# SPACETIME DISCONTINUOUS GALERKIN METHODS FOR SOLVING CONVECTION-DIFFUSION SYSTEMS\*

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Abstract. In this paper, we present two new approaches for solving systems of hyperbolic conservation laws with correct physical viscosity and heat conduction terms such as the compressible Navier–Stokes equations. Our methods are extensions of the spacetime discontinuous Galerkin method for hyperbolic conservation laws developed by Hiltebrand and Mishra [26]. Following this work, we use entropy variables as degrees of freedom and entropy stable fluxes. For the discretization of the diffusion term, we consider two different approaches: the interior penalty approach, resulting in the ST-SIPG and the ST-NIPG method, and a variant of the local discontinuous Galerkin method, resulting in the ST-LDG method. We show entropy stability of the ST-NIPG and the ST-LDG method when applied to the compressible Navier–Stokes equations. For the ST-SIPG method, this result holds under an assumption on the computed solution. All schemes incorporate shock capturing terms. Therefore, the schemes can handle both regimes of underresolved and fully resolved physical diffusion. We present a numerical comparison of the three methods in one space dimension.

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### 1. INTRODUCTION

In this paper, we present new methods for solving convection-diffusion systems. We focus on problems that can be written as systems of hyperbolic conservation laws with physical diffusion terms. In particular, we are interested in solving the compressible Navier–Stokes equations. Our goal is to develop new schemes that satisfy theoretical stability estimates and that are robust and accurate. Therefore, we start with an existing method for conservation laws that has this property and extend it to convection-diffusion systems using two different approaches.

In [26], Hiltebrand and Mishra developed a spacetime (ST) discontinuous Galerkin (DG) method for solving systems of conservation laws based on using entropy variables instead of the standard conserved variables. The scheme uses entropy stable fluxes and features a streamline diffusion and a shock capturing term to handle the shocks and discontinuities occurring in the solution of the system. The scheme is (arbitrarily) high order

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in smooth flow, robust in the presence of shocks, and one can show *a priori* entropy stability estimates for the fully discrete scheme for systems of conservation laws.

In order to extend this scheme to convection-diffusion systems, we need to find a suitable treatment for the viscous terms. We consider two different approaches:

- (1) an approach based on the interior penalty (IP) method introduced by Arnold [1] resulting in the ST-SIPG and the ST-NIPG method;
- (2) an approach in the spirit of the local discontinuous Galerkin (LDG) method introduced by Cockburn and Shu [9] resulting in the ST-LDG method.

For all three methods, we will deduce an entropy stability estimate for convection-diffusion systems. In particular, our result will imply entropy stability of the ST-NIPG and the ST-LDG method for solving the compressible Navier–Stokes equations. Under the assumption of boundedness of the computed solutions, we can also show entropy stability of the ST-SIPG method for solving the compressible Navier–Stokes equations.

Furthermore, we like our methods to be robust for convection-dominated problems. The presence of the physical diffusion term adds a certain level of stability to the system compared to considering the pure system of conservation laws. Therefore, the use of the streamline diffusion and shock capturing terms that were an essential part of the method developed in [26] might seem unnecessary. However, if the physical diffusion is small, then – depending on the grid size used – the physical diffusion might be underresolved. In this case, the system will behave similar to the corresponding system of conservation laws (without physical diffusion terms). In particular, there will probably be oscillations around shocks and discontinuities if no discontinuity capturing operators are used. Especially in higher dimensions this will be a problem as it typically will not be possible to fully resolve all areas of the flow domain. Therefore, we incorporate the streamline diffusion and shock capturing terms used in [26] in our new methods and adjust them appropriately. This guarantees stability of our methods for a wide range of grid sizes.

Often the presence of shock capturing terms results in methods being only first- or second-order accurate, even for smooth flow. We note that for the artificial diffusion terms used in our new schemes this is not the case. For the ST-LDG method, numerical results indicate for smooth flow an ideal convergence rate of  $O(h^{k+1})$  for polynomials of degree  $k \ge 0$  both with and without using the artificial diffusion terms. (For accuracy reasons, we couple the time step length to the mesh width h, even though this is not necessary in terms of stability.) For the ST-SIPG and the ST-NIPG method we observe the classical convergence stalling when using piecewise constant polynomials and a reduced convergence order of  $O(h^k)$  for the ST-NIPG method for k even. Otherwise we observe the ideal convergence rate of  $O(h^{k+1})$  for ST-SIPG and ST-NIPG, when not using the artificial viscosity terms. For using the artificial viscosity terms, we have two sets of results depending on the choice of a parameter in the artificial viscosity terms: for the stronger version, which also works well in the vicinity of strong shocks, we observe the suboptimal convergence rate of  $O(h^k)$  for the ST-SIPG and the ST-NIPG method for  $k \ge 1$ . For a weaker version, which is more suitable for better behaved flow, we observe essentially the same convergence rates as for not using artificial viscosity terms. Therefore, the new methods combine stability in an underresolved regime with good accuracy in smooth flow.

Finally, we note that our schemes are based on using entropy variables. This makes the methods more complicated to implement and somewhat more expensive due to the additional change of variables. In return, we can show a stability result (entropy stability) for the actual problem we like to solve (the compressible Navier–Stokes equations), whereas most other methods only come with theoretical results for simplified model problems.

In the literature, there are various DG methods for solving the compressible Navier–Stokes equations that are based on discretizing the conserved variables of the system, see, *e.g.*, [4-13, 17, 22, 23, 30, 35] and the references cited therein. Most methods use either a variant of the IP or of the LDG approach for discretizing the diffusion term. However, to the best of our knowledge, theoretical stability results (if available) always concern the case of a model problem that has an elliptic diffusion operator, and not the actual compressible Navier–Stokes equations. A unified comparison of several of these methods for the case of elliptic operators can be found

in [2]. Also, many of these methods do not include shock capturing terms that both guarantee stability in an underresolved regime and allow for high-order accuracy.

Though significantly less extensive, there is also some literature for solving the compressible Navier–Stokes equations using entropy variables as degrees of freedom. Early work has been done by Shakib *et al.* [37]. The authors use a finite element formulation that also incorporates a discontinuity capturing term and show entropy stability for their method. However, the authors use continuous elements in space. By using discontinuous elements in space, we guarantee local mass conservation, which is especially important in the underresolved regime, for which shocks may occur. The possibility of choosing suitable numerical fluxes (which is not the case for continuous elements in space) is very favorable in this scenario as well. Barth [3] includes the compressible Navier–Stokes equations in his theoretical considerations and examines DG methods for solving the Euler equations, but he neither suggests a discretization for the physical diffusion term nor shows numerical results for the compressible Navier–Stokes equations. Tadmor and Zhong [39] have developed a difference scheme based on entropy variables. The scheme uses entropy conservative fluxes for the nonlinear term and centered differences for the discretization of the dissipation term. The authors show entropy stability for the semi-discrete form. However, their method does not include shock capturing terms and is at most second-order accurate.

This paper is structured as follows: we start with a review of the original method for conservation laws [26] in Section 2. Then, in Section 3, we discuss the effect of switching to entropy variables on the diffusion matrix. In Sections 4 and 5, we present our two different extensions, the ST-SIPG and the ST-NIPG method, and the ST-LDG method, and prove entropy stability estimates under suitable assumptions. In Section 6, we show numerical results in one space dimension comparing all three methods. In particular, we solve the compressible Navier–Stokes equations. We conclude this work with a comparison of the three methods in Section 7.

### 2. Review of the spacetime DG method for hyperbolic systems

In this section, we review the spacetime DG method developed for hyperbolic conservation laws [26] that our method is based on. We focus on systems of hyperbolic conservation laws in one space dimension given by

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0. \tag{2.1}$$

Here,  $\mathbf{U}: \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$ ,  $\Omega \subset \mathbb{R}$ ,  $m \in \mathbb{N}$ , is the vector of conserved variables and  $\mathbf{F}$  is the flux vector. We use the short-hand notation  $\mathbf{U}_t = \partial_t \mathbf{U}$  and  $\mathbf{F}(\mathbf{U})_x = \partial_x \mathbf{F}(\mathbf{U})$ . As the system is assumed to be hyperbolic, the Jacobian matrix  $\mathbf{F}_{\mathbf{U}} := D\mathbf{F}(\mathbf{U})$  has m real eigenvalues  $\zeta_1(\mathbf{U}) \leq \ldots \leq \zeta_m(\mathbf{U})$  together with a basis of right eigenvectors  $\{r_j(\mathbf{U})\}_{j=1,\ldots,m}$ . We also assume the existence of an entropy pair (S, Q) with strictly convex entropy function S. The weak solution  $\mathbf{U}$  of (2.1) is said to be the entropy solution if it satisfies the following entropy inequality in the sense of distributions

$$S(\mathbf{U})_t + Q(\mathbf{U})_x \le 0.$$

We define the *entropy variables*  $\mathbf{V} = S_{\mathbf{U}}(\mathbf{U}) := \nabla S(\mathbf{U})$ . This is a one-to-one change of variables since S is strictly convex. Therefore, we can equivalently write the system (2.1) as

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x = 0 \tag{2.2}$$

with  $\mathbf{F}(\mathbf{V}) = \mathbf{F}(\mathbf{U}(\mathbf{V}))$  for simplicity. For more information, see LeFloch [31].

