the conductors, $\overline{\Gamma}_{\rm D} := \partial\Omega_{\rm D} \cap \partial\Omega$ that of the dielectrics and $\overline{\Gamma}_{\rm I} := \partial\Omega_{\rm C} \cap \partial\Omega_{\rm D}$ the interface between both domains. We also denote by $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$, as well as other unit vectors normal to particular surfaces that will be deduced from the context.

The connected components of the conducting domain $\Omega_{\rm C}^n$, $n = 1, \ldots, M$, are supposed to be simply connected with connected boundaries. The first N connected components of the conducting domain, the so-called "inductors", are assumed to intersect the boundary of Ω in such a way that the outer boundary of each of them, $\partial\Omega_{\rm C}^n \cap \partial\Omega$, has two disjoint connected components, both being the closure of non-zero measure open surfaces: the "current entrances" $\overline{\Gamma}_{\rm J}^n$ and the "current exits" $\overline{\Gamma}_{\rm E}^n$, where the conductor is connected to an alternating electric source. We denote $\Gamma_{\rm J} := \Gamma_{\rm J}^1 \cup \ldots \cup \Gamma_{\rm J}^N$, $\Gamma_{\rm E} := \Gamma_{\rm E}^1 \cup \ldots \cup \Gamma_{\rm E}^N$ and $\overline{\Gamma}_{\rm I}^n = \partial\Omega_{\rm C}^n \cap \partial\Omega_{\rm D}$, $n = 1, \ldots, N$. Furthermore, we assume that $\overline{\Gamma}_{\rm J}^n \cap \overline{\Gamma}_{\rm J}^m = \emptyset$ and $\overline{\Gamma}_{\rm E}^n \cap \overline{\Gamma}_{\rm E}^m = \emptyset$, $1 \leq m, n \leq N$, $m \neq n$, and $\overline{\Gamma}_{\rm J} \cap \overline{\Gamma}_{\rm E} = \emptyset$. The remaining connected components $\Omega_{\rm C}^n$, $n = N+1, \ldots, M$, are assumed to have their closure included in Ω, and will be referred to as "workpieces".

As illustrated in Figure 1, we assume that for each inductor Ω_{C}^n, $n=1,\dots,N$, there exists a connected "cutting" surface $\Sigma_n\subset\Omega_{\mathrm{D}}$ such that $\partial\Sigma_n\subset\partial\Omega_{\mathrm{D}}$ and $\widetilde{\Omega}_{\mathrm{D}}:=\Omega_{\mathrm{D}}\setminus\bigcup_{n=1}^N\Sigma_n$ is pseudo-Lipschitz and simply connected (see, for instance, [5]). We also assume that $\bar{\Sigma}_n\cap\bar{\Sigma}_m=\emptyset$ for $n\neq m$ and that the boundary of each current entrance surface, $\gamma_n:=\partial\Gamma_{\mathrm{J}}^n$, is a simple closed curve. We denote the two faces of each Σ_n by Σ_n^- and Σ_n^+ and fix a unit normal $\boldsymbol{n}_n$ on Σ_n as the "outer" normal to $\Omega_{\mathrm{D}}\setminus\Sigma_n$ along Σ_n^+. We choose an orientation for each γ_n by taking its initial and end points on Σ_n^- and Σ_n^+, respectively. We denote by $\boldsymbol{\tau}_n$ the unit vector tangent to γ_n according with this orientation. Moreover, let us emphasise that the cutting surfaces Σ_n, $n=1,\dots,N$, will be only a theoretical tool to prove some of the following results. However, there is no need to construct such surfaces to apply the $\boldsymbol{T},\phi-\phi$ formulation of the eddy current problem that we will introduce and analyse in this paper.

We assume that there exist constants $\underline{\mu}$, $\overline{\mu}$, $\underline{\sigma}$ and $\overline{\sigma}$ such that

$$\begin{aligned}&0<\underline{\mu}\le\mu(\boldsymbol{x})\le\overline{\mu}\quad\text{a.e. }\boldsymbol{x}\in\Omega,\\&0<\underline{\sigma}\le\sigma(\boldsymbol{x})\le\overline{\sigma}\quad\text{a.e. }\boldsymbol{x}\in\Omega_{\mathrm{C}}\quad\text{and}\quad\sigma\equiv0\quad\text{in}\quad\Omega_{\mathrm{D}}.\end{aligned}$$

To solve equations (2.1)–(2.4) in a bounded domain, it is necessary to add suitable boundary conditions. We consider the following which will appear as natural boundary conditions of the weak formulation of the problem:

$$\boldsymbol{E}\times\boldsymbol{n}=\boldsymbol{0}\qquad\text{on}\quad\Gamma_{\mathrm{E}}\cup\Gamma_{\mathrm{J}},\tag{2.5}$$

$$\mu\boldsymbol{H}\cdot\boldsymbol{n}=0\qquad\text{on}\quad\partial\Omega.\tag{2.6}$$

The former means that the electric current is normal to the entrance and exit surfaces, whereas the latter means that the magnetic field is tangential to the boundary. These boundary conditions have been proposed in [13]; we refer to [10] for further discussion about them.

Boundary condition (2.6) implies that the tangential component of the electric field $\boldsymbol{E}$ is a surface gradient. Indeed, after integrating $i\omega\mu\boldsymbol{H}\cdot\boldsymbol{n}$ on any surface S contained in $\partial\Omega$, by using (2.2) and Stokes' theorem we obtain

$$0=\int_S i\omega\mu\boldsymbol{H}\cdot\boldsymbol{n}=-\int_S\mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{n}=-\int_{\partial S}\boldsymbol{E}\cdot\boldsymbol{\tau}=-\int_{\partial S}\boldsymbol{n}\times(\boldsymbol{E}\times\boldsymbol{n})\cdot\boldsymbol{\tau},$$

$\boldsymbol{\tau}$ being a unit vector tangent to ∂S. Therefore, since $\partial\Omega$ is simply connected, there exists a sufficiently smooth function V defined on $\partial\Omega$ up to a constant, such that V is a surface potential of the tangential component of $\boldsymbol{E}$; that is, $\boldsymbol{n}\times\boldsymbol{E}\times\boldsymbol{n}=-\,\mathbf{grad}_\tau V$ on $\partial\Omega$, where $\mathbf{grad}_\tau$ denotes the surface gradient. On the other hand, equation (2.5) implies that V must be constant on each connected component of Γ_{E} and Γ_{J}. Let V_{E}^n and V_{J}^n be complex numbers such that $V=V_{\mathrm{E}}^n$ on Γ_{E}^n and $V=V_{\mathrm{J}}^n$ on Γ_{J}^n, $n=1,\dots,N$. The difference $\Delta V_n=V_{\mathrm{E}}^n-V_{\mathrm{J}}^n$ is the potential drop along conductor Ω_{C}^n.

Multiplying Faraday's law (2.2) by $\overline{\boldsymbol{H}}$, integrating over Ω and then applying a Green's formula along with equation (2.1), we obtain

$$\int_\Omega i\omega\mu|\boldsymbol{H}|^2+\int_{\Omega_{\mathrm{C}}}\boldsymbol{E}\cdot\overline{\boldsymbol{J}}=\int_{\partial\Omega}(\boldsymbol{E}\times\boldsymbol{n})\cdot\overline{\boldsymbol{H}}.$$

Using that $\boldsymbol{n}\times\boldsymbol{E}\times\boldsymbol{n}=-\,\mathbf{grad}_\tau V$ on $\partial\Omega$, we write

$$\int_{\partial\Omega}(\boldsymbol{E}\times\boldsymbol{n})\cdot\overline{\boldsymbol{H}}=-\int_{\partial\Omega}(\mathbf{grad}_\tau V\times\boldsymbol{n})\cdot\overline{\boldsymbol{H}}=-\int_{\partial\Omega}\mathbf{curl}_\tau V\cdot\overline{\boldsymbol{H}}=-\int_{\partial\Omega}V\,\mathrm{curl}_\tau\overline{\boldsymbol{H}}=-\int_{\partial\Omega}V\,\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n},\tag{2.7}$$

where $\mathbf{curl}_\tau$ and curl_τ denote the surface vector and scalar curls, respectively.

Now, since $\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J}$ and $\boldsymbol{J} = \mathbf{0}$ in Ω_{D},

$$\int_{\partial\Omega} V\,\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n} = \sum_{n=1}^{N}\left(V_{\mathrm{E}}^{n}\int_{\Gamma_{\mathrm{E}}^{n}}\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n} + V_{\mathrm{J}}^{n}\int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n}\right) = -\sum_{n=1}^{N}\Delta V_n\int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n}, \tag{2.8}$$

the last equality because, for each inductor,

$$0 = \int_{\Omega_{\mathrm{C}}^{n}}\mathrm{div}(\mathbf{curl}\,\overline{\boldsymbol{H}}) = \int_{\partial\Omega_{\mathrm{C}}^{n}}\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n} = \int_{\Gamma_{\mathrm{E}}^{n}}\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n} + \int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\overline{\boldsymbol{H}}\cdot\boldsymbol{n}.$$

Then, from the above equations we derive the energy conservation law:

$$\int_{\Omega} i\omega\mu|\boldsymbol{H}|^2 + \int_{\Omega_{\mathrm{C}}}\boldsymbol{E}\cdot\overline{\boldsymbol{J}} = \sum_{n=1}^{N}\overline{I_n}\,\Delta V_n,$$

with $I_n := \int_{\Gamma_{\mathrm{J}}^{n}}\boldsymbol{J}\cdot\boldsymbol{n} = \int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n}$ being the current intensity through conductor Ω_{C}^{n}, $n = 1,\ldots,N$.

In order to consider sources provided by external circuits we have two possibilities: either the current intensity or the potential drop must be given for each inductor Ω_{C}^{n}, $n = 1,\ldots,N$. We assume that for $n = 1,\ldots,N_I$ $(0 \leq N_I \leq N)$ the current intensity I_n crossing Γ_{J}^{n} is given, in which case the boundary condition reads

$$\int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{n} = I_n, \quad n = 1,\ldots,N_I, \tag{2.9}$$

and, for $n = N_I + 1,\ldots,N$, the potential drop ΔV_n between Γ_{J}^{n} and Γ_{E}^{n} is given, in which case the boundary condition reads

$$\boldsymbol{n}\times\boldsymbol{E}\times\boldsymbol{n} = -\,\mathbf{grad}_{\tau}\,V \qquad \text{on} \quad \partial\Omega, \quad \text{with } V|_{\Gamma_{\mathrm{E}}^{n}} - V|_{\Gamma_{\mathrm{J}}^{n}} = \Delta V_n, \quad n = N_I + 1,\ldots,N. \tag{2.10}$$

The system composed by equations (2.1)–(2.4) subjected to boundary conditions (2.5), (2.6), (2.9) and (2.10) is frequently known as the *eddy current problem with non-local boundary conditions*.

3. $\boldsymbol{T},\phi-\phi$ formulation of the Eddy current problem

Our first goal is to introduce some auxiliary unknowns that will be used to solve the previous set of equations. First of all, note that given a complex vector of currents $(I_n)_{n=1}^{N}\in\mathbb{C}^N$, there exists $\boldsymbol{T}_0\in\mathrm{H}(\mathbf{curl};\Omega)$ such that

$$\int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\boldsymbol{T}_0\cdot\boldsymbol{n} = I_n \quad \text{for } n = 1,\ldots,N,$$
$$\mathbf{curl}\,\boldsymbol{T}_0 = \mathbf{0} \quad \text{in } \Omega_{\mathrm{D}}\cup\left(\cup_{n=N+1}^{M}\Omega_{\mathrm{C}}^{n}\right).$$

Such $\boldsymbol{T}_0$ is usually called an "impressed vector potential". An example is given by $\boldsymbol{T}_0(\boldsymbol{x}) = \sum_{n=1}^{N} I_n\boldsymbol{t}_{0,n}(\boldsymbol{x})$, with $\boldsymbol{t}_{0,n}\in\mathrm{H}(\mathbf{curl};\Omega)$ satisfying

$$\int_{\Gamma_{\mathrm{J}}^{n}}\mathbf{curl}\,\boldsymbol{t}_{0,n}\cdot\boldsymbol{n} = 1, \tag{3.1}$$
$$\mathbf{curl}\,\boldsymbol{t}_{0,n} = \mathbf{0} \qquad \text{in} \quad \Omega\setminus\overline{\Omega_{\mathrm{C}}^{n}}, \tag{3.2}$$

for $n = 1,\ldots,N$. We will refer to these vector fields $\boldsymbol{t}_{0,n}$, $n = 1,\ldots,N$, as "normalised impressed vector potentials". They can be defined in different ways (see, *e.g.*, [11]).

From equations (2.1) and (2.4) we have that $\operatorname{div}\boldsymbol{J}=0$ in $\varOmega_{\mathrm{C}}$ and $\boldsymbol{J}\cdot\boldsymbol{n}=0$ on $\varGamma_{\mathrm{I}}$. Therefore,

$$\begin{aligned}
\operatorname{div}(\boldsymbol{J}-\mathbf{curl}\,\boldsymbol{T}_0)&=0 &&\text{in}\quad \varOmega_{\mathrm{C}},\\
(\boldsymbol{J}-\mathbf{curl}\,\boldsymbol{T}_0)\cdot\boldsymbol{n}&=0 &&\text{on}\quad \varGamma_{\mathrm{I}},\\
\int_{\varGamma_{\mathrm{J}}^n}(\boldsymbol{J}-\mathbf{curl}\,\boldsymbol{T}_0)\cdot\boldsymbol{n}&=0 &&\text{for}\quad n=1,\dots,N.
\end{aligned}$$

Hence, it can be proved that for each connected component of the conducting domain $\varOmega_{\mathrm{C}}^n$, $n=1,\dots,M$, there exists a vector field $\boldsymbol{T}^n$ supported in $\varOmega_{\mathrm{C}}^n$ such that

$$\begin{aligned}
\mathbf{curl}\,\boldsymbol{T}^n&=\boldsymbol{J}-\mathbf{curl}\,\boldsymbol{T}_0 &&\text{in}\quad \varOmega_{\mathrm{C}}^n,\\
\boldsymbol{T}^n\times\boldsymbol{n}&=\boldsymbol{0} &&\text{on}\quad \varGamma_{\mathrm{I}}^n
\end{aligned}$$

(see, for example, Thm. 2.1 in [16]).

Let $\widetilde{\boldsymbol{T}}^n$ be the extension by zero to $\varOmega$ of $\boldsymbol{T}^n$, $n=1,\dots,M$. Let $\widetilde{\boldsymbol{T}}:=\sum_{n=1}^M\widetilde{\boldsymbol{T}}^n$ and $\boldsymbol{T}:=\widetilde{\boldsymbol{T}}|_{\varOmega_{\mathrm{C}}}$. Then, $\boldsymbol{T}$ satisfies $\mathbf{curl}\,\boldsymbol{T}=\boldsymbol{J}-\mathbf{curl}\,\boldsymbol{T}_0$ in $\varOmega_{\mathrm{C}}$ and $\boldsymbol{T}\times\boldsymbol{n}=\boldsymbol{0}$ on $\varGamma_{\mathrm{I}}$. Such a $\boldsymbol{T}$ is called a "current vector potential".

Now, from (2.1), $\mathbf{curl}\,\boldsymbol{H}=\boldsymbol{J}=\mathbf{curl}\,\widetilde{\boldsymbol{T}}+\mathbf{curl}\,\boldsymbol{T}_0$, so that, since $\varOmega$ is simply connected,

$$\boldsymbol{H}=\widetilde{\boldsymbol{T}}+\boldsymbol{T}_0-\mathbf{grad}\,\phi$$

for some $\phi\in\mathrm{H}^1(\varOmega)$; ϕ is usually called a "magnetic scalar potential". Notice that such an $\boldsymbol{H}$ satisfies automatically the constraint $\mathbf{curl}\,\boldsymbol{H}=\boldsymbol{0}$ in $\varOmega_{\mathrm{D}}$, which follows from (2.1) and (2.4).

Taking the previous decomposition into account, the time-harmonic eddy current problem (2.1)–(2.6) leads to:

$$i\omega\mu\,(\boldsymbol{T}+\boldsymbol{T}_0-\mathbf{grad}\,\phi)+\mathbf{curl}\left(\frac{1}{\sigma}\,\mathbf{curl}(\boldsymbol{T}+\boldsymbol{T}_0)\right)=\boldsymbol{0}\qquad\text{in}\quad\varOmega_{\mathrm{C}},\tag{3.3}$$

$$\operatorname{div}\left(\mu(\widetilde{\boldsymbol{T}}+\boldsymbol{T}_0-\mathbf{grad}\,\phi)\right)=0\qquad\text{in}\quad\varOmega,\tag{3.4}$$

$$\left(\frac{1}{\sigma}\,\mathbf{curl}(\boldsymbol{T}+\boldsymbol{T}_0)\right)\times\boldsymbol{n}=\boldsymbol{0}\qquad\text{on}\quad\varGamma_{\mathrm{E}}\cup\varGamma_{\mathrm{J}},\tag{3.5}$$

$$\mu(\widetilde{\boldsymbol{T}}+\boldsymbol{T}_0-\mathbf{grad}\,\phi)\cdot\boldsymbol{n}=0\qquad\text{on}\quad\partial\varOmega.\tag{3.6}$$

Our next goal is to introduce a weak formulation of this problem. If we test (3.3) with a function $\boldsymbol{S}\in\mathrm{H}(\mathbf{curl};\varOmega_{\mathrm{C}})$ such that $\boldsymbol{S}\times\boldsymbol{n}=\boldsymbol{0}$ on $\varGamma_{\mathrm{I}}$, using a Green's formula and (3.5), we obtain

$$\begin{aligned}
\int_{\varOmega_{\mathrm{C}}}i\omega\mu\,(\boldsymbol{T}+\boldsymbol{T}_0-\mathbf{grad}\,\phi)\cdot\overline{\boldsymbol{S}}+\int_{\varOmega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}(\boldsymbol{T}+\boldsymbol{T}_0)\cdot\mathbf{curl}\,\overline{\boldsymbol{S}}&=-\int_{\partial\varOmega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}(\boldsymbol{T}+\boldsymbol{T}_0)\times\boldsymbol{n}\cdot\overline{\boldsymbol{S}}\\
&=-\int_{\varGamma_{\mathrm{E}}\cup\varGamma_{\mathrm{J}}}\frac{1}{\sigma}\,\mathbf{curl}(\boldsymbol{T}+\boldsymbol{T}_0)\times\boldsymbol{n}\cdot\overline{\boldsymbol{S}}+\int_{\varGamma_{\mathrm{I}}}\frac{1}{\sigma}\,\mathbf{curl}(\boldsymbol{T}+\boldsymbol{T}_0)\cdot\overline{\boldsymbol{S}}\times\boldsymbol{n}=0.
\end{aligned}$$

Hence, using that $\boldsymbol{T}_0=\sum_{n=1}^N I_n\boldsymbol{t}_{0,n}$ we write

$$\int_{\varOmega_{\mathrm{C}}}i\omega\mu(\boldsymbol{T}-\mathbf{grad}\,\phi)\cdot\overline{\boldsymbol{S}}+\int_{\varOmega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{T}\cdot\mathbf{curl}\,\overline{\boldsymbol{S}}+\sum_{n=1}^N I_n\left(\int_{\varOmega_{\mathrm{C}}}i\omega\mu\boldsymbol{t}_{0,n}\cdot\overline{\boldsymbol{S}}+\int_{\varOmega_{\mathrm{C}}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{t}_{0,n}\cdot\mathbf{curl}\,\overline{\boldsymbol{S}}\right)=0.\tag{3.7}$$

On the other hand, multiplying (3.4) by $i\omega\overline{\psi}$ with $\psi\in\mathrm{H}^1(\varOmega)$, using a Green's formula and taking (3.6) into account, we obtain

$$\int_{\varOmega}i\omega\mu\left(\widetilde{\boldsymbol{T}}+\boldsymbol{T}_0-\mathbf{grad}\,\phi\right)\cdot\mathbf{grad}\,\overline{\psi}=0.$$

Then, for all $\psi \in \mathrm{H}^1(\Omega)$ we have that

$$\int_\Omega i\omega\mu \left(\widetilde{\boldsymbol{T}} - \mathbf{grad}\,\phi\right) \cdot \mathbf{grad}\,\overline{\psi} + \sum_{n=1}^{N} I_n \int_\Omega i\omega\mu \boldsymbol{t}_{0,n} \cdot \mathbf{grad}\,\overline{\psi} = 0. \tag{3.8}$$

When all the sources are given in terms of the current intensities crossing the conducting subdomains, the problem to solve is (3.7)–(3.8). However, when there are conductors for which the potential drops are given, we need to derive some other equations to determine the corresponding current intensities. To this end, we multiply equation (2.2) by the conjugate of $\boldsymbol{t}_{0,m}$ and integrate over Ω for $m = N_I + 1, \dots, N$, to obtain

$$\int_\Omega i\omega\mu \boldsymbol{H} \cdot \overline{\boldsymbol{t}}_{0,m} + \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \overline{\boldsymbol{t}}_{0,m} = 0.$$

Now, using a Green's formula and the fact that $\mathbf{curl}\,\overline{\boldsymbol{t}}_{0,m} = \mathbf{0}$ out of Ω_{C}^m, we have

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \overline{\boldsymbol{t}}_{0,m} = \int_{\Omega_{\mathrm{C}}^m} \boldsymbol{E} \cdot \mathbf{curl}\,\overline{\boldsymbol{t}}_{0,m} - \int_{\partial\Omega} (\boldsymbol{E} \times \boldsymbol{n}) \cdot \overline{\boldsymbol{t}}_{0,m}.$$

Proceeding as in (2.7)–(2.8) with the test function $\boldsymbol{t}_{0,m}$ instead of $\boldsymbol{H}$, it is easy to check that

$$\int_{\partial\Omega} (\boldsymbol{E} \times \boldsymbol{n}) \cdot \overline{\boldsymbol{t}}_{0,m} = \Delta V_m.$$

Then, from the last three equations, (2.1) and (2.4) we obtain

$$\int_\Omega i\omega\mu \boldsymbol{H} \cdot \overline{\boldsymbol{t}}_{0,m} + \int_{\Omega_{\mathrm{C}}^m} \frac{1}{\sigma} \mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\overline{\boldsymbol{t}}_{0,m} = \Delta V_m.$$

Thus, using again that $\boldsymbol{H} = \widetilde{\boldsymbol{T}} + \boldsymbol{T}_0 - \mathbf{grad}\,\phi$ and $\boldsymbol{T}_0 = \sum_{n=1}^N I_n \boldsymbol{t}_{0,n}$, we write for $m = N_I + 1, \dots, N$

$$\int_\Omega i\omega\mu(\widetilde{\boldsymbol{T}} - \mathbf{grad}\,\phi) \cdot \overline{\boldsymbol{t}}_{0,m} + \int_{\Omega_{\mathrm{C}}^m} \frac{1}{\sigma} \mathbf{curl}\,\boldsymbol{T} \cdot \mathbf{curl}\,\overline{\boldsymbol{t}}_{0,m} + \sum_{n=1}^{N} I_n \int_\Omega i\omega\mu \boldsymbol{t}_{0,n} \cdot \overline{\boldsymbol{t}}_{0,m} + I_m \int_{\Omega_{\mathrm{C}}^m} \frac{1}{\sigma} |\,\mathbf{curl}\,\boldsymbol{t}_{0,m}|^2 = \Delta V_m. \tag{3.9}$$

We define the following closed subspace of $\mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{C}})$:

$$\boldsymbol{\mathcal{Y}} := \left\{ \boldsymbol{S} \in \mathrm{H}(\mathbf{curl}; \Omega_{\mathrm{C}}) \, : \, \boldsymbol{S} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma_{\mathrm{I}} \right\}.$$

Collecting equations (3.7)–(3.9), we derive the following formulation:

Problem 3.1. Let $\boldsymbol{t}_{0,n} \in \mathrm{H}(\mathbf{curl}; \Omega)$, $n = 1, \dots, N$, satisfying (3.1)–(3.2). Given $I_n \in \mathbb{C}$, $n = 1, \dots, N_I$, and $\Delta V_n \in \mathbb{C}$, $n = N_I + 1, \dots, N$, find $\boldsymbol{T} \in \boldsymbol{\mathcal{Y}}$, $\phi \in \mathrm{H}^1(\Omega)$ and $I_n \in \mathbb{C}$ for $n = N_I + 1, \dots, N$ such that

$$\int_{\Omega_{\mathrm{C}}} i\omega\mu(\boldsymbol{T} - \mathbf{grad}\,\phi) \cdot \overline{\boldsymbol{S}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \mathbf{curl}\,\boldsymbol{T} \cdot \mathbf{curl}\,\overline{\boldsymbol{S}} + \sum_{n=N_I+1}^{N} I_n \left(\int_{\Omega_{\mathrm{C}}} i\omega\mu \boldsymbol{t}_{0,n} \cdot \overline{\boldsymbol{S}} + \int_{\Omega_{\mathrm{C}}^n} \frac{1}{\sigma} \mathbf{curl}\,\boldsymbol{t}_{0,n} \cdot \mathbf{curl}\,\overline{\boldsymbol{S}} \right) = -\sum_{n=1}^{N_I} I_n \left(\int_{\Omega_{\mathrm{C}}} i\omega\mu \boldsymbol{t}_{0,n} \cdot \overline{\boldsymbol{S}} + \int_{\Omega_{\mathrm{C}}^n} \frac{1}{\sigma} \mathbf{curl}\,\boldsymbol{t}_{0,n} \cdot \mathbf{curl}\,\overline{\boldsymbol{S}} \right) \quad \forall\, \boldsymbol{S} \in \boldsymbol{\mathcal{Y}}, \tag{3.10}$$

$$-\int_{\Omega_{\rm C}} i\omega\mu \boldsymbol{T}\cdot\mathbf{grad}\,\overline{\psi}+\int_{\Omega} i\omega\mu\,\mathbf{grad}\,\phi\cdot\mathbf{grad}\,\overline{\psi}-\sum_{n=N_I+1}^{N} I_n\int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}\cdot\mathbf{grad}\,\overline{\psi}$$
$$=\sum_{n=1}^{N_I} I_n\int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}\cdot\mathbf{grad}\,\overline{\psi}\quad\forall\,\psi\in \mathrm{H}^1(\Omega),\quad (3.11)$$

$$\left(\int_{\Omega_{\rm C}} i\omega\mu\boldsymbol{T}\cdot\overline{\boldsymbol{t}}_{0,m}-\int_{\Omega} i\omega\mu\,\mathbf{grad}\,\phi\cdot\overline{\boldsymbol{t}}_{0,m}+\int_{\Omega_{\rm C}^m}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{T}\cdot\mathbf{curl}\,\overline{\boldsymbol{t}}_{0,m}+\sum_{n=N_I+1}^{N} I_n\int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}\cdot\overline{\boldsymbol{t}}_{0,m}\right.$$
$$\left.+I_m\int_{\Omega_{\rm C}^m}\frac{1}{\sigma}|\,\mathbf{curl}\,\boldsymbol{t}_{0,m}|^2\right)\overline{K}_m=\Delta V_m\overline{K}_m-\left(\sum_{n=1}^{N_I} I_n\int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}\cdot\overline{\boldsymbol{t}}_{0,m}\right)\overline{K}_m\quad\forall\,K_m\in\mathbb{C},\quad m=N_I+1,\dots,N.$$
$$(3.12)$$

This is the well-known $\boldsymbol{T},\phi-\phi$ formulation of problem (2.1)–(2.4) subjected to boundary conditions (2.5), (2.6), (2.9) and (2.10) (see [12]). Let us notice that this problem is not well-posed; indeed, we will show in the following section that it has infinitely many solutions. Still, we will also show that any of these solutions allows us to solve our original problem.