To discretize, we consider a spacetime grid with each spacetime element being a tensor-product of a spatial grid cell  $K_i = [x_{i-1/2}, x_{i+1/2}] \subset \Omega$  and a time segment  $I_n = [t_n, t_{n+1}] \subset [0, T]$ . For simplicity, we consider equidistant spatial cells with length h. Then, approximations  $\mathbf{V}^h = (v_1^h, \ldots, v_m^h)^T$  (we will use the superscript h when referring to discrete quantities) to the solution  $\mathbf{V}$  are sought in the space

$$\mathbf{V}^{h} \in \mathcal{V}^{k} = \left\{ \left. \mathbf{\Phi}^{h} \in (L^{1}(\Omega \times [0,T]))^{m} : \begin{array}{c} \left. \mathbf{\Phi}^{h} \right|_{K_{i} \times I_{n}} \text{ is a polynomial} \\ \text{ of degree } k \text{ in each component} \end{array} \right\}$$

Multiplying (2.2) with test functions  $\Phi^h \in \mathcal{V}^k$ , integrating in space and time, and doing integration by parts with using numerical fluxes where appropriate results in

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{h}, \mathbf{\Phi}^{h}) = -\sum_{n,i} \int_{I_{n}} \int_{K_{i}} \left( \mathbf{U}(\mathbf{V}^{h}) \cdot \mathbf{\Phi}^{h}_{t} + \mathbf{F}(\mathbf{V}^{h}) \cdot \mathbf{\Phi}^{h}_{x} \right) \, \mathrm{d}x \, \mathrm{d}t \\ + \sum_{n,i} \int_{K_{i}} \left( \mathbb{U}(\mathbf{V}^{h}_{n+1,-}, \mathbf{V}^{h}_{n+1,+}) \cdot \mathbf{\Phi}^{h}_{n+1,-} - \mathbb{U}(\mathbf{V}^{h}_{n,-}, \mathbf{V}^{h}_{n,+}) \cdot \mathbf{\Phi}^{h}_{n,+} \right) \, \mathrm{d}x \\ + \sum_{n,i} \int_{I_{n}} \left( \mathbb{F}(\mathbf{V}^{h}_{i+1/2,L}, \mathbf{V}^{h}_{i+1/2,R}) \cdot \mathbf{\Phi}^{h}_{i+1/2,L} - \mathbb{F}(\mathbf{V}^{h}_{i-1/2,L}, \mathbf{V}^{h}_{i-1/2,R}) \cdot \mathbf{\Phi}^{h}_{i-1/2,R} \right) \, \mathrm{d}t.$$
(2.3)

Here,  $\mathbf{a} \cdot \mathbf{b}$  denotes the standard scalar product  $\mathbf{a} \cdot \mathbf{b} = \sum_{j=1}^{m} a_j b_j$  and the indices +/- and R/L denote the following limits (with  $\varepsilon > 0$ )

$$v_{n+1,\pm}^h(x) = \lim_{\varepsilon \to 0} v^h(x, t_{n+1} \pm \varepsilon), \quad v_{i+1/2, R/L}^h(t) = \lim_{\varepsilon \to 0} v^h(x_{i+1/2} \pm \varepsilon, t).$$

Furthermore,  $\mathbb{U}$  and  $\mathbb{F}$  denote the fluxes in time and space. In order to enable proper time marching, the upwind flux is chosen in time, *i.e.*,

$$\mathbb{U}(\mathbf{V}_{n+1,-}^h,\mathbf{V}_{n+1,+}^h)=\mathbf{U}(\mathbf{V}_{n+1,-}^h).$$

For the numerical flux in space either an entropy conservative flux or an entropy stable flux is used. Entropy conservative fluxes  $\mathbb{F}^*$  have been examined in [38] and satisfy

$$(b-a) \cdot \mathbb{F}^*(a,b) = \psi(b) - \psi(a) \tag{2.4}$$

with  $\psi = \mathbf{V} \cdot \mathbf{F} - Q$  being the entropy potential and (S,Q) being an entropy pair. (We will specify the entropy conservative fluxes that we use for numerical tests in the corresponding parts of Sect. 6).

Entropy stable fluxes are created out of entropy conservative fluxes by adding a diffusion term

$$\mathbb{F}(\mathbf{V}_{a}^{h},\mathbf{V}_{b}^{h}) = \mathbb{F}^{*}(\mathbf{V}_{a}^{h},\mathbf{V}_{b}^{h}) - \frac{1}{2}\mathbb{D}(\mathbf{V}_{a}^{h},\mathbf{V}_{b}^{h})(\mathbf{V}_{b}^{h} - \mathbf{V}_{a}^{h})$$
(2.5)

with

$$\mathbb{D}(\mathbf{a}, \mathbf{b}) = \mathbf{R} \mathbf{P}(\mathbf{Z}; \mathbf{a}, \mathbf{b}) \mathbf{R}^T.$$

Here Z,  $\mathbf{R}$  are the (real) eigenvalue and eigenvector matrices of the Jacobian matrix  $\mathbf{F}_{\mathbf{U}}$  with  $\mathbf{R}$  being scaled such that  $\mathbf{R}\mathbf{R}^T = \mathbf{U}_{\mathbf{V}}$  with  $\mathbf{U}_{\mathbf{V}} := D\mathbf{U}(\mathbf{V})$ . In this work, we use the Rusanov diffusion operator given by

$$\mathbf{P}(\mathbf{Z}; \mathbf{a}, \mathbf{b}) = \max \{\zeta_{\max}(\mathbf{a}), \zeta_{\max}(\mathbf{b})\} \mathbf{I}_m$$

with  $\zeta_{\max}(\mathbf{U}) = \max(|\zeta_1(\mathbf{U})|, |\zeta_m(\mathbf{U})|)$  denoting the maximum wave speed. Several other choices are possible, see [14, 15, 26, 38].

In the simplest form of the scheme, the numerical solution  $\mathbf{V}^h \in \mathcal{V}^k$  to (2.2) is then found as the solution of the system

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{\Phi}^h) = 0 \quad \forall \, \mathbf{\Phi}^h \in \mathcal{V}^k.$$
(2.6)

For problems involving shocks or discontinuities, however, this scheme exhibits oscillations and overshoot. Therefore, in the scheme developed in [26], a streamline diffusion operator  $\mathcal{B}_{SD}$  and a shock capturing operator  $\mathcal{B}_{SC}$ are added. Then, the discrete solution  $\mathbf{V}^h \in \mathcal{V}^k$  is found as the solution to the system

$$\mathcal{B}(\mathbf{V}^h, \mathbf{\Phi}^h) = \mathcal{B}_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{\Phi}^h) + \mathcal{B}_{\mathrm{SD}}(\mathbf{V}^h, \mathbf{\Phi}^h) + \mathcal{B}_{\mathrm{SC}}(\mathbf{V}^h, \mathbf{\Phi}^h) = 0$$
(2.7)

for all  $\Phi^h \in \mathcal{V}^k$ .

A slightly improved version of the streamline diffusion term used in [26] is given by (compare [25])

$$\mathcal{B}_{\rm SD}(\mathbf{V}^h, \mathbf{\Phi}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} \left( \mathbf{U}_{\mathbf{V}}(\mathbf{V}^h) \mathbf{\Phi}_t^h + \mathbf{F}_{\mathbf{V}}(\mathbf{V}^h) \mathbf{\Phi}_x^h \right) \cdot \left( \mathbf{D}^{SD} \operatorname{Res} \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.8}$$

with

$$\operatorname{Res} = \mathbf{U}(\mathbf{V}^h)_t + \mathbf{F}(\mathbf{V}^h)_x \tag{2.9}$$

and

$$\mathbf{D}^{SD} = C^{SD} \Delta t_n \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V}^h).$$
(2.10)

Typically,  $C^{SD} = 10$  is used. The shock capturing term in [25, 26] uses both an inner-element residual (based on (2.9), the residual within the cell) and a boundary residual (based on measuring jumps along the boundaries of the discontinuous spacetime elements) in order to adjust the amount of viscosity that is added to the system. As the boundary residual is very complicated and has only fairly small impact, we will only use the inner residual for our extensions of the method. Therefore, we only present a reduced form of the shock capturing term used in [25], which is given by

$$\mathcal{B}_{\rm SC}(\mathbf{V}^h, \mathbf{\Phi}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} D_{n,i}^{SC} \left( \mathbf{\Phi}_t^h \cdot \left( \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_t^h \right) + \frac{h^2}{\Delta t_n^2} \mathbf{\Phi}_x^h \cdot \left( \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_x^h \right) \right) \,\mathrm{d}x \,\mathrm{d}t \tag{2.11}$$

with

$$\tilde{\mathbf{V}}_{n,i} = \frac{1}{h\Delta t_n} \int_{I_n} \int_{K_i} \mathbf{V}^h(x,t) \, \mathrm{d}x \, \mathrm{d}t \,,$$

and

$$D_{n,i}^{SC} = \frac{\Delta t_n C^{SC} \overline{\operatorname{Res}}_{n,i}}{\sqrt{\int_{I_n} \int_{K_i} \left( \mathbf{V}_t^h \cdot \left( \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_t^h \right) + \frac{h^2}{\Delta t_n^2} \mathbf{V}_x^h \cdot \left( \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_x^h \right) \right) \, \mathrm{d}x \, \mathrm{d}t + \varepsilon} ,$$
(2.12)

and

$$\overline{\operatorname{Res}}_{n,i} = \sqrt{\int_{I_n} \int_{K_i} \operatorname{Res} \cdot \left( \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V}^h) \operatorname{Res} \right) \, \mathrm{d}x \, \mathrm{d}t} \,.$$
(2.13)

Here,  $\varepsilon = \frac{h^{3/2}}{\varDelta t_n^{1/2} \operatorname{diam}(\Omega)}$  and typically  $C^{SC} = 1$  is used.

Remark 2.1. In a first attempt of extending the original scheme from conservation laws to convection-diffusion systems, we did not include the streamline diffusion and shock capturing operators  $\mathcal{B}_{SD}$  and  $\mathcal{B}_{SC}$ . We expected the natural viscosity introduced by the additional diffusion term to take over the role of the streamline diffusion and shock capturing operators in terms of avoiding overshoot. Numerical tests indicated that this is the case if the grid size is chosen fine enough to resolve the additional viscosity in the system. If the grid is too coarse, however, this is not the case and we observed overshoot and oscillations. Therefore, to ensure robustness of our methods both in a fully resolved and in an underresolved regime, we also extend the streamline diffusion and shock capturing operators to convection-diffusion systems. The modified operators will be presented in Sections 4 and 5, respectively.

Among other properties, one can show that the method described in (2.7) (as well as the method in (2.6)) produces entropy stable discrete solutions.

**Theorem 2.2** (Part of Theorem 3.1 in [26]). Consider the system of conservation laws (2.1) with strictly convex entropy function S and entropy flux function Q. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain  $\Omega$ . Let the final time be denoted by  $t_N$ . Then the approximate solutions produced by (2.7) satisfy

$$\int_{\Omega} S(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))) \, \mathrm{d}x \le \int_{\Omega} S(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))) \, \mathrm{d}x.$$

**Remark 2.3.** Analogously to this result, we will prove entropy stability for the extensions to convectiondiffusion systems under the assumption of compact support of the solution. This is discussed in more detail in Remark 4.3.

This concludes our summary of the method for hyperbolic conservation laws and of the features that are relevant for our new methods. We will present the details of our extensions ST-SIPG, ST-NIPG, and ST-LDG in Sections 4 and 5. First, however, we will examine the effect of switching to entropy variables on the convection-diffusion systems that we consider.

#### 3. Convection-diffusion systems written in entropy variables

We consider a system of hyperbolic conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$$

and add a diffusion matrix  $\mathbf{D}: \mathbb{R}^m \to \mathbb{R}^m$ , to be thought of as, *e.g.*, physical viscosity and heat conduction terms resulting in

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = (\mathbf{D}(\mathbf{U})\mathbf{U}_x)_x$$

We assume that we are given a strictly convex entropy function S with corresponding entropy flux Q. Using entropy variables  $\mathbf{V} = S_{\mathbf{U}}(\mathbf{U})$  as degrees of freedom results in

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x = (\mathbf{A}(\mathbf{V})\mathbf{V}_x)_x \tag{3.1}$$

with  $\mathbf{A}(\mathbf{V}) = \mathbf{D}(\mathbf{U}(\mathbf{V}))\mathbf{U}_{\mathbf{V}}(\mathbf{V})$ . It is well-known that such a change to entropy variables symmetrizes a hyperbolic system of conservation laws [16, 18, 20, 32]. Additionally, this change of variable can have a positive effect on the properties of the matrix  $\mathbf{A}$ . We will demonstrate this on our main application, the compressible Navier–Stokes equations.

### 3.1. The compressible Navier–Stokes equations

We shortly describe the situation for the compressible Navier–Stokes equations. We focus on introducing the entropy S that we use and on discussing the properties of **A**. Further information for using entropy variables in the context of the compressible Navier–Stokes equations can be found in the literature [3, 19, 28, 37, 39].

The compressible Navier–Stokes equations in one space dimension are given by

$$\rho_t + (\rho u)_x = 0,$$

$$(\rho u)_t + (\rho u^2 + p)_x = \nu u_{xx},$$

$$E_t + ((E+p)u)_x = \nu \left(\frac{u^2}{2}\right)_{xx} + \kappa \theta_{xx},$$
(3.2)

with  $\rho = \rho(x,t) > 0$  denoting the density, u = u(x,t) the velocity, p = p(x,t) > 0 the pressure, and  $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2$  the total energy with  $\gamma > 1$  being the adiabatic exponent. We will also use  $m = m(x,t) = \rho(x,t)u(x,t)$  for the momentum. Additionally, R > 0 is the gas constant,  $C_v > 0$  is the specific heat at constant volume, and  $\theta = \frac{p}{R\rho}$  refers to the temperature. We assume the viscosity  $\nu > 0$  and the heat conductivity  $\kappa > 0$  to be constant. We further assume the relation between  $\nu$  and  $\kappa/R$  to be given by the Prandtl number  $P_r = 4\gamma/(9\gamma - 5)$  via

$$\frac{\kappa}{R} = \frac{\gamma C_v}{RP_r}\nu = \frac{9\gamma - 5}{4(\gamma - 1)}\nu \cdot$$

Writing the right hand side of (3.2) in the form  $(\mathbf{D}(\mathbf{U})\mathbf{U}_x)_x$  with  $\mathbf{U} = (\rho, m, E)^T$  results in

$$\mathbf{D}(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 \\ -\nu \frac{m}{\rho^2} & \frac{\nu}{\rho} & 0 \\ -\nu \frac{m^2}{\rho^3} + \frac{\kappa}{R}(\gamma - 1) \left(\frac{m^2}{\rho^3} - \frac{E}{\rho^2}\right) \nu \frac{m}{\rho^2} - \frac{\kappa}{R}(\gamma - 1) \frac{m}{\rho^2} \frac{\kappa}{R} \frac{\gamma - 1}{\rho} \end{pmatrix}$$

We note that the matrix  $\mathbf{D}(\mathbf{U})$  is not symmetric.

For the transformation to entropy variables, we use the entropy and the entropy flux given by

$$S = \frac{-\rho s}{\gamma - 1}, \quad Q = \frac{-\rho u s}{\gamma - 1}, \quad s = \log(p) - \gamma \log(\rho).$$

This results in the entropy variables (written in terms of primitive variables and s for simplicity)

$$\mathbf{V} = \left(\frac{\gamma - s}{\gamma - 1} - \frac{\rho u^2}{2p}, \quad \frac{\rho u}{p}, \quad -\frac{\rho}{p}\right)^T$$

The matrix  $\mathbf{A}(\mathbf{V})$  is then given by

$$\mathbf{A}(\mathbf{V}) = \begin{pmatrix} 0 & 0 & 0\\ 0 - \nu \frac{1}{v_3} & \nu \frac{v_2}{v_3^2} \\ 0 & \nu \frac{v_2}{v_3^2} & -\nu \frac{v_2^2}{v_3^3} + \frac{\kappa}{R} \frac{1}{v_3^2} \end{pmatrix}.$$
(3.3)

Since  $\rho, p > 0$ , this matrix is symmetric positive semi-definite. Furthermore, the reduced matrix

$$\tilde{\mathbf{A}}(\mathbf{V}) = \begin{pmatrix} -\nu \frac{1}{v_3} & \nu \frac{v_2}{v_3^2} \\ \nu \frac{v_2}{v_3^2} & -\nu \frac{v_2^2}{v_3^3} + \frac{\kappa}{R} \frac{1}{v_3^2} \end{pmatrix}$$
(3.4)

is symmetric positive definite with eigenvalues

$$\lambda_{1,2} = \frac{1}{2} \left( -b \mp \sqrt{b^2 + 4\nu \frac{\kappa}{R} \frac{1}{v_3^3}} \right) > 0, \quad b = \nu \frac{v_2^2}{v_3^3} - \frac{\kappa}{R} \frac{1}{v_3^2} + \frac{\nu}{v_3}, \tag{3.5}$$

since  $\nu, \kappa > 0$  and  $\rho, p > 0$ .

#### 3.2. Assumptions on the matrix A

We develop our methods for general convection-diffusion systems given in the form (3.1), for which the left hand side corresponds to a hyperbolic system of conservation laws. In order to prove entropy stability for the methods that we will present later in this contribution we need to make some assumptions on the matrix **A**, which will be formulated in the following. Additionally, we will examine in this section whether these assumptions are satisfied for our main application, the compressible Navier–Stokes equations.

Assumption 3.1 (Ass. for ST-NIPG and ST-LDG). We assume that the matrix  $\mathbf{A}(\mathbf{V}) : \mathbb{R}^m \to \mathbb{R}^m$  in (3.1) is symmetric positive semi-definite.

We note that this is the case for the compressible Navier–Stokes equations as discussed above. In other words, our results will imply that the ST-NIPG and the ST-LDG method are entropy stable for the compressible Navier–Stokes equations.

To prove entropy stability for the ST-SIPG method, we need a stronger assumption.

Assumption 3.2 (Ass. for ST-SIPG). We assume that the matrix  $\mathbf{A}(\mathbf{V}) : \mathbb{R}^m \to \mathbb{R}^m$  in (3.1) is symmetric positive definite. We further assume that there are lower and upper bounds  $(\lambda, \Lambda)$  on the eigenvalues of  $\mathbf{A}$  (corresponding to the discrete solution) such that  $0 < \lambda \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \Lambda$ .

Based on our considerations above, this is *not* the case for the compressible Navier–Stokes equations as  $\mathbf{A}$  is only positive semi-definite. However, the matrix  $\mathbf{A}$  for the compressible Navier–Stokes equations has the structure

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathbf{A}} \end{bmatrix}. \tag{3.6}$$

It is sufficient (see the explanation in Remark 4.11 after having presented the ST-SIPG method) to satisfy Assumption 3.2 for the matrix  $\tilde{\mathbf{A}}$ , which is positive definite. In order to satisfy Assumption 3.2, it remains to provide uniform bounds  $\lambda$ ,  $\Lambda$  for the eigenvalues  $\lambda_{1,2}$  given in (3.5). We are not aware of general, problem-independent bounds for the continuous problem. However, we only need these bounds for the eigenvalues corresponding to the numerical solution, for which such bounds might be checked *a posteriori*.

Lemma 3.3. Define

$$\lambda^{h} = min_{x} \left( \nu \frac{\kappa}{R} \frac{\frac{p^{h}(x)}{\rho^{h}(x)}}{\nu(u^{h}(x)^{2} + 1)\frac{\rho^{h}(x)}{p^{h}(x)} + \frac{\kappa}{R}} \right) \quad and \quad \Lambda^{h} = \max_{x} \left( (u^{h}(x)^{2} + 1)\nu \frac{p^{h}(x)}{\rho^{h}(x)} + \frac{\kappa}{R} \frac{p^{h}(x)^{2}}{\rho^{h}(x)^{2}} \right)$$
(3.7)

Here,  $\rho^h$ ,  $u^h$ , and  $p^h$  correspond to the computed solution for density, velocity, and pressure, respectively. Then, there holds  $0 < \lambda^h \le \lambda_1^h < \lambda_2^h \le \Lambda^h$  for the eigenvalues of  $\tilde{\mathbf{A}}$  for the compressible Navier–Stokes equations.