4. Mathematical analysis of the $\boldsymbol{T},\phi-\phi$ formulation

Now, we recall the magnetic field formulation considered in [10] of the same eddy current problem that will be used to analyse the $\boldsymbol{T},\phi-\phi$ formulation. To this end, we define

$$\boldsymbol{\mathcal{X}}:=\{\boldsymbol{G}\in\mathrm{H}(\mathbf{curl};\Omega)\,:\,\mathbf{curl}\,\boldsymbol{G}=\mathbf{0}\quad\text{in }\Omega_{\rm D}\}$$

and, given $\boldsymbol{K}\in\mathbb{C}^{N_I}$,

$$\boldsymbol{\mathcal{V}}(\boldsymbol{K}):=\left\{\boldsymbol{G}\in\boldsymbol{\mathcal{X}}\,:\,\int_{\Gamma_{\rm J}^n}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}=K_n,\ n=1,\dots,N_I\right\},$$

which is a closed linear manifold of $\boldsymbol{\mathcal{X}}$.

Remark 4.1. For all $\boldsymbol{G}\in\boldsymbol{\mathcal{X}}$, $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}\in\mathrm{H}^{-1/2}(\partial\Omega)$ and $\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}=0$ on $\Gamma_{\rm D}$. Then, $\int_{\Gamma_{\rm J}^n}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}$ is well defined. Indeed, let δ be any smooth function defined in $\partial\Omega$, such that $\delta=1$ on $\Gamma_{\rm J}^n$ and $\delta=0$ on $\Gamma_{\rm E}$ and on $\Gamma_{\rm J}^m$, $m=1,\dots,N$, $m\neq n$. Then $\int_{\Gamma_{\rm J}^n}\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n}:=\langle\mathbf{curl}\,\boldsymbol{G}\cdot\boldsymbol{n},\delta\rangle_{\mathrm{H}^{-1/2}(\partial\Omega)\times\mathrm{H}^{1/2}(\partial\Omega)}$ is well defined and its value does not depend on the particular choice of δ.

The following magnetic field formulation is derived by using the same arguments from [10], where a similar problem but only with current intensity source terms has been considered.

Problem 4.2. Given $I_n\in\mathbb{C}$, $n=1,\dots,N_I$, and $\Delta V_n\in\mathbb{C}$, $n=N_I+1,\dots,N$, find $\boldsymbol{H}\in\boldsymbol{\mathcal{V}}(\boldsymbol{I})$ such that

$$\int_{\Omega} i\omega\mu\boldsymbol{H}\cdot\overline{\boldsymbol{G}}+\int_{\Omega_{\rm C}}\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\overline{\boldsymbol{G}}=\sum_{n=N_I+1}^{N}\Delta V_n\int_{\Gamma_{\rm J}^n}\mathbf{curl}\,\overline{\boldsymbol{G}}\cdot\boldsymbol{n}\quad\forall\,\boldsymbol{G}\in\boldsymbol{\mathcal{V}}(\mathbf{0}).\qquad(4.1)$$

We have the following results:

Theorem 4.3. *Problem 4.2 has a unique solution.*

Proof. The result follows immediately from the fact that $\boldsymbol{\mathcal{V}}(\boldsymbol{I}) \neq \emptyset$ (see Lem. 2 in [10] or Sect. 3.2 in [8]), the $\boldsymbol{\mathcal{X}}$-ellipticity of the continuous sesquilinear form in the left hand side of (4.1) and the continuity of the linear functional $\boldsymbol{G} \longmapsto \sum_{n=N_I+1}^{N} \Delta V_n \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\overline{\boldsymbol{G}} \cdot \boldsymbol{n}$. □

Theorem 4.4. *Given $I_n \in \mathbb{C}$, $n = 1, \ldots, N_I$, and $\Delta V_n \in \mathbb{C}$, $n = N_I + 1, \ldots, N$, let $\boldsymbol{H} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$ be the solution to Problem 4.2. Let $\boldsymbol{J} := \mathbf{curl}\,\boldsymbol{H}$ and $\boldsymbol{E} := \left(\frac{1}{\sigma}\boldsymbol{J}\right)|_{\Omega_{\mathrm{C}}}$. Then, the following equations hold true:*

$$i\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \qquad \text{in} \qquad \Omega_{\mathrm{C}}, \tag{4.2}$$

$$\operatorname{div}(\mu\boldsymbol{H}) = 0 \qquad \text{in} \qquad \Omega, \tag{4.3}$$

$$\boldsymbol{J} = \mathbf{0} \qquad \text{in} \qquad \Omega_{\mathrm{D}}, \tag{4.4}$$

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\,\boldsymbol{H} \cdot \boldsymbol{n} = I_n \qquad n = 1, \ldots, N_I, \tag{4.5}$$

$$\mu\boldsymbol{H} \cdot \boldsymbol{n} = 0 \qquad \text{on} \qquad \partial\Omega, \tag{4.6}$$

$$\boldsymbol{E} \times \boldsymbol{n} = -\,\mathbf{grad}\,V_* \times \boldsymbol{n} \qquad \text{in} \qquad \mathrm{H}_{00}^{-1/2}(\Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}})^3, \tag{4.7}$$

for some $V_* \in \mathrm{H}^1(\Omega_{\mathrm{C}})$ constant on each connected component of $\Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}}$ and satisfying $V_*|_{\Gamma_{\mathrm{E}}^n} - V_*|_{\Gamma_{\mathrm{J}}^n} = \Delta V_n$, $n = N_I + 1, \ldots, N$. Hence, in particular,

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on} \qquad \Gamma_{\mathrm{J}} \cup \Gamma_{\mathrm{E}}.$$

Proof. The proof is quite similar to that of Theorem 3.8 in [8]. For the sake of completeness, we include it here.

Given $\delta \in \mathcal{D}(\Omega)$, $\mathbf{grad}\,\delta \in \boldsymbol{\mathcal{V}}(\mathbf{0})$. Then, (4.1) yields

$$\int_{\Omega} i\omega\mu\boldsymbol{H} \cdot \mathbf{grad}\,\overline{\delta} = 0.$$

Consequently, (4.3) holds true.

Now, let $\boldsymbol{G} \in \mathcal{D}(\Omega)^3$ be such that $\operatorname{supp}\boldsymbol{G} \subset \Omega_{\mathrm{C}}$. Then, $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\mathbf{0})$ too, and (4.1) yields

$$\int_{\Omega_{\mathrm{C}}} i\omega\mu\boldsymbol{H} \cdot \overline{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H} \cdot \mathbf{curl}\,\overline{\boldsymbol{G}} = 0.$$

Hence, $\boldsymbol{E} := \left(\frac{1}{\sigma}\,\mathbf{curl}\,\boldsymbol{H}\right)|_{\Omega_{\mathrm{C}}}$ satisfies (4.2).

Equation (4.4) follows from the definition of $\boldsymbol{J}$ and the fact that $\boldsymbol{H} \in \boldsymbol{\mathcal{X}}$, whereas equation (4.5) follows from the fact that $\boldsymbol{H} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$.

To prove (4.6), notice that $\mu\boldsymbol{H} \in \mathrm{H}(\operatorname{div}, \Omega)$ because of (4.3). Then, $\mu\boldsymbol{H} \cdot \boldsymbol{n} \in \boldsymbol{H}^{-1/2}(\partial\Omega)$ and, given $\delta \in \mathrm{H}^1(\Omega)$, we have that

$$\langle \mu\boldsymbol{H} \cdot \boldsymbol{n}, \delta \rangle_{\partial\Omega} = \int_{\Omega} \operatorname{div}(\mu\boldsymbol{H})\overline{\delta} + \int_{\Omega} \mu\boldsymbol{H} \cdot \mathbf{grad}\,\overline{\delta} = 0,$$

the last equality because of (4.3) and (4.1), since $\mathbf{grad}\,\delta \in \boldsymbol{\mathcal{V}}(\mathbf{0})$. Then $\mu\boldsymbol{H} \cdot \boldsymbol{n} = 0$ in $\mathrm{H}^{-1/2}(\partial\Omega)$ and thus (4.6) holds true.

Finally, notice that $\boldsymbol{E} \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{C}})$ because of (4.2), and consequently $\boldsymbol{E} \times \boldsymbol{n} \in \mathrm{H}^{-1/2}(\partial\Omega_{\mathrm{C}})^3$. Hence, to prove (4.7), it is enough to show that there exists $V_* \in \mathrm{H}^1(\Omega_{\mathrm{C}})$ constant on each connected component of $\Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}}$, satisfying $V_*|_{\Gamma_{\mathrm{E}}^n} - V_*|_{\Gamma_{\mathrm{J}}^n} = \Delta V_n$, $n = N_I + 1, \dots, N$ and such that $\langle \boldsymbol{E} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega_{\mathrm{C}}} = -\langle \mathbf{grad}\, V_* \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega_{\mathrm{C}}}$ for every $\boldsymbol{v} \in \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{J}} \cup \Gamma_{\mathrm{E}})^3$.

Given one such $\boldsymbol{v}$, there exits $\boldsymbol{G} \in \mathrm{H}^1(\Omega)^3$ vanishing in Ω_{D} and such that $\boldsymbol{G}|_{\partial\Omega_{\mathrm{C}}} = \boldsymbol{v}$. Clearly, $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$. In what follows we prove that $\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\mathrm{J}}^n} = 0$, $n = 1, \dots, N_I$. With this aim, let ζ_n be a smooth function defined in Ω such that $\zeta_n|_{\Gamma_{\mathrm{J}}^m} = \delta_{nm}$, $m = 1, \dots, N$, and $\zeta_n|_{\Gamma_{\mathrm{E}}} = 0$. Then, using a Green's formula and the fact that $\boldsymbol{G}$ vanishes in Ω_{D}, we obtain

$$\langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\mathrm{J}}^n} = \langle \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}, \zeta_n \rangle_{\partial\Omega} = \int_{\Omega_{\mathrm{C}}} \mathbf{curl}\, \boldsymbol{G} \cdot \mathbf{grad}\, \overline{\zeta}_n.$$

Moreover, since $\boldsymbol{G}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^1(\Omega_{\mathrm{C}})^3$ and $\mathbf{grad}\, \zeta_n|_{\Omega_{\mathrm{C}}} \in \mathrm{H}(\mathbf{curl}, \Omega_{\mathrm{C}})$, using a Green's formula,

$$\int_{\Omega_{\mathrm{C}}} \mathbf{curl}\, \boldsymbol{G} \cdot \mathbf{grad}\, \overline{\zeta}_n = \int_{\partial\Omega_{\mathrm{C}}} \mathbf{grad}\, \overline{\zeta}_n \times \boldsymbol{n} \cdot \overline{\boldsymbol{G}} = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{grad}\, \overline{\zeta}_n \times \boldsymbol{n} \cdot \overline{\boldsymbol{G}} = 0,$$

the last equality because ζ_n is constant on Γ_{J}^n. Therefore, $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\mathbf{0})$ and we can use it to test (4.1).

Since $\boldsymbol{G}$ is null outside Ω_{C} and $\boldsymbol{E} = \frac{1}{\sigma}\, \mathbf{curl}\, \boldsymbol{H}$ in Ω_{C}, using a Green's formula and (4.2) we obtain

$$\sum_{n=N_I+1}^{N} \Delta V_n \langle \mathbf{curl}\, \overline{\boldsymbol{G}} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_{\mathrm{J}}^n} = \int_{\Omega_{\mathrm{C}}} i\omega\mu \boldsymbol{H} \cdot \overline{\boldsymbol{G}} + \int_{\Omega_{\mathrm{C}}} \boldsymbol{E} \cdot \mathbf{curl}\, \overline{\boldsymbol{G}} = \langle \boldsymbol{E} \times \boldsymbol{n}, \boldsymbol{G} \rangle_{\partial\Omega_{\mathrm{C}}} = \langle \boldsymbol{E} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega_{\mathrm{C}}}.$$

On the other hand, for $n = N_I + 1, \dots, N$, let $\zeta_n \in \mathrm{H}^1(\Omega)$ be such that ζ_n vanishes in $\Omega_{\mathrm{C}} \setminus \Omega_{\mathrm{C}}^n$, $\zeta_n = 1$ on Γ_{J}^n and $\zeta_n = 0$ on Γ_{E}^n. If we define $V_* := \sum_{n=N_I+1}^{N} (-\Delta V_n)\, \overline{\zeta}_n|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^1(\Omega_{\mathrm{C}})$, taking into account that $\boldsymbol{G}$ is null in Ω_{D} and applying Green's formulas, we obtain

$$\begin{aligned} \sum_{n=N_I+1}^{N} \Delta V_n \left\langle \mathbf{curl}\, \overline{\boldsymbol{G}} \cdot \boldsymbol{n}, 1 \right\rangle_{\Gamma_{\mathrm{J}}^n} &= \left\langle \mathbf{curl}\, \overline{\boldsymbol{G}} \cdot \boldsymbol{n}, \sum_{n=N_I+1}^{N} \overline{\Delta V_n} \zeta_n \right\rangle_{\partial\Omega} \\ &= -\int_{\Omega_{\mathrm{C}}} \mathbf{curl}\, \overline{\boldsymbol{G}} \cdot \mathbf{grad}\, V_* = -\left\langle \mathbf{grad}\, V_* \times \boldsymbol{n}, \boldsymbol{v} \right\rangle_{\partial\Omega_{\mathrm{C}}}. \end{aligned}$$

Thus, from the last two equations, we derive (4.7). □

Remark 4.5. Let us notice that the current intensities through Γ_{J}^n, $n = N_I + 1, \dots, N$, can be computed from $\boldsymbol{H}$ as follows:

$$I_n = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{H} \cdot \boldsymbol{n}, \qquad n = N_I + 1, \dots, N.$$

Our next goal is to prove that Problems 3.1 and 4.2 are equivalent, for what the following lemma will be the main tool. Here and thereafter, for any $\boldsymbol{S} \in \boldsymbol{\mathcal{Y}}$, $\widetilde{\boldsymbol{S}}$ will denote the extension of $\boldsymbol{S}$ by zero to Ω. Notice that $\widetilde{\boldsymbol{S}} \in \boldsymbol{\mathcal{X}}$.