*Proof.* (We drop the superscript h for simplicity.) The eigenvalues  $\lambda_{1,2}$  of  $\tilde{\mathbf{A}}$  are given by (3.5) with  $\lambda_2$  corresponding to the '+' sign. We can bound  $\lambda_2$  from above by  $\lambda_2 \leq |b|$ , which we express in terms of the primitive variables. For the lower bound, we write

$$\lambda_1 = \frac{1}{2} |b| \left( 1 - \sqrt{1 - \varepsilon} \right), \quad \varepsilon = -\frac{4\nu \frac{\kappa}{R} \frac{1}{v_3^3}}{b^2}.$$

We note that  $0 < \varepsilon < 1$ . We use the Taylor expansion of the square root function to bound

$$\sqrt{1-\varepsilon} \le 1 - \frac{1}{2}\varepsilon.$$

This implies  $\lambda_1 \geq \frac{1}{4}|b| \varepsilon$ , which we again express in terms of the primitive variables.

To summarize, with the bounds  $\lambda^h$  and  $\Lambda^h$  provided by (3.7), the compressible Navier–Stokes equations satisfy Assumption 3.2, which will be the main assumption for showing entropy stability for the ST-SIPG method.

### 4. The ST-NIPG and the ST-SIPG method

In this section, we present our extensions ST-NIPG and ST-SIPG that are based on the interior penalty (IP) approach by Arnold [1] as well as on earlier work by Nitsche [33]. For more information about IP methods, we refer to Oden *et al.* [34] and Rivière [36]. For both methods, we seek the discrete solution  $\mathbf{V}^h \in \mathcal{V}^k$  that satisfies for all  $\Phi^h \in \mathcal{V}^k$  the quasilinear form

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{h}, \mathbf{\Phi}^{h}) + \mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^{h}, \mathbf{\Phi}^{h}) + \mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^{h}, \mathbf{\Phi}^{h}) + \mathcal{B}_{\mathrm{IP}}(\vartheta; \mathbf{V}^{h}, \mathbf{\Phi}^{h}) = \mathcal{B}_{\mathrm{bdy}}(\mathbf{\Phi}^{h}).$$
(4.1)

Here,  $\mathcal{B}_{DG}$  is given by (2.3),  $\mathcal{B}_{SD}^{IP}$  and  $\mathcal{B}_{SC}^{IP}$  are modifications of  $\mathcal{B}_{SD}$  and  $\mathcal{B}_{SC}$  that will be described below, and  $\mathcal{B}_{bdy}$  incorporates a suitable discretization of boundary terms. Finally,  $\mathcal{B}_{IP}(\vartheta; \mathbf{V}^h, \Phi^h)$ , which depends on the parameter  $\vartheta$ , represents the discretization of the diffusion term  $-(\mathbf{A}(\mathbf{V})\mathbf{V}_x)_x$  and is given by

$$\mathcal{B}_{\mathrm{IP}}(\vartheta; \mathbf{V}^{h}, \mathbf{\Phi}^{h}) = \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \left( \mathbf{A}(\mathbf{V}^{h}) \mathbf{V}_{x}^{h} \right) \cdot \mathbf{\Phi}_{x}^{h} \, \mathrm{d}x \, \mathrm{d}t - \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{A}(\mathbf{V}^{h}) \right\}_{i+1/2} \left\{ \mathbf{V}_{x}^{h} \right\}_{i+1/2} \right) \cdot \left[ \mathbf{\Phi}^{h} \right]_{i+1/2} \, \mathrm{d}t + \vartheta \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{A}(\mathbf{V}^{h}) \right\}_{i+1/2} \left[ \mathbf{V}^{h} \right]_{i+1/2} \right) \cdot \left\{ \mathbf{\Phi}_{x}^{h} \right\}_{i+1/2} \, \mathrm{d}t + \sum_{n,i} \int_{I_{n}} \frac{\sigma}{h} \left( \left\{ \mathbf{A}(\mathbf{V}^{h}) \right\}_{i+1/2} \left[ \mathbf{V}^{h} \right]_{i+1/2} \right) \cdot \left[ \mathbf{\Phi}^{h} \right]_{i+1/2} \, \mathrm{d}t$$

$$(4.2)$$

for interior edges  $i \pm 1/2$ . We use the standard notation for average and jump given by

$$\left\{v^{h}\right\}_{i+1/2} = \frac{1}{2} \left(v^{h}_{i+1/2,R} + v^{h}_{i+1/2,L}\right), \quad [v^{h}]_{i+1/2} = v^{h}_{i+1/2,L} - v^{h}_{i+1/2,R},$$

with  $\{\mathbf{A}(\mathbf{V}^h)\}_{i+1/2} = \left(\mathbf{A}(\mathbf{V}^h_{i+1/2,L}) + \mathbf{A}(\mathbf{V}^h_{i+1/2,R})\right)/2$ . We refer to the new methods

- as ST-NIPG for  $\vartheta = 1$ ,
- and as ST-SIPG for  $\vartheta = -1$ .

We note that the edge terms in the second line result from integration by parts using central fluxes. The edge terms in the third line are added to make the form (anti-)symmetric. Finally, the jump terms in the fourth line are stabilization terms that enforce coercivity of the form  $\mathcal{B}_{IP}$  for the ST-SIPG method, with the parameter  $\sigma > 0$  being the penalty or stabilization parameter.

Finally, we need to adjust the streamline diffusion and shock capturing terms given by (2.8) and (2.11) for the original method. We first adjust the definition of the residual. Instead of (2.9), we use

$$\operatorname{Res}^{\mathrm{IP}} = \mathbf{U}(\mathbf{V}^{h})_{t} + \mathbf{F}(\mathbf{V}^{h})_{x} - \mathbf{A}(\mathbf{V}^{h})_{x}\mathbf{V}_{x}^{h} - \mathbf{A}(\mathbf{V}^{h})\mathbf{V}_{xx}^{h}.$$
(4.3)

Note that now second derivatives need to be evaluated. This can lead to suboptimal convergence rates for smooth flow (cmp Sect. 6). We therefore make an additional change and use with  $\alpha \ge 0$ 

$$\overline{\operatorname{Res}}_{n,i}^{\operatorname{IP}} = h^{\alpha} \sqrt{\int_{I_n} \int_{K_i} \operatorname{Res}^{\operatorname{IP}} \cdot \left( \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V}^h) \operatorname{Res}^{\operatorname{IP}} \right) \, \mathrm{d}x \, \mathrm{d}t} \tag{4.4}$$

instead of the definition given by (2.13). For  $\alpha = 0$ , the resulting shock capturing term is similar to the original term given by equations (2.11)–(2.13), just with Res replaced by Res<sup>IP</sup>. For  $\alpha = 0.9$ , the shock capturing term has similarities with the shock capturing term used in [21]. We will compare the behavior of the resulting shock capturing terms for these two values of  $\alpha$  in the numerical results section. As a general rule, a bigger value of  $\alpha$  should be more favorable for optimal convergence rates for smooth flow, whereas a smaller value of  $\alpha$  should lead to more stability around shocks. This completes the changes for the shock capturing term. For the streamline diffusion term, additional changes are necessary. The form  $\mathcal{B}_{SD}$  given by (2.8) has been constructed such that the left hand side of the inner product corresponds to a linearized residual. Consequently, we make the following adjustment

$$\mathcal{B}_{\rm SD}^{\rm IP}(\mathbf{V}^h, \mathbf{\Phi}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} \left( \mathbf{U}_{\mathbf{V}}(\mathbf{V}^h) \mathbf{\Phi}_t^h + \mathbf{F}_{\mathbf{V}}(\mathbf{V}^h) \mathbf{\Phi}_x^h - \mathbf{A}(\mathbf{V}^h)_x \mathbf{\Phi}_x^h - \mathbf{A}(\mathbf{V}^h) \mathbf{\Phi}_{xx}^h \right) \cdot \left( \mathbf{D}^{SD, \rm IP} \operatorname{Res}^{\rm IP} \right) \, \mathrm{d}x \, \mathrm{d}t \,.$$

$$\tag{4.5}$$

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Also, to be consistent with the shock capturing term, we replace the definition of  $\mathbf{D}^{SD}$  given by (2.10) by

$$\mathbf{D}^{SD,\mathrm{IP}} = C^{SD} h^{\alpha} \Delta t_n \mathbf{U}_{\mathbf{V}}^{-1}(\mathbf{V}^h).$$
(4.6)

This concludes the description of the ST-NIPG and the ST-SIPG method.

We can show the following theorems concerning the entropy stability of the ST-NIPG and ST-SIPG method.

**Theorem 4.1** (Entropy stability for ST-NIPG). Let Assumption 3.1 hold true. Consider the system (3.1) with strictly convex entropy function S and entropy flux function Q. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain  $\Omega$ . Let the final time be denoted by  $t_N$ . Then, the approximate solutions generated by the scheme (4.1) with  $\vartheta = 1$  and  $\sigma > 0$  satisfy

$$\int_{\Omega} S(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))) \, \mathrm{d}x \le \int_{\Omega} S(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))) \, \mathrm{d}x.$$

**Theorem 4.2** (Entropy stability for ST-SIPG). Let Assumption 3.2 hold true. Consider the system (3.1) with strictly convex entropy function S and entropy flux function Q. For simplicity, assume that the exact and approximate solutions have compact support inside the spatial domain  $\Omega$ . Let the final time be denoted by  $t_N$ . Then, the approximate solutions generated by the scheme (4.1) with  $\vartheta = -1$  satisfy

$$\int_{\Omega} S(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))) \, \mathrm{d}x \le \int_{\Omega} S(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))) \, \mathrm{d}x,$$

provided  $\sigma$  is chosen sufficiently large such that

$$\sigma > \frac{c_{\rm inv}\Lambda}{\lambda} \tag{4.7}$$

where  $\lambda, \Lambda$  are defined in Assumption 3.2 and  $c_{inv}$  will be specified below in Lemma 4.4.

**Remark 4.3.** In this work, which considers one-dimensional convection-diffusion systems, we will focus on the interior of the flow domain and not consider discretizations of boundary conditions. Finding a suitable discretization of inflow, outflow, and solid wall with no heat flux boundary conditions for the compressible Navier–Stokes equations in two and three dimensions is often not trivial. Incorporating the boundary conditions in the entropy considerations helps to single out physically relevant formulations and strengthens results such as Theorems 4.1 and 4.2. However, as there are no physically relevant boundary conditions for the compressible Navier–Stokes equations in one dimension, we postpone these considerations to future work about multi-dimensional extensions.

In order to prove Theorems 4.1 and 4.2, we will need several auxiliary results, which we will provide in the following.

Lemma 4.4. For the discrete spacetime polynomials there holds

$$\int_{I_n} (v_{x,B}^h)^2 \, \mathrm{d}t \le \frac{c_{\mathrm{inv}}}{h} \int_{I_n} \int_{K_i} (v_x^h)^2 \, \mathrm{d}x \, \mathrm{d}t$$

with  $v_{x,B}^h = v_{x,i+1/2,L}^h$  or  $v_{x,B}^h = v_{x,i-1/2,R}^h$ .

This trace inverse inequality can be shown with  $c_{inv} \sim (k+1)^2 [24, 40]$ .

Lemma 4.5. Under the conditions of Theorem 4.1 and of Theorem 4.2, respectively, there holds

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{h}, \mathbf{V}^{h}) \ge \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{h}_{N, -}(x))) \,\mathrm{d}x - \int_{\Omega} S(\mathbf{U}(\mathbf{V}^{h}_{0, -}(x))) \,\mathrm{d}x.$$
(4.8)

The proof follows directly from the proof of the original method given in [26] and is summarized in Appendix A. Lemma 4.6. For the ST-NIPG method, under the assumptions of Theorem 4.1, there holds

$$\mathcal{B}_{\mathrm{IP}}(1; \mathbf{V}^h, \mathbf{V}^h) \ge 0.$$

*Proof.* By definition and due to the symmetry of  $\mathbf{A}$ , there holds

$$\mathcal{B}_{\mathrm{IP}}(1;\mathbf{V}^{h},\mathbf{V}^{h}) = \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \left(\mathbf{A}(\mathbf{V}^{h})\mathbf{V}_{x}^{h}\right) \cdot \mathbf{V}_{x}^{h} \,\mathrm{d}x \,\mathrm{d}t + \sum_{n,i} \int_{I_{n}} \frac{\sigma}{h} \left(\{\mathbf{A}(\mathbf{V}^{h})\}_{i+1/2} [\mathbf{V}^{h}]_{i+1/2}\right) \cdot [\mathbf{V}^{h}]_{i+1/2} \,\mathrm{d}t.$$

As **A** is assumed to be positive semi-definite and  $\sigma > 0$ , this implies the claim.

In order to show a similar result for the ST-SIPG method, we need the following lemma, which generalizes Young's inequality to matrices.

**Lemma 4.7.** Let the matrix  $\mathbf{C} : \mathbb{R}^m \to \mathbb{R}^m$  be symmetric positive definite. Then there holds for arbitrary vectors  $v, w \in \mathbb{R}^m$  and  $\delta > 0$ 

$$2w^T \mathbf{C} v \le \delta w^T \mathbf{C} w + \frac{1}{\delta} v^T \mathbf{C} v.$$

*Proof.* The proof follows directly from

$$0 \le \frac{1}{\delta} \left( (\delta w - v)^T \mathbf{C} (\delta w - v) \right) = \delta w^T \mathbf{C} w - 2w^T \mathbf{C} v + \frac{1}{\delta} v^T \mathbf{C} v.$$

For the ST-SIPG method, we have the following results.

Lemma 4.8. For the ST-SIPG method, under the assumptions of Theorem 4.2, there holds

$$\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^h, \mathbf{V}^h) \ge (\lambda - c_{\mathrm{inv}} \delta \Lambda) \sum_{n,i} \int_{I_n} \int_{K_i} \mathbf{V}_x^h \cdot \mathbf{V}_x^h \mathrm{d}x \mathrm{d}t + \frac{\sigma - \frac{1}{\delta}}{h} \sum_{n,i} \int_{I_n} \left( \{ \mathbf{A}(\mathbf{V}^h) \}_{i+1/2} [\mathbf{V}^h]_{i+1/2} \right) \cdot [\mathbf{V}^h]_{i+1/2} \mathrm{d}t,$$

with  $c_{inv}$  being the constant from the inverse estimate from Lemma 4.4 with  $\delta > 0$  arbitrary.

Corollary 4.9. Under the assumptions of Theorem 4.2, there holds

$$\mathcal{B}_{\mathrm{IP}}(-1;\mathbf{V}^h,\mathbf{V}^h) \ge 0.$$

*Proof.* The proof follows directly from Lemma 4.8, the assumption that **A** is positive definite, and the assumption on  $\sigma$  given by (4.7) by defining  $\delta = \frac{1}{\sigma}$ .

*Proof of Lemma* 4.8. Due to the solutions having compact support, there holds with  $\vartheta = -1$ 

$$\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^{h}, \mathbf{V}^{h}) = \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \underbrace{\left(\mathbf{A}(\mathbf{V}^{h})\mathbf{V}_{x}^{h}\right) \cdot \mathbf{V}_{x}^{h}}_{\Gamma_{1}} \, \mathrm{d}x \, \mathrm{d}t \\ - 2 \sum_{n,i} \int_{I_{n}} \left(\left\{\mathbf{A}(\mathbf{V}^{h})\right\}_{i+1/2} \left\{\mathbf{V}_{x}^{h}\right\}_{i+1/2}\right) \cdot [\mathbf{V}^{h}]_{i+1/2} \, \mathrm{d}t \\ + \sum_{n,i} \int_{I_{n}} \underbrace{\frac{\sigma}{h} \left(\left\{\mathbf{A}(\mathbf{V}^{h})\right\}_{i+1/2} [\mathbf{V}^{h}]_{i+1/2}\right) \cdot [\mathbf{V}^{h}]_{i+1/2}}_{\Gamma_{2}} \, \mathrm{d}t.$$

Based on Assumption 3.2, the matrix A is positive definite. Applying Lemma 4.7 with arbitrary  $\delta > 0$  to the middle term on the right hand side results in

$$2\left(\{\mathbf{A}(\mathbf{V}^{h})\}_{i+1/2}\left\{\mathbf{V}_{x}^{h}\right\}_{i+1/2}\right) \cdot [\mathbf{V}^{h}]_{i+1/2} \\ \leq \underbrace{\delta h\left(\{\mathbf{A}(\mathbf{V}^{h})\}_{i+1/2}\left\{\mathbf{V}_{x}^{h}\right\}_{i+1/2}\right) \cdot \{\mathbf{V}_{x}^{h}\}_{i+1/2}}_{\Pi_{1}} + \underbrace{\frac{1}{\delta h}\left(\{\mathbf{A}(\mathbf{V}^{h})\}_{i+1/2}[\mathbf{V}^{h}]_{i+1/2}\right) \cdot [\mathbf{V}^{h}]_{i+1/2}}_{\Pi_{2}}.$$

There holds

$$-\sum_{n,i}\int_{I_n}\Pi_2\,\mathrm{d}t + \sum_{n,i}\int_{I_n}\Gamma_2\,\mathrm{d}t = \frac{\sigma - \frac{1}{\delta}}{h}\sum_{n,i}\int_{I_n}\left(\{\mathbf{A}(\mathbf{V}^h)\}_{i+1/2}[\mathbf{V}^h]_{i+1/2}\right)\cdot[\mathbf{V}^h]_{i+1/2}\,\mathrm{d}t.$$

Let us consider the remaining terms. By assumption, the eigenvalues of the matrix **A** are uniformly bounded, *i.e.*, there holds  $0 < \lambda \leq \lambda_1 \leq \ldots \leq \lambda_m \leq \Lambda$ . Therefore,

$$\sum_{n,i} \int_{I_n} \int_{K_i} \Gamma_1 \, \mathrm{d}x \, \mathrm{d}t - \sum_{n,i} \int_{I_n} \Pi_1 \, \mathrm{d}t \ge \sum_{n,i} \int_{I_n} \int_{K_i} \lambda \mathbf{V}_x^h \cdot \mathbf{V}_x^h \, \mathrm{d}x \, \mathrm{d}t - \sum_{n,i} \int_{I_n} \delta h \Lambda \left\{ \mathbf{V}_x^h \right\}_{i+1/2} \cdot \left\{ \mathbf{V}_x^h \right\}_{i+1/2} \, \mathrm{d}t.$$

As

$$\left\{\mathbf{V}_{x}^{h}\right\}_{i+1/2} \cdot \left\{\mathbf{V}_{x}^{h}\right\}_{i+1/2} = \sum_{j=1}^{m} \left(\left\{(v_{j}^{h})_{x}\right\}_{i+1/2}\right)^{2},$$

we can apply an inverse trace inequality to each component. Using

$$\left(\left\{v_x^h\right\}_{i+1/2}\right)^2 = \left(\frac{1}{2}(v_{x,i+1/2,L}^h + v_{x,i+1/2,R}^h)\right)^2 \le \frac{1}{2}(v_{x,i+1/2,L}^h)^2 + \frac{1}{2}(v_{x,i+1/2,R}^h)^2$$

and Lemma 4.4, we get

$$\sum_{n,i} \int_{I_n} \delta h \Lambda \left\{ \mathbf{V}_x^h \right\}_{i+1/2} \cdot \left\{ \mathbf{V}_x^h \right\}_{i+1/2} \, \mathrm{d}t \le \sum_{n,i} \int_{I_n} \int_{K_i} c_{\mathrm{inv}} \delta \Lambda \mathbf{V}_x^h \cdot \mathbf{V}_x^h \, \mathrm{d}x \, \mathrm{d}t.$$

This implies

$$\sum_{n,i} \int_{I_n} \int_{K_i} \Gamma_1 \, \mathrm{d}x \, \mathrm{d}t - \sum_{n,i} \int_{I_n} \Pi_1 \, \mathrm{d}t \ge (\lambda - c_{\mathrm{inv}} \delta A) \sum_{n,i} \int_{I_n} \int_{K_i} \mathbf{V}_x^h \cdot \mathbf{V}_x^h \, \mathrm{d}x \, \mathrm{d}t.$$

Summarizing all results, there holds

$$\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^h, \mathbf{V}^h) \ge (\lambda - c_{\mathrm{inv}} \delta A) \sum_{n,i} \int_{I_n} \int_{K_i} \mathbf{V}_x^h \cdot \mathbf{V}_x^h \mathrm{d}x \mathrm{d}t + \frac{\sigma - \frac{1}{\delta}}{h} \sum_{n,i} \int_{I_n} \left( \{\mathbf{A}(\mathbf{V}^h)\}_{i+1/2} [\mathbf{V}^h]_{i+1/2} \right) \cdot [\mathbf{V}^h]_{i+1/2} \mathrm{d}t,$$

which concludes the proof.