Lemma 4.6. *Let $\boldsymbol{t}_{0,n} \in \mathrm{H}(\mathbf{curl}; \Omega)$, $n = 1, \dots, N$, satisfying (3.1)–(3.2). Given $K_n \in \mathbb{C}$, $n = 1, \dots, N_I$, $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\boldsymbol{K})$ if and only if there exist $\boldsymbol{S} \in \boldsymbol{\mathcal{Y}}$, $\psi \in \mathrm{H}^1(\Omega)$ and $K_n \in \mathbb{C}$, $n = N_I + 1, \dots, N$, such that $\boldsymbol{G} = \widetilde{\boldsymbol{S}} + \sum_{n=1}^{N} K_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \psi$. Moreover, in such a case, $K_n = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}$, $n = N_I + 1, \dots, N$.*

Proof. Given $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\boldsymbol{K})$, let $K_n := \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}$, $n = N_I + 1, \ldots, N$, and $\widehat{\boldsymbol{G}} := \boldsymbol{G} - \sum_{n=1}^{N} K_n \boldsymbol{t}_{0,n}$. We have that $\widehat{\boldsymbol{G}} \in \mathrm{H}(\mathbf{curl}; \Omega)$ and it satisfies

$$\begin{aligned} \operatorname{div}(\mathbf{curl}\, \widehat{\boldsymbol{G}}) &= \mathbf{0} && \text{in} \quad \Omega, \\ \mathbf{curl}\, \widehat{\boldsymbol{G}} \cdot \boldsymbol{n} &= 0 && \text{on} \quad \Gamma_{\mathrm{I}}, \\ \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \widehat{\boldsymbol{G}} \cdot \boldsymbol{n} &= 0 && \text{for} \quad n = 1, \ldots, N. \end{aligned}$$

The equations above allow us to use again Theorem 2.1 from [16] as in the derivation of the $\boldsymbol{T}, \phi - \phi$ formulation to obtain $\boldsymbol{S} \in \boldsymbol{\mathcal{Y}}$ which satisfies

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{S} &= \mathbf{curl}\, \widehat{\boldsymbol{G}} && \text{in} \quad \Omega_{\mathrm{C}}, \\ \boldsymbol{S} \times \boldsymbol{n} &= \mathbf{0} && \text{on} \quad \Gamma_{\mathrm{I}}. \end{aligned}$$

Then, $\mathbf{curl}(\widehat{\boldsymbol{G}} - \widetilde{\boldsymbol{S}}) = \mathbf{0}$ in Ω, so that, since Ω is simply connected, there exists $\psi \in \mathrm{H}^1(\Omega)$ such that $\widehat{\boldsymbol{G}} = \widetilde{\boldsymbol{S}} - \mathbf{grad}\, \psi$. Thus, $\boldsymbol{G} = \widehat{\boldsymbol{G}} + \sum_{n=1}^{N} K_n \boldsymbol{t}_{0,n} = \widetilde{\boldsymbol{S}} + \sum_{n=1}^{N} K_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \psi$ in Ω.

Conversely, let $\boldsymbol{G} = \widetilde{\boldsymbol{S}} + \sum_{n=1}^{N} K_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \psi$, with $\boldsymbol{S} \in \boldsymbol{\mathcal{Y}}$, $\psi \in \mathrm{H}^1(\Omega)$ and $K_n \in \mathbb{C}$, $n = N_I + 1, \ldots, N$. Clearly $\boldsymbol{G} \in \mathrm{H}(\mathbf{curl}; \Omega)$ and $\mathbf{curl}\, \boldsymbol{G} = \mathbf{0}$ in Ω_{D}, so that $\boldsymbol{G} \in \boldsymbol{\mathcal{X}}$.

Moreover, for $n = 1, \ldots, N_I$, we have that

$$\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n} = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{S} \cdot \boldsymbol{n} + \sum_{m=1}^{N} K_m \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{t}_{0,m} \cdot \boldsymbol{n} = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{S} \cdot \boldsymbol{n} + K_n.$$

Let $\delta \in \mathcal{C}^\infty(\overline{\Omega})$ be as in Remark 4.1. Then, using a Green's formula, the divergence theorem and Proposition 3.3 from [15], we have that

$$\begin{aligned} \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{S} \cdot \boldsymbol{n} &:= \langle \mathbf{curl}\, \widetilde{\boldsymbol{S}} \cdot \boldsymbol{n}, \delta \rangle_{\mathrm{H}^{-1/2}(\partial\Omega) \times \mathrm{H}^{1/2}(\partial\Omega)} \\ &= \int_{\Omega} \mathbf{curl}\, \widetilde{\boldsymbol{S}} \cdot \mathbf{grad}\, \overline{\delta} = -\langle \widetilde{\boldsymbol{S}} \times \boldsymbol{n}, \mathbf{grad}_\tau\, \delta \rangle_{\mathrm{H}^{-1/2}(\partial\Omega)^3 \times \mathrm{H}^{1/2}(\partial\Omega)^3} \\ &= -\langle \widetilde{\boldsymbol{S}} \times \boldsymbol{n}, \mathbf{grad}_\tau\, \delta \rangle_{\mathrm{H}^{-1/2}(\Gamma_{\mathrm{D}})^3 \times \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})^3} - \langle \widetilde{\boldsymbol{S}}^n \times \boldsymbol{n}, \mathbf{grad}_\tau\, \delta \rangle_{\mathrm{H}^{-1/2}(\Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}})^3 \times \mathrm{H}^{1/2}(\Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}})^3} = 0, \end{aligned}$$

where for the last equality we have used that $\widetilde{\boldsymbol{S}} = \mathbf{0}$ in Ω_{D} and δ is constant on each connected component of $\Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{J}}$ (see Rem. 4.1). Hence, $K_n = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G} \cdot \boldsymbol{n}$, $n = 1, \ldots, N$, so that, in particular, $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\boldsymbol{K})$. ☐

Taking the previous decomposition into account, we have that solving Problem 3.1 is equivalent to solving Problem 4.2. In fact, we have the following result:

Theorem 4.7. *Let $\boldsymbol{t}_{0,n} \in \mathrm{H}(\mathbf{curl}; \Omega)$, $n = 1, \ldots, N$, satisfying (3.1)–(3.2). Let $I_n \in \mathbb{C}$, $n = 1, \ldots, N_I$, and $\Delta V_n \in \mathbb{C}$, $n = N_I + 1, \ldots, N$. Any solution $(\boldsymbol{T}, \phi, I_{N_I+1}, \ldots, I_N)$ to Problem 3.1 leads to the unique magnetic field $\boldsymbol{H} := \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \phi$ that solves Problem 4.2. Conversely, the solution $\boldsymbol{H}$ to Problem 4.2 can be written as $\boldsymbol{H} = \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \phi$, with $(\boldsymbol{T}, \phi, I_{N_I+1}, \ldots, I_N)$ being a solution to Problem 3.1.*

Proof. Let $(\boldsymbol{T}, \phi, I_{N_I+1}, \ldots, I_N)$ be a solution to Problem 3.1 and $\boldsymbol{H} := \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \phi$. According to Lemma 4.6, $\boldsymbol{H} \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$. Let $\boldsymbol{G} \in \boldsymbol{\mathcal{V}}(\mathbf{0})$. We use Lemma 4.6 to write $\boldsymbol{G} = \widetilde{\boldsymbol{S}} + \sum_{n=N_I+1}^{N} K_n \boldsymbol{t}_{0,n} - \mathbf{grad}\, \psi$ with $\boldsymbol{S} \in \boldsymbol{\mathcal{Y}}$ and $\psi \in \mathrm{H}^1(\Omega)$. Hence, (4.1) follows by adding equalities (3.10), (3.11) and (3.12). Then, $\boldsymbol{H}$ is the solution to Problem 4.2.

Conversely, let $\boldsymbol{H}$ be the unique solution to Problem 4.2. According to Lemma 4.6, we write $\boldsymbol{H} = \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \operatorname{\mathbf{grad}} \phi$ with $\boldsymbol{T} \in \boldsymbol{\mathcal{Y}}$, $\phi \in \mathrm{H}^1(\Omega)$ and $I_n \in \mathbb{C}$, $n = N_I + 1, \ldots, N$. Then, by substituting this expression in (4.1) and taking separately test functions $\boldsymbol{G} = \widetilde{\boldsymbol{S}}$ for $\boldsymbol{S} \in \boldsymbol{\mathcal{Y}}$, $\boldsymbol{G} = \operatorname{\mathbf{grad}} \psi$ for $\psi \in \mathrm{H}^1(\Omega)$ and $\boldsymbol{G} = \boldsymbol{t}_{0,n}$, $n = N_I + 1, \ldots, N$, we check that $(\boldsymbol{T}, \phi, I_{N_I+1}, \ldots, I_N)$ is a solution to Problem 3.1. □

Remark 4.8. The decomposition $\boldsymbol{H} = \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \operatorname{\mathbf{grad}} \phi$ is not unique. Therefore, Problem 3.1 is not well-posed since it has multiple solutions $(\boldsymbol{T}, \phi, I_{N_I+1}, \ldots, I_N)$; however, $\boldsymbol{H} := \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \operatorname{\mathbf{grad}} \phi$ is uniquely determined for all of them. Furthermore, from the computational point of view, it could be interesting to obtain one particular solution to this underdetermined problem because, to do this, the more expensive vector unknown has to be computed only in conductors. Moreover, another advantage of the $\boldsymbol{T}, \phi - \phi$ formulation with respect to an $\boldsymbol{H}, \phi$ formulation is that it does not involve a multivalued potential, what would require the construction of cutting surfaces.

5. Finite element discretisation

In this section we will introduce a discretisation of Problem 3.1 and proceed as in the previous section for its analysis. From now on, we assume that Ω, Ω_{C} and Ω_{D} are Lipschitz polyhedra and consider regular tetrahedral meshes $\mathcal{T}_h$ of Ω such that each element $T \in \mathcal{T}_h$ is contained either in $\overline{\Omega_{\mathrm{C}}}$ or in $\overline{\Omega_{\mathrm{D}}}$ (h stands as usual for the corresponding mesh-size). Therefore, $\mathcal{T}_h(\Omega_{\mathrm{D}}) := \left\{T \in \mathcal{T}_h : \ T \subset \overline{\Omega_{\mathrm{D}}}\right\}$ and $\mathcal{T}_h(\Omega_{\mathrm{C}}) := \left\{T \in \mathcal{T}_h : \ T \subset \overline{\Omega_{\mathrm{C}}}\right\}$ are meshes of Ω_{D} and Ω_{C}, respectively.

We employ edge finite elements to approximate the current vector potential $\boldsymbol{T}$, more precisely, lowest-order Nédélec finite elements:

$$\boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{C}}) := \{\boldsymbol{G}_h \in \mathrm{H}(\operatorname{\mathbf{curl}}; \Omega_{\mathrm{C}}) : \ \boldsymbol{G}_h|_T \in \boldsymbol{\mathcal{N}}(T) \ \forall T \in \mathcal{T}_h(\Omega_{\mathrm{C}})\},$$

where, for each tetrahedron T,

$$\boldsymbol{\mathcal{N}}(T) := \left\{\boldsymbol{G}_h \in \mathbb{P}_1^3(T) : \ \boldsymbol{G}_h(\boldsymbol{x}) = \mathbf{a} \times \boldsymbol{x} + \mathbf{b}, \ \mathbf{a}, \mathbf{b} \in \mathbb{C}^3, \ \boldsymbol{x} \in T\right\}.$$

For the magnetic potential ϕ, we use standard finite elements:

$$\mathcal{L}_h(\Omega) := \left\{\psi_h \in \mathrm{H}^1(\Omega) \ : \ \psi_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h\right\}.$$

We introduce the discrete subspace of $\boldsymbol{\mathcal{Y}}$

$$\boldsymbol{\mathcal{Y}}_h := \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega_{\mathrm{C}}) \ : \ \boldsymbol{G}_h \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{I}}\}.$$

We also introduce discrete normalised impressed vector potentials $\boldsymbol{t}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$, $n = 1, \ldots, N$, satisfying

$$\int_{\Gamma_{\mathrm{J}}^n} \operatorname{\mathbf{curl}} \boldsymbol{t}_{0,n}^h \cdot \boldsymbol{n} = 1, \tag{5.1}$$

$$\operatorname{\mathbf{curl}} \boldsymbol{t}_{0,n}^h = \mathbf{0} \qquad \text{in} \qquad \overline{\Omega \setminus \Omega_{\mathrm{C}}^n} \tag{5.2}$$

and a discrete impressed vector potential $\boldsymbol{T}_0^h := \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$. We describe in next section how such $\boldsymbol{t}_{0,n}^h$, $n = 1, \ldots, N$, can be computed in practice.