**Remark 4.10.** For solving the scalar equation  $u_t + f(u)_x = (au_x)_x$  with  $0 < \underline{a} \leq a \leq \overline{a}$  and bounds  $m \leq S_{uu} \leq M$ , with  $S_{uu}$  denoting the second derivative of the entropy S with respect to the conserved variable u, the condition (4.7) on  $\sigma$  reduces to

$$\sigma > \frac{c_{\rm inv}(\overline{a}/m)}{\underline{a}/M}.$$
(4.9)

Now, we have collected all auxiliary results needed to prove Theorems 4.1 and 4.2.

Proof of Theorems 4.1 and 4.2. Using the compact support of the discrete solution, we can drop the boundary terms  $\mathcal{B}_{bdy}$  in (4.1). Then, testing with  $\Phi^h = \mathbf{V}^h$  results in

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{V}^h) + \mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^h, \mathbf{V}^h) + \mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^h, \mathbf{V}^h) + \mathcal{B}_{\mathrm{IP}}(\vartheta; \mathbf{V}^h, \mathbf{V}^h) = 0.$$

We consider each of the four terms separately:

(1) Term  $\mathcal{B}_{DG}(\mathbf{V}^h, \mathbf{V}^h)$ : According to Lemma 4.5, there holds

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{V}^h) \ge \int_{\Omega} S(\mathbf{U}(\mathbf{V}^h_{N, -}(x))) \, \mathrm{d}x - \int_{\Omega} S(\mathbf{U}(\mathbf{V}^h_{0, -}(x))) \, \mathrm{d}x.$$

(2) Term  $\mathcal{B}_{SD}^{IP}(\mathbf{V}^h, \mathbf{V}^h)$ : Claim: There holds

$$\mathcal{B}_{\mathrm{SD}}^{\mathrm{IP}}(\mathbf{V}^h, \mathbf{V}^h) \ge 0.$$

*Proof.* We essentially follow the proof of Theorem 3.1 in [26]. Based on our new definition of the streamline diffusion term, there holds by chain rule

$$\mathcal{B}_{\rm SD}^{\rm IP}(\mathbf{V}^h, \mathbf{V}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} \operatorname{Res}^{\rm IP} \cdot \left( \mathbf{D}^{SD, \rm IP} \operatorname{Res}^{\rm IP} \right) \, \mathrm{d}x \, \mathrm{d}t.$$

With the definition of  $\mathbf{D}^{SD,\text{IP}}$  given by (4.6) and due to the assumption that the entropy S is strictly convex, this implies  $\mathcal{B}_{\text{SD}}^{\text{IP}}(\mathbf{V}^h, \mathbf{V}^h) \geq 0$ . We note that our adjustment of the term  $\mathcal{B}_{\text{SD}}^{\text{IP}}$  compared to the original term  $\mathcal{B}_{\text{SD}}$  was essential for proving this claim.

(3) Term  $\mathcal{B}_{SC}^{IP}(\mathbf{V}^h, \mathbf{V}^h)$ :

*Claim*: There holds

$$\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^h, \mathbf{V}^h) \ge 0$$

*Proof.* There holds (compare (2.11))

$$\mathcal{B}_{\mathrm{SC}}^{\mathrm{IP}}(\mathbf{V}^h, \mathbf{V}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} D_{n,i}^{SC} \left( \mathbf{V}_t^h \cdot \left( \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_t^h \right) + \frac{h^2}{(\Delta t_n)^2} \mathbf{V}_x^h \cdot \left( \mathbf{U}_{\mathbf{V}}(\tilde{\mathbf{V}}_{n,i}) \mathbf{V}_x^h \right) \right) \,\mathrm{d}x \,\mathrm{d}t$$

with  $D_{n,i}^{SC}$  being given by (2.12) but with  $\overline{\text{Res}}_{n,i}$  replaced by  $\overline{\text{Res}}_{n,i}^{\text{IP}}$ , which is given by (4.4). Due to the strict convexity of the entropy function S, both  $\mathbf{U}_{\mathbf{V}}$  and  $\mathbf{U}_{\mathbf{V}}^{-1}$  are strictly positive definite. This implies  $D_{n,i}^{SC} \geq 0$ . This also directly implies  $\mathcal{B}_{\text{SC}}^{\text{IP}}(\mathbf{V}^h, \mathbf{V}^h) \geq 0$ .

(4) Term  $\mathcal{B}_{IP}(\vartheta; \mathbf{V}^h, \mathbf{V}^h)$ : Based on Lemma 4.6 and on Corollary 4.9 there holds for both the ST-NIPG and the ST-SIPG method (under the respective assumptions)

$$\mathcal{B}_{\mathrm{IP}}(\vartheta; \mathbf{V}^h, \mathbf{V}^h) \ge 0.$$

Summarizing the estimates for the four terms results in

$$0 \ge \int_{\Omega} S(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))) \, \mathrm{d}x - \int_{\Omega} S(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))) \, \mathrm{d}x + 0 + 0 + 0,$$

which concludes the proof.

**Remark 4.11.** We now like to return to our claim made at the end of Section 3, which said that for the compressible Navier–Stokes equations it is sufficient to satisfy Assumption 3.2 for the reduced matrix  $\tilde{A}$  instead

of having to deal with the full matrix **A**. Based on the structure of **A** given by (3.6) and the fact that all terms in the form  $\mathcal{B}_{\text{IP}}$  contain the matrix **A**, there holds for  $\widetilde{\mathbf{V}}^h = (v_2^h, v_3^h)$  and  $\widetilde{\mathbf{\Phi}}^h = (\Phi_2^h, \Phi_3^h)$ ,

$$\mathcal{B}_{\mathrm{IP}}(-1; \mathbf{V}^h, \mathbf{\Phi}^h) = \widetilde{\mathcal{B}_{\mathrm{IP}}}(-1; \widetilde{\mathbf{V}}^h, \widetilde{\mathbf{\Phi}}^h) \quad \forall \, \mathbf{V}^h, \mathbf{\Phi}^h \in \mathcal{V}^k$$

with  $\widetilde{\mathcal{B}_{IP}}$  defined as  $\mathcal{B}_{IP}$  (compare (4.2)) but with **A** replaced by  $\widetilde{\mathbf{A}}$  and the scalar product being taken only over 2 components. Due to this equivalence, it is sufficient to implement the shortened form  $\widetilde{\mathcal{B}_{IP}}$ . For entropy stability, one now needs to show

$$\widetilde{\mathcal{B}}_{\mathrm{IP}}(-1; \widetilde{\mathbf{V}}^h, \widetilde{\mathbf{V}}^h) \ge 0.$$

In order to apply the proof of Lemma 4.8, it is therefore sufficient to satisfy Assumption 3.2 for the matrix  $\dot{\mathbf{A}}$ .

### 5. The ST-LDG Method

Our second approach for extending the original spacetime DG method is a variant of the LDG method introduced by Cockburn and Shu [9]. For the ST-LDG method, we take Assumption 3.1 for granted (which assumes that **A** is symmetric positive semi-definite). This implies that there exists a symmetric positive semi-definite matrix  $\mathbf{B} = \mathbf{B}(\mathbf{V})$  such that  $\mathbf{B}^2 = \mathbf{A}$ .

**Remark 5.1.** We refer to Appendix B for a description of the matrix B that we use in our numerical algorithm for solving the compressible Navier–Stokes equations.

Using this definition, we now consider the convection-diffusion system

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x = (\mathbf{B}^2(\mathbf{V})\mathbf{V}_x)_x.$$

We define  $\mathbf{P} = \mathbf{B}(\mathbf{V})\mathbf{V}_x$  and rewrite the second-order system as

$$\mathbf{U}(\mathbf{V})_t + \mathbf{F}(\mathbf{V})_x = (\mathbf{B}(\mathbf{V})\mathbf{P})_x,$$
$$\mathbf{P} = \mathbf{B}(\mathbf{V})\mathbf{V}_x.$$

We note that we keep the second set of equations in *non-conservative* form. In the original LDG method, all equations are written in conservation form. To achieve this, an auxiliary function  $g(\mathbf{V})$  is introduced with the property that  $g(\mathbf{V})_x = -\mathbf{B}(\mathbf{V})\mathbf{V}_x$ . This is very practical in terms of theory. Numerically, however, it is very complicated to evaluate g if the matrix **B** depends nonlinearly on **V**, which is the case for us. Therefore, we use the non-conservative form. Note though that the first set of equations, which contains the conserved quantity  $\mathbf{U}$ , is written in conservation form.