Then, the discretisation of Problem 3.1 reads as follows:

Problem 5.1. Let $\boldsymbol{t}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$, $n = 1,\dots,N$, satisfying (5.1)–(5.2). Given $I_n \in \mathbb{C}$, $n = 1,\dots,N_I$, and $\Delta V_n \in \mathbb{C}$, $n = N_I+1,\dots,N$, find $\boldsymbol{T}_h \in \boldsymbol{\mathcal{Y}}_h$, $\phi_h \in \mathcal{L}_h(\Omega)$ and $I_n^h \in \mathbb{C}$ for $n = N_I+1,\dots,N$ such that

$$\int_{\Omega_{\mathrm{C}}} i\omega\mu(\boldsymbol{T}_h - \operatorname{\mathbf{grad}}\phi_h)\cdot\overline{\boldsymbol{S}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{T}_h\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{S}}_h + \sum_{n=N_I+1}^{N} I_n^h \left(\int_{\Omega_{\mathrm{C}}} i\omega\mu\boldsymbol{t}_{0,n}^h\cdot\overline{\boldsymbol{S}}_h + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{t}_{0,n}^h\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{S}}_h\right)$$
$$= -\sum_{n=1}^{N_I} I_n \left(\int_{\Omega_{\mathrm{C}}} i\omega\mu\boldsymbol{t}_{0,n}^h\cdot\overline{\boldsymbol{S}}_h + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{t}_{0,n}^h\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{S}}_h\right) \quad \forall\, \boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h, \quad (5.3)$$

$$-\int_{\Omega_{\mathrm{C}}} i\omega\mu\boldsymbol{T}_h\cdot\operatorname{\mathbf{grad}}\overline{\psi}_h + \int_{\Omega} i\omega\mu\operatorname{\mathbf{grad}}\phi_h\cdot\operatorname{\mathbf{grad}}\overline{\psi}_h - \sum_{n=N_I+1}^{N} I_n^h \int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}^h\cdot\operatorname{\mathbf{grad}}\overline{\psi}_h$$
$$= \sum_{n=1}^{N_I} I_n \int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}^h\cdot\operatorname{\mathbf{grad}}\overline{\psi}_h \quad \forall\, \psi_h \in \mathcal{L}_h(\Omega), \quad (5.4)$$

$$\left(\int_{\Omega_{\mathrm{C}}} i\omega\mu\boldsymbol{T}_h\cdot\overline{\boldsymbol{t}}_{0,m}^h - \int_{\Omega} i\omega\mu\operatorname{\mathbf{grad}}\phi_h\cdot\overline{\boldsymbol{t}}_{0,m}^h + \int_{\Omega_{\mathrm{C}}^m}\frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{T}_h\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{t}}_{0,m}^h + \sum_{n=N_I+1}^{N} I_n^h \int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}^h\cdot\overline{\boldsymbol{t}}_{0,m}^h\right.$$
$$\left. + I_m^h \int_{\Omega_{\mathrm{C}}^m}\frac{1}{\sigma}|\operatorname{\mathbf{curl}}\boldsymbol{t}_{0,m}^h|^2\right)\overline{K}_m = \Delta V_m\overline{K}_m - \left(\sum_{n=1}^{N_I} I_n\int_{\Omega} i\omega\mu\boldsymbol{t}_{0,n}^h\cdot\overline{\boldsymbol{t}}_{0,m}^h\right)\overline{K}_m \quad \forall\, K_m\in\mathbb{C}, \quad m = N_I+1,\dots,N. \quad (5.5)$$

As in the continuous case, Problem 5.1 has infinitely many solutions, but any of them will be useful for our purpose.

Again, we will perform the mathematical analysis of the above problem by proving its equivalence with a discrete version of Problem 4.2. To this end, let us consider the following discrete subspaces:

$$\boldsymbol{\mathcal{X}}_h := \{\boldsymbol{G}_h \in \boldsymbol{\mathcal{N}}_h(\Omega) : \operatorname{\mathbf{curl}}\boldsymbol{G}_h = \boldsymbol{0} \quad \text{in } \Omega_{\mathrm{D}}\} \subset \boldsymbol{\mathcal{X}},$$
$$\boldsymbol{\mathcal{V}}_h(\boldsymbol{K}) := \left\{\boldsymbol{G}_h \in \boldsymbol{\mathcal{X}}_h : \int_{\Gamma_{\mathrm{J}}^n}\operatorname{\mathbf{curl}}\boldsymbol{G}_h\cdot\boldsymbol{n} = K_n,\ n = 1,\dots,N_I\right\} \subset \boldsymbol{\mathcal{V}}(\boldsymbol{K}).$$

Then, Problem 4.2 is discretised as follows:

Problem 5.2. Given $I_n \in \mathbb{C}$, $n = 1,\dots,N_I$, and $\Delta V_n \in \mathbb{C}$, $n = N_I+1,\dots,N$, find $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$ such that

$$\int_{\Omega} i\omega\mu\boldsymbol{H}_h\cdot\overline{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}}\frac{1}{\sigma}\operatorname{\mathbf{curl}}\boldsymbol{H}_h\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{G}}_h = \sum_{n=N_I+1}^{N}\Delta V_n\int_{\Gamma_{\mathrm{J}}^n}\operatorname{\mathbf{curl}}\overline{\boldsymbol{G}}_h\cdot\boldsymbol{n} \quad \forall\, \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{0}). \quad (5.6)$$

We have the following result.

Theorem 5.3. *Problem 5.2 has a unique solution $\boldsymbol{H}_h$. Moreover, if the solution to Problem 4.2 satisfies $\boldsymbol{H}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^r(\operatorname{\mathbf{curl}},\Omega_{\mathrm{C}})$ and $\boldsymbol{H}|_{\Omega_{\mathrm{D}}} \in \mathrm{H}^r(\Omega_{\mathrm{D}})^3$ with $r \in \left(\frac{1}{2},1\right]$, then the following error estimate holds*

$$\|\boldsymbol{H}-\boldsymbol{H}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le Ch^r\left[\|\boldsymbol{H}\|_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\boldsymbol{H}\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3}\right],$$

where C is a strictly positive constant independent of h and $\boldsymbol{H}$.

Proof. Since the sesquilinear form in the left hand side of (5.6) is continuous and $\boldsymbol{\mathcal{X}}_h-$elliptic, the only thing to prove in order to obtain the well-posedness of Problem 5.2 is that $\boldsymbol{\mathcal{V}}_h(\boldsymbol{I}) \neq \emptyset$. This proof is very similar to the one appearing in Section 4 of [8], but we include it here for the sake of completeness. Indeed, for $n = 1, \ldots, N_I$, let $\widehat{\boldsymbol{H}}_h^n \in \boldsymbol{\mathcal{X}}_h$ be such that $\int_{\partial \Gamma_{\mathrm{J}}^j} \widehat{\boldsymbol{H}}_h^n \cdot \boldsymbol{\tau}_j = \delta_{jn}$ for $j = 1, \ldots, N$. In Remark 5.3 from [8], a basis of $\boldsymbol{\mathcal{X}}_h$ under our geometrical assumptions is given, from which such $\widehat{\boldsymbol{H}}_h^n$ are easy to construct. Then we define

$$\widehat{\boldsymbol{H}}_h := \sum_{n=1}^{N_I} I_n \widehat{\boldsymbol{H}}_h^n.$$

Hence,

$$\int_{\Gamma_{\mathrm{J}}^j} \mathbf{curl}\, \widehat{\boldsymbol{H}}_h \cdot \boldsymbol{n} = \int_{\partial \Gamma_{\mathrm{J}}^j} \widehat{\boldsymbol{H}}_h \cdot \boldsymbol{\tau}_j = \sum_{n=1}^{N_I} I_n \int_{\partial \Gamma_{\mathrm{J}}^j} \widehat{\boldsymbol{H}}_h^n \cdot \boldsymbol{\tau}_j = I_j.$$

Thus, $\widehat{\boldsymbol{H}}_h \in \boldsymbol{\mathcal{V}}(\boldsymbol{I})$.

On the other hand, concerning the error estimate, it follows from the ellipticity of the sesquilinear form, Céa's lemma and standard error estimates for edge elements. □

To the best of the authors knowledge, the regularity of $\boldsymbol{H}$ assumed in the second part of Theorem 5.3 has not been proved. Indeed, as shown in Theorem 4.4, the solution $\boldsymbol{H}$ to Problem 4.2 satisfies $\operatorname{div}(\mu H) = 0$ in Ω and $\mu \boldsymbol{H} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Therefore, when μ is constant in the whole domain Ω, according to [5, Prop. 3.7], $\boldsymbol{H} \in \mathrm{H}^r(\Omega)^3$ for some $r > 1/2$. However, even in this case, it is not known whether $\mathbf{curl}\, \boldsymbol{H} \in \mathrm{H}^r(\Omega_{\mathrm{C}})^3$.

The following characterization is the discrete analogue to Lemma 4.6.

Lemma 5.4. *Let* $\boldsymbol{t}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$, $n = 1, \ldots, N$, *satisfying* (5.1)–(5.2). *Given* $K_n \in \mathbb{C}$, $n = 1, \ldots, N_I$, *a discrete field* $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{K})$ *if and only if there exist* $\boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h$, $\psi_h \in \mathcal{L}_h(\Omega)$ *and* $K_n^h \in \mathbb{C}$, $n = N_I + 1, \ldots, N$, *such that* $\boldsymbol{G}_h = \widetilde{\boldsymbol{S}}_h + \sum_{n=1}^{N_I} K_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} K_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\, \psi_h$. *Moreover,* $K_n^h = \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{n}$, $n = N_I + 1, \ldots, N$.

Proof. Given $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{K})$, let $K_n^h := \int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \boldsymbol{G}_h \cdot \boldsymbol{n}$, $n = N_I + 1, \ldots, N$, and $\widehat{\boldsymbol{G}}_h := \boldsymbol{G}_h - \sum_{n=1}^{N_I} K_n \boldsymbol{t}_{0,n}^h - \sum_{n=N_I+1}^{N} K_n^h \boldsymbol{t}_{0,n}^h$. Then, $\widehat{\boldsymbol{G}}_h \in \boldsymbol{\mathcal{N}}_h(\Omega)$, $\mathbf{curl}\, \widehat{\boldsymbol{G}}_h = \mathbf{0}$ in Ω_{D} and $\int_{\Gamma_{\mathrm{J}}^n} \mathbf{curl}\, \widehat{\boldsymbol{G}}_h \cdot \boldsymbol{n} = 0$, $n = 1, \ldots, N$.

Let us recall that we denote $\widetilde{\Omega}_{\mathrm{D}} := \Omega_{\mathrm{D}} \setminus \bigcup_{n=1}^{N} \Sigma_n$ the simply connected domain obtained by removing the cut surfaces Σ_n, $n = 1, \ldots, N$, from Ω_{D}. We assume that surfaces Σ_n are polyhedral and the meshes are compatible with them in the sense that each Σ_n is a union of faces of tetrahedra $T \in \mathcal{T}_h$. Therefore, $\mathcal{T}_h(\Omega_{\mathrm{D}})$ can also be seen as a mesh of $\widetilde{\Omega}_{\mathrm{D}}$. Each function $\widehat{\psi} \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}})$ has, in general, different traces on each side of Σ_n and we denote by

$$[\![\widehat{\psi}]\!]_{\Sigma_n} := \widehat{\psi}|_{\Sigma_n^-} - \widehat{\psi}|_{\Sigma_n^+}$$

the jump of $\widehat{\psi}$ through Σ_n along $\boldsymbol{n}_n$. Moreover, the gradient of $\widehat{\psi}$ in $\mathcal{D}'(\widetilde{\Omega}_{\mathrm{D}})$ can be extended to $\mathrm{L}^2(\Omega_{\mathrm{D}})^3$ and will be denoted by $\widetilde{\mathbf{grad}}\, \widehat{\psi}$.

Let us introduce the space:

$$\mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) := \left\{ \widehat{\psi}_h \in \mathrm{H}^1(\widetilde{\Omega}_{\mathrm{D}}) : \; \widehat{\psi}_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h(\Omega_{\mathrm{D}}) \right\},$$

and the subspace

$$\Theta_h := \left\{ \widehat{\psi}_h \in \mathcal{L}_h(\widetilde{\Omega}_{\mathrm{D}}) : \; [\![\widehat{\psi}_h]\!]_{\Sigma_n} = \text{constant}, n = 1, \ldots, N \right\}.$$

Since $\widehat{\boldsymbol{G}}_h|_{\Omega_{\rm D}} \in \boldsymbol{\mathcal{N}}_h(\Omega_{\rm D})$ is such that $\mathbf{curl}\,\widehat{\boldsymbol{G}}_h|_{\Omega_{\rm D}} = \mathbf{0}$, according to Lemma 5.5 from [9], there exists $\widehat{\psi}_h \in \Theta_h$ such that $\widehat{\boldsymbol{G}}_h|_{\Omega_{\rm D}} = -\widetilde{\mathbf{grad}}\,\widehat{\psi}_h$. Moreover, by using Stokes' theorem,

$$0 = \int_{\Gamma_{\rm J}^n} \mathbf{curl}\,\widehat{\boldsymbol{G}}_h \cdot \boldsymbol{n} = \int_{\gamma_n} \widehat{\boldsymbol{G}}_h \cdot \boldsymbol{\tau}_n = \int_{\gamma_n} \widetilde{\mathbf{grad}}\,\widehat{\psi}_h \cdot \boldsymbol{\tau}_n = [\![\widehat{\psi}_h]\!]_{\Sigma_n},$$

which implies that $\widehat{\psi}_h$ does not have jumps across the cut interfaces Σ_n, $n = 1, \ldots, N$, and hence $\widehat{\psi}_h \in \mathcal{L}_h(\Omega_{\rm D})$ and $\widehat{\boldsymbol{G}}_h|_{\Omega_{\rm D}} = -\mathbf{grad}\,\widehat{\psi}_h$. Let $\psi_h \in \mathcal{L}_h(\Omega)$ be any extension of $\widehat{\psi}_h$ to Ω and $\boldsymbol{S}_h := \widehat{\boldsymbol{G}}_h|_{\Omega_{\rm C}} + \mathbf{grad}\,\psi_h|_{\Omega_{\rm C}} \in \boldsymbol{\mathcal{N}}_h(\Omega_{\rm C})$. Since $\widehat{\boldsymbol{G}}_h = -\mathbf{grad}\,\psi_h$ in $\Omega_{\rm D}$, we have that $\widehat{\boldsymbol{G}}_h \times \boldsymbol{n} = -\mathbf{grad}\,\psi_h \times \boldsymbol{n}$ on $\Gamma_{\rm I}$. Therefore,

$$\boldsymbol{S}_h \times \boldsymbol{n} = \widehat{\boldsymbol{G}}_h \times \boldsymbol{n} + \mathbf{grad}\,\psi_h \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma_{\rm I}.$$

Then, $\boldsymbol{G}_h = \widetilde{\boldsymbol{S}}_h + \sum_{n=1}^{N_I} K_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} K_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\psi_h$, with $\boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h$ and $\psi_h \in \mathcal{L}_h(\Omega)$.

Conversely, let $\boldsymbol{G}_h = \widetilde{\boldsymbol{S}}_h + \sum_{n=1}^{N_I} K_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} K_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\psi_h$ with $\boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h$, $\psi_h \in \mathcal{L}_h(\Omega)$ and $K_n \in \mathbb{C}$, $n = N_I + 1, \ldots, N$. Clearly, $\boldsymbol{G}_h \in \boldsymbol{\mathcal{X}}_h$. Moreover, since $\boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h$, by Stokes' theorem

$$\int_{\Gamma_{\rm J}^n} \mathbf{curl}\,\boldsymbol{S}_h \cdot \boldsymbol{n} = \int_{\gamma_n} \boldsymbol{S}_h \cdot \boldsymbol{\tau}_n = 0, \quad n = 1, \ldots, N.$$

Therefore,

$$\int_{\Gamma_{\rm J}^m} \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n} = \sum_{n=1}^{N_I} K_n \int_{\Gamma_{\rm J}^m} \mathbf{curl}\,\boldsymbol{t}_{0,n}^h \cdot \boldsymbol{n} + \sum_{n=N_I+1}^{N} K_n^h \int_{\Gamma_{\rm J}^m} \mathbf{curl}\,\boldsymbol{t}_{0,n}^h \cdot \boldsymbol{n} = K_m, \quad m = 1, \ldots, N_I,$$
$$\int_{\Gamma_{\rm J}^m} \mathbf{curl}\,\boldsymbol{G}_h \cdot \boldsymbol{n} = \sum_{n=1}^{N_I} K_n \int_{\Gamma_{\rm J}^m} \mathbf{curl}\,\boldsymbol{t}_{0,n}^h \cdot \boldsymbol{n} + \sum_{n=N_I+1}^{N} K_n^h \int_{\Gamma_{\rm J}^m} \mathbf{curl}\,\boldsymbol{t}_{0,n}^h \cdot \boldsymbol{n} = K_m^h, \quad m = N_I + 1, \ldots, N.$$

Consequently, $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{K})$ and we finish the proof. □

Taking the previous decomposition into account, we conclude that solving Problem 5.1 is equivalent to solving Problem 5.2.

Theorem 5.5. *Let $\boldsymbol{t}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$, $n = 1, \ldots, N$, satisfying (5.1)–(5.2). Let $I_n \in \mathbb{C}$, $n = 1, \ldots, N_I$, and $\Delta V_n \in \mathbb{C}$, $n = N_I + 1, \ldots, N$. If $(\boldsymbol{T}_h, \phi_h, I_{N_I+1}^h, \ldots, I_N^h)$ is a solution to Problem 5.1, then $\boldsymbol{H}_h := \widetilde{\boldsymbol{T}}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} I_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\phi_h$ solves Problem 5.2. Conversely, if $\boldsymbol{H}_h$ is the solution to Problem 5.2, then it can be written as $\boldsymbol{H}_h = \widetilde{\boldsymbol{T}}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} I_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\phi_h$, with $(\boldsymbol{T}_h, \phi_h, I_{N_I+1}^h, \ldots, I_N^h)$ being a solution to Problem 5.1.*

Proof. Let $(\boldsymbol{T}_h, \phi_h, I_{N_I+1}^h, \ldots, I_N^h)$ be a solution to Problem 5.1 and $\boldsymbol{H}_h := \widetilde{\boldsymbol{T}}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} I_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\phi_h$. According to Lemma 5.4, $\boldsymbol{H}_h \in \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$. Let $\boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0})$. Using again Lemma 5.4, we have that there exist $\boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h$, $\psi_h \in \mathcal{L}_h(\Omega)$ and $K_n^h \in \mathbb{C}$, $n = N_I + 1, \ldots, N$ such that $\boldsymbol{G}_h = \widetilde{\boldsymbol{S}}_h + \sum_{n=N_I+1}^{N} K_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\psi_h$. Then, by testing equations (5.3), (5.4) and (5.5) with $\boldsymbol{S}_h$, ψ_h and $K_{N_I+1}^h, \ldots, K_N^h$, respectively, and adding the resulting equations, it is easy to check (5.6). Thus, $\boldsymbol{H}_h$ is the solution to Problem 5.2.

Conversely, let $\boldsymbol{H}_h$ be the solution to Problem 5.2. According to Lemma 5.4, there exist $\boldsymbol{T}_h \in \boldsymbol{\mathcal{Y}}_h$, $\phi_h \in \mathcal{L}_h(\Omega)$ and $I_n^h \in \mathbb{C}$, $n = N_I + 1, \ldots, N$, such that $\boldsymbol{H}_h = \widetilde{\boldsymbol{T}}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} I_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\phi_h$. Moreover, $I_n^h = \int_{\Gamma_{\rm J}^n} \mathbf{curl}\,\boldsymbol{H}_h \cdot \boldsymbol{n}$, $n = N_I + 1, \ldots, N$. Substituting $\boldsymbol{H}_h$ by this expression in (5.6) and testing the resulting equation successively with $\boldsymbol{G}_h = \widetilde{\boldsymbol{S}}_h$ for $\boldsymbol{S}_h \in \boldsymbol{\mathcal{Y}}_h$, $\boldsymbol{G}_h = \mathbf{grad}\,\psi_h$ for $\psi_h \in \mathcal{L}_h(\Omega)$ and $\boldsymbol{G}_h = \boldsymbol{t}_{0,m}^h$, $m = N_I + 1, \ldots, N$, we obtain equations (5.3), (5.4) and (5.5), respectively. Thus $(\boldsymbol{T}_h, \phi_h, I_{N_I+1}, \ldots, I_N)$ is a solution to Problem 5.1. □

Remark 5.6. The decomposition of the solution to Problem 5.2, $\boldsymbol{H}_h = \widetilde{\boldsymbol{T}}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} I_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\phi_h$, is not unique and, therefore, Problem 5.1 is not well posed. Actually, Problem 5.1 has infinitely many solutions, all of them leading to the same approximated magnetic field $\boldsymbol{H}_h$. In order to obtain a particular solution to this problem one could use an iterative method like biconjugate gradient, which is the one that we have used in our numerical tests.

Theorem 5.7. *Let* $(\boldsymbol{T}, \phi, I_{N_I+1}, \ldots, I_N)$ *and* $(\boldsymbol{T}_h, \phi_h, I_{N_I+1}^h, \ldots, I_N^h)$ *be solutions to Problems 3.1 and 5.1, respectively. Let* $\boldsymbol{H} := \widetilde{\boldsymbol{T}} + \sum_{n=1}^{N} I_n \boldsymbol{t}_{0,n} - \mathbf{grad}\,\phi$ *and* $\boldsymbol{H}_h := \widetilde{\boldsymbol{T}}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h + \sum_{n=N_I+1}^{N} I_n^h \boldsymbol{t}_{0,n}^h - \mathbf{grad}\,\phi_h$. *If* $\boldsymbol{H}|_{\Omega_{\mathrm{C}}} \in \mathrm{H}^r(\mathbf{curl}, \Omega_{\mathrm{C}})$ *and* $\boldsymbol{H}|_{\Omega_{\mathrm{D}}} \in \mathrm{H}^r(\Omega_{\mathrm{D}})^3$ *with* $r \in \left(\frac{1}{2}, 1\right]$, *then*

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C h^r \left[\|\boldsymbol{H}\|_{\mathrm{H}^r(\mathbf{curl},\Omega_{\mathrm{C}})} + \|\boldsymbol{H}\|_{\mathrm{H}^r(\Omega_{\mathrm{D}})^3} \right],$$

where C *is a strictly positive constant independent of* h *and* $\boldsymbol{H}$.

Proof. It is a straightforward consequence of Theorem 5.3 since, according to Theorem 4.7, $\boldsymbol{H}$ is the solution to Problem 4.2 and, according to Theorem 5.5, $\boldsymbol{H}_h$ is the solution to Problem 5.2. □

6. Computation of the normalised impressed vector potentials

The aim of this section is to introduce some numerical procedures to compute the discrete normalised impressed vector potential $\boldsymbol{t}_{0,n}^h$ that do not make use of cutting surfaces.

First, by following the ideas in [12], we propose a numerical method based on the Biot−Savart law. For each $n = 1, \ldots, N$, let L_n be a polygonal filament (namely, a closed simple polygonal curve) going across Ω_{C}^n as shown in Fig. 2. We assume that $L_n \cap \Omega_{\mathrm{C}}^n$ is made of edges of thetrahedra (as in Fig. 2, again). Let $\boldsymbol{H}_{\mathrm{BS}}^n$ be the Biot−Savart field in Ω corresponding to L_n and carrying a unit current intensity:

$$\boldsymbol{H}_{\mathrm{BS}}^n(\boldsymbol{x}) := \frac{1}{4\pi} \int_{L_n} \boldsymbol{\tau}_{L_n} \times \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, d\boldsymbol{x}', \tag{6.1}$$

where $\boldsymbol{\tau}_{L_n}$ is the unit vector tangent to L_n. It is easy to check that $\boldsymbol{H}_{\mathrm{BS}}^n$ has no singularities outside inductor Ω_{C}^n, since the current filament L_n does not intersect $\overline{\Omega \setminus \Omega_{\mathrm{C}}^n}$. In fact, since the integrand is infinitely smooth outside of L_n, it is immediate to check by differentiating under the integral sign that $\boldsymbol{H}_{\mathrm{BS}}^n \in \mathcal{C}^\infty(\overline{\Omega \setminus \Omega_{\mathrm{C}}^n})^3$. Then, we can take as discrete normalised impressed vector potential $\boldsymbol{t}_{0,n}^h$ the field in $\boldsymbol{\mathcal{N}}_h(\Omega)$ with its degrees of freedom defined for each edge ℓ of the mesh $\mathcal{T}_h$ as follows:

$$\int_\ell \boldsymbol{t}_{0,n}^h \cdot \boldsymbol{\tau}_\ell := \begin{cases} \int_\ell \boldsymbol{H}_{\mathrm{BS}}^n \cdot \boldsymbol{\tau}_\ell, & \text{if } \ell \subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}, \\ 0, & \text{if } \ell \not\subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}, \end{cases} \tag{6.2}$$

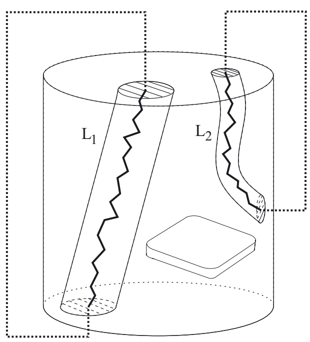


Figure 2. Current filaments for the domain in Figure 1.

where $\boldsymbol{\tau}_\ell$ is the unit vector tangent to the edge ℓ. Therefore, $\boldsymbol{t}_{0,n}^h|_{\overline{\Omega\setminus\Omega_C^n}}$ is the Nédélec interpolant of $\boldsymbol{H}_{\mathrm{BS}}^n|_{\overline{\Omega\setminus\Omega_C^n}}$. Thus, since $\mathbf{curl}\,\boldsymbol{H}_{\mathrm{BS}}^n = \mathbf{0}$ in $\overline{\Omega\setminus\Omega_C^n}$, we have that $\mathbf{curl}\,\boldsymbol{t}_{0,n}^h = \mathbf{0}$ in $\overline{\Omega\setminus\Omega_C^n}$, too. On the other hand, since $\int_{\gamma_n}\boldsymbol{H}_{\mathrm{BS}}^n\cdot\boldsymbol{\tau}_n = 1$, we also have that $\int_{\gamma_n}\boldsymbol{t}_{0,n}^h\cdot\boldsymbol{\tau}_n = \int_{\gamma_n}\boldsymbol{H}_{\mathrm{BS}}^n\cdot\boldsymbol{\tau}_n = 1$. Thus, $\boldsymbol{t}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ satisfies (5.1)–(5.2).