Let the discrete solution be given by the pair  $\mathbf{W}^h = (\mathbf{V}^h, \mathbf{P}^h) \in \mathcal{V}^k \times \mathcal{V}^k$ . To deduce the variational formulation, we multiply the first set of equations with a test function  $\Phi^h \in \mathcal{V}^k$ , integrate over spacetime elements, and do integration by parts using central fluxes for the diffusion terms. This results in

$$\mathcal{B}^{1}(\mathbf{W}^{h}, \mathbf{\Phi}^{h}) = \mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{h}, \mathbf{\Phi}^{h}) + \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \left( \mathbf{B}(\mathbf{V}^{h})\mathbf{P}^{h} \right) \cdot \mathbf{\Phi}_{x}^{h} \, \mathrm{d}x \, \mathrm{d}t - \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{B}(\mathbf{V}^{h}) \right\}_{i+1/2} \left\{ \mathbf{P}^{h} \right\}_{i+1/2} \right) \cdot \mathbf{\Phi}_{i+1/2,L}^{h} \, \mathrm{d}t + \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{B}(\mathbf{V}^{h}) \right\}_{i-1/2} \left\{ \mathbf{P}^{h} \right\}_{i-1/2} \right) \cdot \mathbf{\Phi}_{i-1/2,R}^{h} \, \mathrm{d}t$$
(5.1)

with  $\mathcal{B}_{\text{DG}}$  defined by (2.3) and  $\{\mathbf{B}(\mathbf{V}^h)\}_{i+1/2} = \left(\mathbf{B}(\mathbf{V}^h_{i+1/2,L}) + \mathbf{B}(\mathbf{V}^h_{i+1/2,R})\right)/2$ . For the discretization of the non-conservative equations  $\mathbf{P} = \mathbf{B}(\mathbf{V})\mathbf{V}_x$ , we multiply both sides with a test function  $\Psi^h \in \mathcal{V}^k$ , integrate over spacetime elements, and then add stability terms resulting in

$$\mathcal{B}^{2}(\mathbf{W}^{h}, \mathbf{\Psi}^{h}) = \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \mathbf{P}^{h} \cdot \mathbf{\Psi}^{h} \, \mathrm{d}x \, \mathrm{d}t$$

$$- \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \left( \mathbf{B}(\mathbf{V}^{h}) \mathbf{V}_{x}^{h} \right) \cdot \mathbf{\Psi}^{h} \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \frac{1}{2} \sum_{n,i} \int_{I_{n}} \left( \{ \mathbf{B}(\mathbf{V}^{h}) \}_{i+1/2} [\mathbf{V}^{h}]_{i+1/2} \right) \cdot \mathbf{\Psi}_{i+1/2,L}^{h} \, \mathrm{d}t$$

$$+ \frac{1}{2} \sum_{n,i} \int_{I_{n}} \left( \{ \mathbf{B}(\mathbf{V}^{h}) \}_{i-1/2} [\mathbf{V}^{h}]_{i-1/2} \right) \cdot \mathbf{\Psi}_{i-1/2,R}^{h} \, \mathrm{d}t.$$
(5.2)

We note that we did not apply integration by parts in this case. The method without streamline diffusion and shock capturing terms then reads: find  $\mathbf{W}^h = (\mathbf{V}^h, \mathbf{P}^h) \in \mathcal{V}^k \times \mathcal{V}^k$  such that

$$\mathcal{B}^{1}(\mathbf{W}^{h}, \mathbf{\Phi}^{h}) + \mathcal{B}^{2}(\mathbf{W}^{h}, \mathbf{\Psi}^{h}) = 0$$
(5.3)

for all  $\Sigma^h = (\Phi^h, \Psi^h) \in \mathcal{V}^k \times \mathcal{V}^k$ .

**Remark 5.2.** The method (5.3) is *consistent* for solving equation (3.1) if the exact solution V is continuous and if  $\mathbf{B}(\mathbf{V})$  depends continuously on V.

It remains to adjust  $\mathcal{B}_{SD}$  and  $\mathcal{B}_{SC}$  appropriately. We first modify the residual (given by (2.9) for the original method) to

$$\operatorname{Res}^{\operatorname{LDG}} = \mathbf{U}(\mathbf{V}^h)_t + \mathbf{F}(\mathbf{V})_x - \mathbf{B}(\mathbf{V}^h)_x \mathbf{P}^h - \mathbf{B}(\mathbf{V}^h) \mathbf{P}^h_x.$$

For the shock capturing term  $\mathcal{B}_{SC}^{LDG}$ , which is only added to the first set of equations, no further adjustment is necessary, *i.e.*,  $\mathcal{B}_{SC}^{LDG}(\mathbf{W}^h, \mathbf{\Phi}^h)$  is given by (2.11) and the equations following (2.11) with Res replaced by Res<sup>LDG</sup>. The streamline diffusion term  $\mathcal{B}_{SD}^{LDG}$  is split up in two parts:

(1) in the first set of equations, we use

$$\mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},1}(\mathbf{W}^h, \mathbf{\Phi}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} \left( \mathbf{U}_{\mathbf{V}}(\mathbf{V}^h) \mathbf{\Phi}_t^h + \mathbf{F}_{\mathbf{V}}(\mathbf{V}^h) \mathbf{\Phi}_x^h \right) \cdot \left( \mathbf{D}^{SD} \operatorname{Res}^{\mathrm{LDG}} \right) \, \mathrm{d}x \, \mathrm{d}t,$$

(2) in the second set of equations, we use

$$\mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},2}(\mathbf{W}^h, \mathbf{\Psi}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} \left( -\mathbf{B}(\mathbf{V}^h)_x \mathbf{\Psi}^h - \mathbf{B}(\mathbf{V}^h) \mathbf{\Psi}_x^h \right) \cdot \left( \mathbf{D}^{SD} \operatorname{Res}^{\mathrm{LDG}} \right) \, \mathrm{d}x \, \mathrm{d}t.$$

Then, the complete ST-LDG method is given by: find  $\mathbf{W}^h = (\mathbf{V}^h, \mathbf{P}^h) \in \mathcal{V}^k \times \mathcal{V}^k$  such that

$$\mathcal{B}_{\mathrm{ST-LDG}}(\mathbf{W}^h, \mathbf{\Sigma}^h) = \mathcal{B}^1(\mathbf{W}^h, \mathbf{\Phi}^h) + \mathcal{B}^2(\mathbf{W}^h, \mathbf{\Psi}^h) + \mathcal{B}^{\mathrm{LDG},1}_{\mathrm{SD}}(\mathbf{W}^h, \mathbf{\Phi}^h) + \mathcal{B}^{\mathrm{LDG},2}_{\mathrm{SD}}(\mathbf{W}^h, \mathbf{\Psi}^h) + \mathcal{B}^{\mathrm{LDG},2}_{\mathrm{SD}}(\mathbf{W}^h, \mathbf{W}^h) + \mathcal{B}^{\mathrm{LDG},2}_{\mathrm{SD}}(\mathbf{W$$

for all  $\Sigma^h = (\Phi^h, \Psi^h) \in \mathcal{V}^k \times \mathcal{V}^k$ .

We can show the following result concerning entropy stability.

**Theorem 5.3** (Entropy stability for ST-LDG). Let Assumption 3.1 hold true. Consider the system (3.1) with strictly convex entropy function S and entropy flux function Q. For simplicity, also assume that the exact and approximate solutions have compact support inside the spatial domain. Let the final time be denoted by  $t_N$ . Then, the approximate solutions generated by the scheme (5.4) satisfy

$$\int_{\Omega} S(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))) \, \mathrm{d}x + \int_{t_0}^{t_N} \int_{\Omega} (\mathbf{P}^{h})^2 \, \mathrm{d}x \, \mathrm{d}t \le \int_{\Omega} S(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))) \, \mathrm{d}x.$$

*Proof.* We set  $(\mathbf{\Phi}^h, \mathbf{\Psi}^h) = (\mathbf{V}^h, \mathbf{P}^h)$  to get

$$\begin{split} 0 &= \mathcal{B}_{\mathrm{ST}-\mathrm{LDG}}(\mathbf{W}^{h}, \mathbf{W}^{h}) \\ &= \mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},1}(\mathbf{W}^{h}, \mathbf{V}^{h}) + \mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},2}(\mathbf{W}^{h}, \mathbf{P}^{h}) + \mathcal{B}_{\mathrm{SC}}^{\mathrm{LDG}}(\mathbf{W}^{h}, \mathbf{V}^{h}) \\ &+ \mathcal{B}_{\mathrm{DG}}(\mathbf{V}^{h}, \mathbf{V}^{h}) + \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \mathbf{P}^{h} \cdot \mathbf{P}^{h} \, \mathrm{d}x \, \mathrm{d}t \\ &+ \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \left( \left( \mathbf{B}(\mathbf{V}^{h})\mathbf{P}^{h} \right) \cdot \mathbf{V}_{x}^{h} - \left( \mathbf{B}(\mathbf{V}^{h})\mathbf{V}_{x}^{h} \right) \cdot \mathbf{P}^{h} \right) \, \mathrm{d}x \, \mathrm{d}t \\ &- \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{B}(\mathbf{V}^{h}) \right\}_{i+1/2} \left\{ \mathbf{P}^{h} \right\}_{i+1/2} \right) \cdot \mathbf{V}_{i+1/2,L}^{h} \, \mathrm{d}t \\ &+ \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{B}(\mathbf{V}^{h}) \right\}_{i-1/2} \left\{ \mathbf{P}^{h} \right\}_{i-1/2} \right) \cdot \mathbf{V}_{i-1/2,R}^{h} \, \mathrm{d}t \\ &+ \frac{1}{2} \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{B}(\mathbf{V}^{h}) \right\}_{i+1/2} [\mathbf{V}^{h}]_{i+1/2} \right) \cdot \mathbf{P}_{i+1/2,L}^{h} \, \mathrm{d}t \\ &+ \frac{1}{2} \sum_{n,i} \int_{I_{n}} \left( \left\{ \mathbf{B}(\mathbf{V}^{h}) \right\}_{i-1/2} [\mathbf{V}^{h}]_{i-1/2} \right) \cdot \mathbf{P}_{i-1/2,R}^{h} \, \mathrm{d}t. \end{split}$$

We examine the single terms, starting from below. Taking the symmetry of  $\mathbf{B}$  and the compact support of the discrete solution into account, the boundary terms in the last four lines cancel each other. Also, the domain terms in the line above the boundary terms cancel each other. For the remaining terms, there holds

(1) for the streamline diffusion term

$$\mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},1}(\mathbf{W}^h, \mathbf{V}^h) + \mathcal{B}_{\mathrm{SD}}^{\mathrm{LDG},2}(\mathbf{W}^h, \mathbf{P}^h) = \sum_{n,i} \int_{I_n} \int_{K_i} \mathrm{Res}^{\mathrm{LDG}} \cdot \left(\mathbf{D}^{SD} \operatorname{Res}^{\mathrm{LDG}}\right) \, \mathrm{d}x \, \mathrm{d}t \ge 0$$

(2) for the shock capturing term

$$\mathcal{B}_{\rm SC}^{\rm LDG}(\mathbf{W}^h, \mathbf{V}^h) \ge 0,$$

(3) and for the  $\mathcal{B}_{DG}$  term due to Lemma 4.5

$$\mathcal{B}_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{V}^h) \ge \int_{\Omega} S(\mathbf{U}(\mathbf{V}^h_{N, -}(x))) \, \mathrm{d}x - \int_{\Omega} S(\mathbf{U}(\mathbf{V}^h_{0, -}(x))) \, \mathrm{d}x.$$

This then directly implies the claim.