In order to compute (6.1) for $\boldsymbol{x}\in\overline{\Omega\setminus\Omega_C^n}$, we add the contribution of each edge ℓ lying on the current filament L_n (note that this includes tehtrahedra edges as well as the segments out of Ω added to close the curve L_n; see Fig. 2). Thus, we write

$$\boldsymbol{H}_{BS}^n(\boldsymbol{x}) = \sum_{\ell\subset L_n}\boldsymbol{H}_{BS}^{n,\ell}(\boldsymbol{x})$$

with

$$\boldsymbol{H}_{BS}^{n,\ell}(\boldsymbol{x}) := \frac{1}{4\pi}\int_\ell \boldsymbol{\tau}_{L_n}\times\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,d\boldsymbol{x}' = \int_0^1\frac{\boldsymbol{v}_\ell\times(\boldsymbol{x}-\boldsymbol{x}_{1,\ell}-s\boldsymbol{v}_\ell)}{|\boldsymbol{x}-\boldsymbol{x}_{1,\ell}-s\boldsymbol{v}_\ell|^3}ds = \frac{\boldsymbol{v}_\ell\times(\boldsymbol{x}-\boldsymbol{x}_{1,\ell})}{4\pi}\int_0^1\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_{1,\ell}-s\boldsymbol{v}_\ell|^3}ds,$$

with $\boldsymbol{x}_{1,\ell}$, $\boldsymbol{x}_{2,\ell}$ the end-points of the edge ℓ and $\boldsymbol{v}_\ell := \boldsymbol{x}_{2,\ell}-\boldsymbol{x}_{1,\ell}$. If we denote by L_ℓ the straight line in $\mathbb{R}^3$ containing the edge ℓ, we notice that the integrand in the above expression is ill defined if $\boldsymbol{x}\in\ell$ and that $\boldsymbol{H}_{BS}^{n,\ell}(\boldsymbol{x}) = \mathbf{0}$ for every $\boldsymbol{x}\in L_\ell\setminus\ell$. Let us define

$$\boldsymbol{a}_1 := \boldsymbol{x}-\boldsymbol{x}_{1,\ell} \qquad \text{and} \qquad \boldsymbol{a}_2 := \boldsymbol{x}-\boldsymbol{x}_{2,\ell}.$$

Then, it can be shown that the integral in the previous expression reduces to:

$$\boldsymbol{H}_{BS}^{n,\ell}(\boldsymbol{x}) = \begin{cases} \dfrac{(\boldsymbol{a}_1-\boldsymbol{a}_2)\times\boldsymbol{a}_1}{4\pi}\,\dfrac{\boldsymbol{a}_2\cdot(\boldsymbol{a}_1-\boldsymbol{a}_2)\,|\boldsymbol{a}_1| - \boldsymbol{a}_1\cdot(\boldsymbol{a}_1-\boldsymbol{a}_2)\,|\boldsymbol{a}_2|}{|\boldsymbol{a}_1|\,|\boldsymbol{a}_2|\,|\boldsymbol{a}_1\times\boldsymbol{a}_2|^2}, & \text{if } \boldsymbol{x}\notin L_\ell, \\ \mathbf{0}, & \text{if } \boldsymbol{x}\in L_\ell\setminus\ell. \end{cases} \tag{6.3}$$

Remark 6.1. The above formula was developed following the ideas proposed by Urankar in [22], where he establishes an expression for the Biot−Savart field created by a straight current filament oriented in the $\boldsymbol{e}_z$ direction. Other alternatives can be found, for example, in [17] and the references therein.

Even though (6.3) are analytical expressions to evaluate the integrals in (6.1), we compute numerically the integrals on the right hand side of (6.2) by means of the mid-point quadrature rule. In the next theorem we prove that the errors that arise from this numerical quadrature do not spoil the rate of convergence of the method in the case where all sources are given in terms of the current intensities.

Let $\widehat{\boldsymbol{t}}_{0,n}^h$ be the approximate discrete normalised impressed vector potential, obtained by using the mid-point rule for computing the integrals in (6.2); namely, $\widehat{\boldsymbol{t}}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ and

$$\int_\ell \widehat{\boldsymbol{t}}_{0,n}^h\cdot\boldsymbol{\tau}_\ell := \begin{cases} \left(\boldsymbol{H}_{\mathrm{BS}}^n(\boldsymbol{x}_\ell)\cdot\boldsymbol{\tau}_\ell\right)|\ell|, & \text{if } \ell\subset\overline{\Omega\setminus\Omega_C^n}, \\ 0, & \text{if } \ell\not\subset\overline{\Omega\setminus\Omega_C^n}, \end{cases} \tag{6.4}$$

where $|\ell|$ denotes the length of the edge ℓ and $\boldsymbol{x}_\ell$ is its middle point. When $\widehat{\boldsymbol{t}}_{0,n}^h$ are used instead of $\boldsymbol{t}_{0,n}^h$ in Problem 5.1, we obtain an approximate discrete solution $(\widehat{\boldsymbol{T}}_h,\widehat{\phi}_h)$ instead of $(\boldsymbol{T}_h,\phi_h)$, from which we compute the approximate discrete magnetic field $\widehat{\boldsymbol{H}}_h := \widetilde{\widehat{\boldsymbol{T}}}_h + \sum_{n=1}^{N_I} I_n\widehat{\boldsymbol{t}}_{0,n}^h - \mathbf{grad}\,\widehat{\phi}_h$.

The following result shows that using the computed values $\widehat{\boldsymbol{H}}_h$ instead of the exact ones $\boldsymbol{H}_h$ does not deteriorate the order of convergence.

Theorem 6.2. *Let $\boldsymbol{H}_h$ and $\widehat{\boldsymbol{H}}_h$ be as defined above. Then, there exists a constant $C>0$ such that*

$$\left\|\boldsymbol{H}_h-\widehat{\boldsymbol{H}}_h\right\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le Ch.$$

Proof. As shown in Theorem 5.5, $\boldsymbol{H}_h$ satisfies

$$\int_{\Omega} i\omega\mu \boldsymbol{H}_h \cdot \overline{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{H}_h \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{G}}_h = 0 \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0}). \tag{6.5}$$

Notice that $\widehat{\boldsymbol{t}}_{0,n}^h \in \boldsymbol{\mathcal{N}}_h(\Omega)$ satisfy (5.2) but, in general,

$$\int_{\Gamma_{\mathrm{J}}^n} \operatorname{\mathbf{curl}} \widehat{\boldsymbol{t}}_{0,n}^h \cdot \boldsymbol{n} \neq 1.$$

As a consequence, $\widehat{\boldsymbol{H}}_h$ is not a solution to Problem 5.2 because, in general, $\widehat{\boldsymbol{H}}_h \notin \boldsymbol{\mathcal{V}}_h(\boldsymbol{I})$. However, the same arguments used in the proof of Theorem 5.5 allow us to show that $\widehat{\boldsymbol{H}}_h$ satisfies the same equation:

$$\int_{\Omega} i\omega\mu \widehat{\boldsymbol{H}}_h \cdot \overline{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \widehat{\boldsymbol{H}}_h \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{G}}_h = 0 \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0}). \tag{6.6}$$

Let $\boldsymbol{F}_h := \widetilde{\boldsymbol{T}}_h - \operatorname{\mathbf{grad}} \phi_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0})$ and $\widehat{\boldsymbol{F}}_h := \widehat{\widetilde{\boldsymbol{T}}}_h - \operatorname{\mathbf{grad}} \widehat{\phi}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0})$. Then, $\boldsymbol{H}_h = \boldsymbol{F}_h + \sum_{n=1}^{N_I} I_n \boldsymbol{t}_{0,n}^h$ and $\widehat{\boldsymbol{H}}_h = \widehat{\boldsymbol{F}}_h + \sum_{n=1}^{N_I} I_n \widehat{\boldsymbol{t}}_{0,n}^h$. Substituting these expressions into (6.5) and (6.6) and subtracting we obtain

$$\int_{\Omega} i\omega\mu \Delta\boldsymbol{F}_h \cdot \overline{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \Delta\boldsymbol{F}_h \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{G}}_h$$
$$+ \sum_{n=1}^{N_I} I_n \left(\int_{\Omega_{\mathrm{C}}} i\omega\mu \Delta\boldsymbol{t}_{0,n}^h \cdot \overline{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \Delta\boldsymbol{t}_{0,n}^h \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{G}}_h \right) = 0 \quad \forall \boldsymbol{G}_h \in \boldsymbol{\mathcal{V}}_h(\mathbf{0}), \tag{6.7}$$

where $\Delta\boldsymbol{F}_h := \boldsymbol{F}_h - \widehat{\boldsymbol{F}}_h$ and $\Delta\boldsymbol{t}_{0,n}^h := \boldsymbol{t}_{0,n}^h - \widehat{\boldsymbol{t}}_{0,n}^h$. Since $a(\boldsymbol{F}_h, \boldsymbol{G}_h) := \int_{\Omega} i\omega\mu \boldsymbol{F}_h \cdot \overline{\boldsymbol{G}}_h + \int_{\Omega_{\mathrm{C}}} \frac{1}{\sigma} \operatorname{\mathbf{curl}} \boldsymbol{F}_h \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{G}}_h$ is a continuous and elliptic bilinear form in $\boldsymbol{\mathcal{X}}_h \times \boldsymbol{\mathcal{X}}_h$ (see [10]), by taking $\boldsymbol{G}_h = \Delta\boldsymbol{F}_h$, we obtain

$$||\Delta\boldsymbol{F}_h||^2_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C\, a(\Delta\boldsymbol{F}_h, \Delta\boldsymbol{F}_h) \leq C \sum_{n=1}^{N_I} |I_n|\; ||\Delta\boldsymbol{F}_h||_{\mathrm{H}(\mathbf{curl};\Omega)}\; ||\Delta\boldsymbol{t}_{0,n}^h||_{\mathrm{H}(\mathbf{curl};\Omega)},$$

and, then,

$$\left\|\boldsymbol{H}_h - \widehat{\boldsymbol{H}}_h\right\|_{\mathrm{H}(\mathbf{curl};\Omega)} \leq ||\Delta\boldsymbol{F}_h||_{\mathrm{H}(\mathbf{curl};\Omega)} + \sum_{n=1}^{N_I} |I_n|\; ||\Delta\boldsymbol{t}_{0,n}^h||_{\mathrm{H}(\mathbf{curl};\Omega)} \leq C \sum_{n=1}^{N_I} |I_n|\; ||\Delta\boldsymbol{t}_{0,n}^h||_{\mathrm{H}(\mathbf{curl};\Omega)}.$$

Let $\boldsymbol{\phi}_\ell$ be the basis function of the lowest-order Nédélec finite element space $\boldsymbol{\mathcal{N}}_h(\Omega)$ corresponding to the edge ℓ. Then, $\boldsymbol{t}_{0,n}^h = \sum_{\ell \subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}} \left(\int_\ell \boldsymbol{H}_{\mathrm{BS}}^n \cdot \boldsymbol{\tau}_{\boldsymbol{\ell}} \right) \boldsymbol{\phi}_\ell$ and $\widehat{\boldsymbol{t}}_{0,n}^h = \sum_{\ell \subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}} \left(\boldsymbol{H}_{\mathrm{BS}}^n(\boldsymbol{x}_\ell) \cdot \boldsymbol{\tau}_\ell |\ell| \right) \boldsymbol{\phi}_\ell$, $n = 1, \ldots, N$. Consequently, using the classical error formula for the mid-point rule leads to

$$||\Delta\boldsymbol{t}_{0,n}^h||_{\mathrm{L}^2(\Omega)^3} \leq \sum_{\ell \subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}} \left| \int_\ell (\boldsymbol{H}_{\mathrm{BS}} - \boldsymbol{H}_{\mathrm{BS}}(\boldsymbol{x}_\ell)) \cdot \boldsymbol{\tau}_\ell \right| ||\boldsymbol{\phi}_\ell||_{\mathrm{L}^2(\Omega)^3} \leq \sum_{\ell \subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}} \frac{||\boldsymbol{H}_{\mathrm{BS}} \cdot \boldsymbol{\tau}_\ell||_{\mathrm{W}^{2,\infty}(\ell)} |\ell|^3}{24} ||\boldsymbol{\phi}_\ell||_{\mathrm{L}^2(\Omega)^3}$$

and, analogously,

$$||\operatorname{\mathbf{curl}} \Delta\boldsymbol{t}_{0,n}^h||_{\mathrm{L}^2(\Omega)^3} \leq \sum_{\ell \subset \overline{\Omega \setminus \Omega_{\mathrm{C}}^n}} \frac{||\boldsymbol{H}_{\mathrm{BS}} \cdot \boldsymbol{\tau}_\ell||_{\mathrm{W}^{2,\infty}(\ell)} |\ell|^3}{24} ||\operatorname{\mathbf{curl}} \boldsymbol{\phi}_\ell||_{\mathrm{L}^2(\Omega)^3},$$

Now, scaling arguments (see, for instance, [21]) and the regularity of the meshes lead to

$$||\boldsymbol{\phi}_\ell||_{\mathrm{L}^2(\Omega)^3} \le \frac{C}{|\ell|} \qquad \text{and} \qquad ||\,\mathbf{curl}\,\boldsymbol{\phi}_\ell||_{\mathrm{L}^2(\Omega)^3} \le \frac{C}{|\ell|^2}.$$

Therefore,

$$||\Delta \boldsymbol{t}^h_{0,n}||_{\mathrm{L}^2(\Omega)^3} \le Ch^2 \qquad \text{and} \qquad ||\,\mathbf{curl}\,\Delta \boldsymbol{t}^h_{0,n}||_{\mathrm{L}^2(\Omega)^3} \le Ch,$$

where C is a strictly positive constant independent of h. Then,

$$\left\|\boldsymbol{H}_h - \widehat{\boldsymbol{H}}_h\right\|_{\mathrm{H}(\mathbf{curl};\Omega)} \le Ch. \qquad \square$$

For problems in which the potential drop is given as source data instead of the current intensity, the proof above is no longer valid. However, we have numerically checked that this procedure for approximating the discrete normalised impressed vector potentials does not spoil the convergence rate.

Alternatively, the procedure introduced in [2] to construct the so-called loop fields allows for constructing a normalised impressed vector potential that exactly meets conditions (5.1)–(5.2). This algorithm, like the previous one, does not make use of cutting surfaces, but is slightly more involved as it requires the use of some graph theory concepts. For completeness, we include here a brief description of the construction of a normalised impressed vector potential based on the one appearing in [2] and refer to this paper for further details.

Let us denote by V and E the set of vertices and edges of the mesh $\mathcal{T}_h(\overline{\Omega \setminus \Omega^n_{\mathrm{C}}}) := \left\{T \in \mathcal{T}_h :\ T \subset \overline{\Omega \setminus \Omega^n_{\mathrm{C}}}\right\}$, respectively. Moreover, let $\mathcal{S}_h = (V, L)$ be a spanning tree of the graph (V, E) (that is, a subgraph of (V, E) that includes all of the vertices and for which every two vertices are connected by exactly one path) and let v_1 be one of its vertices. Then, given a vertex $v \in V$, there exists a unique path C that connects v_1 to v. Furthermore, given a path C_v, let us denote by $-C_v$ the path that connects v to v_1. Finally, given an edge $e \in E$, with extremities $v_{e,1}$ and $v_{e,2}$,we define $D_e := C_{v_{e,1}} + e - C_{v_{e,2}}$. The Nédélec degrees of freedom of the normalised impressed vector potential can be computed as follows:

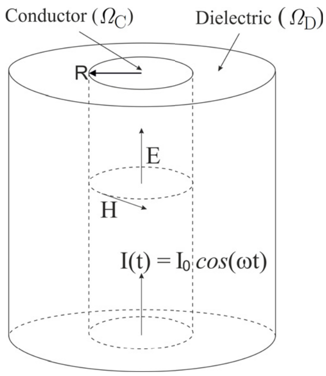
$$\int_\ell \boldsymbol{t}^h_{0,n} \cdot \boldsymbol{\tau}_\ell := \begin{cases} \mathrm{lk}(D_\ell, L_n), & \text{if } \ell \subset \overline{\Omega \setminus \Omega^n_{\mathrm{C}}}, \\ 0, & \text{if } \ell \not\subset \overline{\Omega \setminus \Omega^n_{\mathrm{C}}}, \end{cases}$$

where $\mathrm{lk}(D_\ell, L_n)$ is the so-called linking number of the oriented curves D_e and L_n. To compute this linking number we have used the algorithm described in [6]. As shown in [2], the field $\boldsymbol{t}^h_{0,n} \in \boldsymbol{\mathcal{N}}_h(\Omega)$ computed in this way satisfies (5.1)–(5.2).

7. Numerical results

In this section we report some numerical results obtained with a Matlab code which implements the numerical methods described above. In particular, since the linear system matrix arising from Problem 5.1 is complex non-Hermitian, we have used the command `bicg` to solve it, with a tolerance threshold of 10^{-8}. Let us remark that when this command is applied to an underdetermined linear system with a singular square matrix, it yields one particular solution. (Let us recall that obtaining one particular solution is enough, since any solution of this problem leads to the same magnetic field.)

First, we will study the convergence of the method by using an example with known analytical solution and only one connected component in the conducting part. Next, we will report the results for a domain with several connected components in the conducting domain.



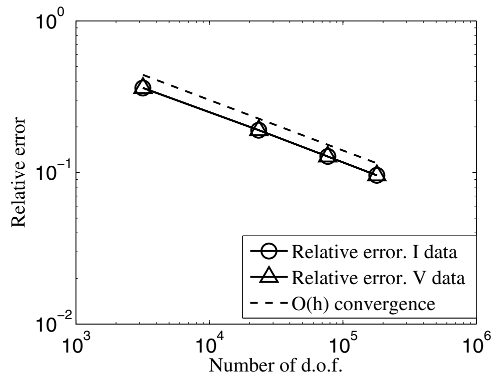


FIGURE 3. Infinite cylinder carrying an alternating current (*left*). Convergence order in $\mathrm{H}(\mathbf{curl};\Omega)$ (*right*).

7.1. Test with analytical solution

In this section we report the numerical results obtained for an academic test that confirm the results stated in the previous sections and the convergence of the proposed methodology.

We take as conducting domain a section of height $L = 0.5$ m of an infinite cylinder with radius $R = 0.5$ m, as shown in Figure 3(*left*). This cylinder is composed by a conducting material with electric conductivity $\sigma = 151\,565.8\ (\Omega\mathrm{m})^{-1}$ and magnetic permeability $\mu = \mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm}^{-1}$, and it carries an alternating current $I(t) = I_0 \cos(\omega t)$, where $I_0 = 10^4$ A and $\omega = 2\pi f$, with $f = 50$ Hz, surrounded by dielectric material having an outer radius $R_\infty = 1$ m. We can obtain the analytical solution to the associated eddy current problem, which is (see [4], Sect. 8.1.5):

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{cases} \dfrac{I_0 \mathcal{I}_1(\sqrt{i\omega\mu\sigma}\rho)}{2\pi R \mathcal{I}_1(\sqrt{i\omega\mu\sigma}R)}\boldsymbol{e}_\theta, & \text{if} \quad \rho \leq R, \\[2ex] \dfrac{I_0}{2\pi\rho}\boldsymbol{e}_\theta, & \text{if} \quad \rho > R, \end{cases}$$

where $\mathcal{I}_1$ is the modified Bessel function of the first kind and order 1, and $\rho = \sqrt{x_1^2 + x_2^2}$ and $\boldsymbol{e}_\theta := (-x_2, x_1, 0)/\rho$ are the radial coordinate and the angular unit vector in cylindrical coordinates, respectively. Notice that the solution to the problem does not depend on the values of L or R_∞ because the magnetic field $\boldsymbol{H}$ is independent of the z-coordinate and it exactly satisfies the boundary condition (2.6).

When comparing the numerical solution obtained from Problem 5.1 with the exact one, we obtain the error curves in Figure 3(*right*), which show that an order of convergence $O(h)$ is clearly attained in this case, in agreement with the theoretical results. This test has been separately performed with current and potential drop as source data; the solutions showed the expected order of convergence with very similar relative errors with respect to the analytical solution, as it is deduced from the overlapping of the error curves in Figure 3(*right*). In particular, for the case in which the potential drop is given, we have used the analytical expression (see again [4], Sect. 8.1.5):

$$\mathit{\Delta} V = \frac{\sqrt{i\omega\mu\sigma}LI_0}{2\pi\sigma R}\frac{\mathcal{I}_0(\sqrt{i\omega\mu\sigma}R)}{\mathcal{I}_1(\sqrt{i\omega\mu\sigma}R)} + i\omega\mu\frac{LI_0}{2\pi}\log\left(\frac{R_\infty}{R}\right),$$

where $\mathcal{I}_0$ is the modified Bessel function of the first kind and order 0.

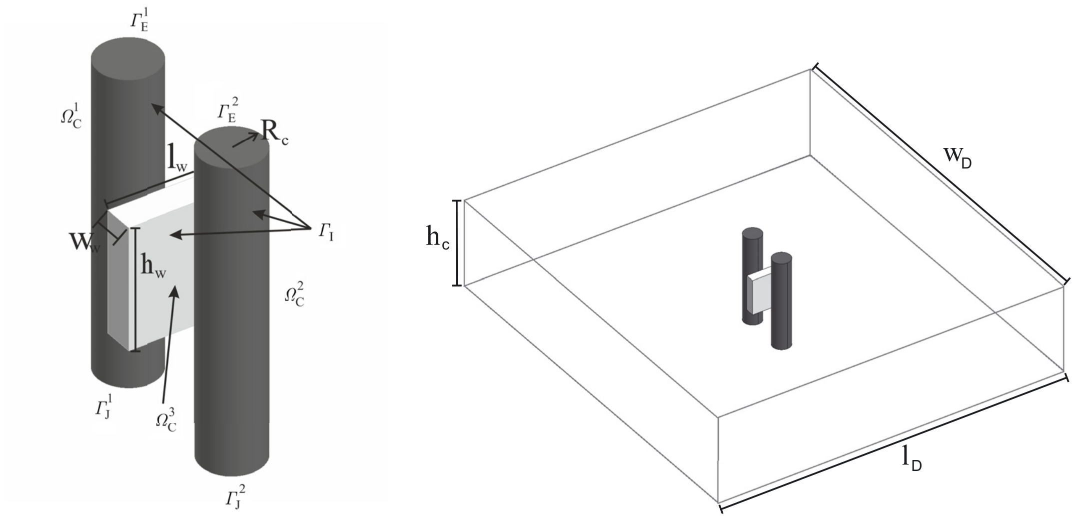


FIGURE 4. Geometrical setting for the conducting domain (*left*) and for the whole computational domain (*right*).

7.2. Example with several connected components in the conductor

In this section we report the results of a numerical test with a topological setting closer to that appearing in applications, where the conducting domain usually comprises several connected components. Indeed, the proposed geometry consists of two cylindrical inductors, Ω_C^1 and Ω_C^2, with radius $R_c = 0.05$ m and height $h_c = 0.5$ m, and a parallelepipedic workpiece, Ω_C^3, with dimensions $h_w = 0.2$ m, $l_w = 0.2$ m and $w_w = 0.05$ m (see Fig. 4). The distance between each inductor and the workpiece is 0.05 m. Both inductors have electrical conductivity $\sigma = 100\ (\Omega\mathrm{m})^{-1}$ and vacuum magnetic permeability μ_0, while the workpiece has electrical conductivity $\sigma = 10^6\ (\Omega\mathrm{m})^{-1}$ and vacuum magnetic permeability μ_0. The inductors are connected to an external current source and surrounded by a box of dielectric material of dimensions $w_D = 2.2$ m and $l_D = 2$ m (and with the same height as the inductors). The main goal is to compute the current density induced within the workpiece. The currents entering the inductors are $I_1 = I_2 = 10^3$ A, the frequency of the problem being $f = 50$ Hz.

In Figure 5 we show the complex modulus of the current density in the workpiece. Notice that the mesh was generated taking into account the "skin effect", which usually appears in this kind of problems, making the current density in the workpiece to be highly concentrated near its boundary. This can be clearly seen in Figure 5.

In order to validate the method in this topological setting, we have compared the magnetic field obtained from the solution to Problem 5.1 and the solution to the same problem obtained with the commercial software Altair Flux®, which makes use of another variant of the $\boldsymbol{T}$,ϕ–ϕ with second order elements. This comparison is presented in terms of the active power in each connected component of the conducting domain and the potential drop in the inductors (see Tab. 1). Moreover, in Figure 6, we compare the value of the current density modulus along straight lines in the workpiece Ω_C^3 passing through the center of the piece in the z- and y-directions.

As it can be seen in Figure 6, the current density presents an important rate of change both along the z- and the y-directions in the workpiece. Anyway, the agreement between the values of the density computed with both methods can be clearly seen from this figure. Let us remark that this agreement is even better in the inductors, as is suggested by Table 1.

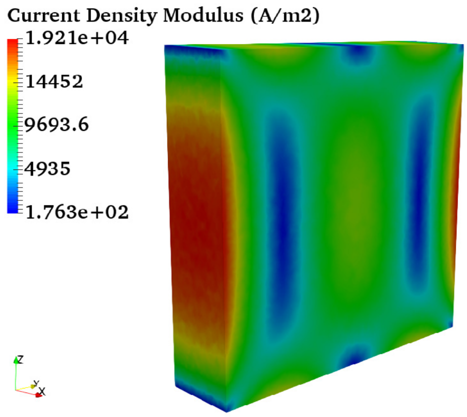


FIGURE 5. Current density modulus ($\mathrm{A/m^2}$) in the workpiece Ω^3_{C}.

TABLE 1. Active power and potential drop comparison.

		Problem 5.1	Flux	Relative Error
Active Power	Ω^1_{C}	3.2105e5 W	3.1924e5 W	0.5666%
$\displaystyle\int_{\Omega^j_{\mathrm{C}}} \frac{\lvert \boldsymbol{J} \rvert^2}{2\sigma},\ j=1,2,3$	Ω^2_{C}	3.2105e5 W	3.1955e5 W	0.4691%
	Ω^3_{C}	0.0685 W	0.0632 W	6.6718%
Potential Drop	ΔV_1	$642.1028 + 0.1574i$ V	$636.6797 + 0.1528i$ V	0.5666%
	ΔV_2	$642.1005 + 0.1572i$ V	$636.6760 + 0.1526i$ V	0.4691%

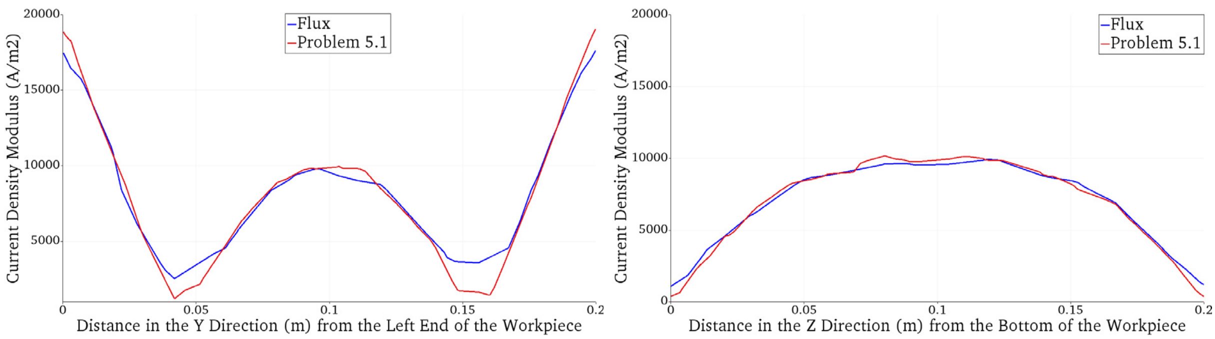


FIGURE 6. Current density modulus ($\mathrm{A/m^2}$) comparison in the workpiece Ω^3_{C} along different directions.

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