### 6. Numerical results

In this section we present various numerical results, comparing the ST-NIPG, the ST-SIPG, and the ST-LDG method. In all our tests, the choice of the time step is purely based on the convection term, *i.e.*, it does not take

the presence of a diffusion term into account. We note that for stability reasons it is not necessary to restrict the time step due to the spacetime DG approach. But for accuracy reasons, we typically use the CFL condition

$$\Delta t_n \le C_{\text{CFL}} \min_{x \in \Omega} \frac{h}{\zeta_{\max}(\mathbf{U}_{n,-}^h(x))}$$

with  $\Delta t_n$  denoting the time step from  $t_n$  to  $t_{n+1}$  and  $C_{CFL} = 0.5$ . We also use equidistant grid cells in our tests, but this is not necessary.

Our code is an extension of the one-dimensional version of SPARCCLE - the software package developed for the original scheme for hyperbolic conservation laws [26]. We refer to [26, 27] for more detailed information concerning the implementation and only give a short summary here.

The approximate solution  $\mathbf{V}^h = (v_1^h, \dots, v_m^h)^T$  is sought in  $\mathcal{V}^k$  with each component  $v_j^h$  being of the form

$$v^h_j = \sum_{n,i,l} \hat{v}^n_{i,j,l} \phi^n_{i,l}$$

with n indicating the time segment, i the spatial cell, and  $1 \leq l \leq n_f$  the degree of freedom depending on the choice of  $\mathcal{V}^k$ . Further,  $\phi_{il}^n$  are basis functions with finite support given by

$$\phi_{i,l}^n\big|_{K_i \times I_n} = \left(\frac{t - t_{n+1}}{\Delta t_n}\right)^{k_{t,l}} \left(\frac{x - x_i}{h}\right)^{k_{x,l}}$$

with  $x_i$  denoting the centroid of the spatial cell *i*, and  $k_{t,l} + k_{x,l} \leq k$ . All spacetime and boundary integrals appearing in the numerical methods are evaluated using Gaussian quadrature formulae of the appropriate order.

While the form  $\mathcal{B}_{\text{DG}}$  is nonlinear in the discrete solution, it is linear in the test function. The same holds true for  $\mathcal{B}_{\text{SD}}^{(\cdot)}$  and  $\mathcal{B}_{\text{SC}}^{(\cdot)}$ . As a consequence, all terms appearing in the ST-SIPG, the ST-NIPG, and the ST-LDG methods are linear in the test function. Therefore, it is sufficient to satisfy (4.1) and (5.4) for all basis functions of  $\mathcal{V}^k$ . Due to the choice of the upwind flux in time, one can solve each time step separately. All in all, in each time step, one needs to solve a nonlinear system with  $N_c \times n_f \times m$  unknowns,  $N_c$  denoting the number of spatial cells. Newton method with an analytically computed Jacobian is used for this purpose. For test problems in one dimension, it is typically sufficient to use a sparse LU decomposition in order to solve the linear problem in each Newton iteration.

In the following, we will first show results for the scalar linear advection diffusion equation. Then, in Section 6.2, we will present results for the compressible Navier–Stokes equations.

#### 6.1. Numerical results for the linear advection diffusion equation

We start with the linear advection diffusion equation

$$u_t + cu_x = au_{xx}$$

with c and a constant (compare [9]). The initial data are chosen as

$$u(t=0,x) = \sin(x)$$

on the domain  $[0, 2\pi]$  with periodic boundary conditions. The exact solution is  $u(t, x) = e^{-at} \sin(x - ct)$ . Since we are interested in convection-dominated problems, we use the parameters c = 1 and  $a = 10^{-5}$ . We compute the  $L^1$  error at T = 2. We use the quadratic entropy function  $S(u) = \frac{1}{2}u^2$  and use central flux for the entropy conservative flux  $\mathbb{F}^*$ , *i.e.*,  $\mathbb{F}^*(a, b) = \frac{1}{2}(a + b)$ . We add Rusanov diffusion, which results in the numerical flux  $\mathbb{F}$ corresponding to standard upwind flux. The entropy stability condition for scalar equations for the ST-SIPG method is given by (4.9). In this fairly simple test, there holds  $\underline{a} = \overline{a} = a$  and m = M = 1. Therefore,



FIGURE 1. Lin. adv. diff. eqn. for  $a = 10^{-5}$  without  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms: The *x*-axis denotes the mesh width *h*, the *y*-axis the  $L^1$  error. All plots use the same axes scaling.



FIGURE 2. Lin. adv. diff. eqn. for a = 0.1 without  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms: The *x*-axis denotes the mesh width *h*, the *y*-axis the  $L^1$  error. All plots use the same axes scaling.

the stability condition requires  $\sigma > c_{inv}$ . We use  $\sigma = 20$  for the ST-SIPG method and  $\sigma = 10$  for the ST-NIPG method in the following tests.

As artificial diffusion terms are known to typically decrease the accuracy of a smooth solution, we initially drop the streamline diffusion and shock capturing terms in our methods. Figure 1 shows the results for the ST-SIPG method, the ST-NIPG method, and the ST-LDG for varying spaces  $\mathcal{V}^k$ , k = 0, 1, 2, 3. The result is (almost) identical for all three methods. We observe convergence orders of  $O(h^{k+1})$  for all three methods.

This result is unexpectedly good. For example, the ST-SIPG method and the ST-NIPG method are not consistent for  $\mathcal{V}^0$ , which should lead to a convergence stalling. This indicates that the diffusion term might be too small relative to the mesh width. Indeed, on the finest grid tested, we use the mesh width  $h \approx 10^{-2}$  which is significantly bigger than the diffusion coefficient  $a = 10^{-5}$ . Instead of using extremely fine grids (as the errors are already very small), we repeat the test with a = 0.1. For this test, the different discretizations of the diffusion operator should certainly make a difference.

Figure 2 shows the results for the ST-SIPG method, the ST-NIPG method, and the ST-LDG for a = 0.1, again without the artificial viscosity terms. For the ST-LDG method we observe the same convergence orders as before: the method converges with  $O(h^{k+1})$  for the spaces  $\mathcal{V}^k$ . Both the ST-SIPG and the ST-NIPG method do not converge for  $\mathcal{V}^0$ . This is consistent with the fact that for piecewise constant polynomials, most terms in the  $\mathcal{B}_{\rm IP}$  form drop out and only the penalty term is left. In addition, the ST-NIPG method shows the fairly well-known phenomenon (see Rivière [36]) of a reduced convergence order  $O(h^k)$  for k even.

Next, we repeat the test with a = 0.1 using the complete methods, *i.e.*, with streamline diffusion and shock capturing terms. We use  $\alpha = 0$  for the ST-IP methods. For this value, the artificial viscosity terms correspond



FIGURE 3. Lin. adv. diff. eqn. for a = 0.1 with  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms, using  $\alpha = 0$  for the ST-IP methods: The *x*-axis denotes the mesh width *h*, the *y*-axis the  $L^1$  error. All plots use the same axes scaling for better comparison.



FIGURE 4. Lin. adv. diff. eqn. for a = 0.1 with  $\mathcal{B}_{SD}^{IP}$  and  $\mathcal{B}_{SC}^{IP}$  terms,  $\alpha = 0.9$ : The x-axis denotes the mesh width h, the y-axis the  $L^1$  error. Both plots use the same axes scaling.

to a natural expansion of the terms used in the original method for conservation laws. The result is shown in Figure 3. For the ST-LDG method, we observe almost ideal convergence rates. For both the ST-SIPG and the ST-NIPG method, we observe suboptimal convergence rates of  $O(h^k)$ . We believe that this is due to the fact that  $\text{Res}^{\text{IP}}$  (different to Res and  $\text{Res}^{\text{LDG}}$ ) now involves second derivatives, see (4.3). This reduces the convergence order of the residual. Note that the complete ST-NIPG method (with  $\mathcal{B}_{\text{SD}}^{\text{IP}}$  and  $\mathcal{B}_{\text{SC}}^{\text{IP}}$  included) performs very similar to the complete ST-SIPG method for this test.

We repeat the test for the ST-IP methods, using both streamline diffusion and shock capturing terms but with  $\alpha = 0.9$ . The result is shown in Figure 4. For this value of  $\alpha$ , we observe almost the same convergence rates as in the test without using artificial viscosity terms (see Fig. 2).

#### 6.2. Numerical results for the compressible Navier–Stokes equations

In this section, we solve the compressible Navier–Stokes equations in one dimension given by (3.2) with Dirichlet boundary conditions and  $\gamma = 1.4$ . We use entropy stable flux for our experiments. We follow [29] for the entropy conservative flux  $\mathbb{F}^*$  for the convection terms. Entropy stable flux is attained by adding the Rusanov diffusion operator. We use  $\sigma = 10$  for the ST-NIPG and  $\sigma = 20$  for the ST-SIPG method.



FIGURE 5. Manufactured solution: The x-axis denotes the mesh width h, the y-axis the  $L^1$  error. All plots use the same axes scaling for better comparison.

#### 6.2.1. Manufactured solution

We start with a test that has a manufactured solution. To assess the accuracy of our methods, we like the solution to be given by the smooth functions

$$\rho(x,t) = \sin(x^2 + 5t) + 1.5,$$
  

$$u(x,t) = 2 \left[ \sin(x^2 + 5t) + 0.1 \right],$$
  

$$e(x,t) = 3 \left[ \cos(x^2 + 5t) + 1.5 \right].$$

with  $e = \frac{E}{\rho}$ . We insert this solution into the compressible Navier–Stokes equations and compute the corresponding source terms that need to be added on the right hand side of the equations to render the above triple  $(\rho, u, e)$  a solution of the resulting equations. For this test, we do not use our artificial viscosity terms  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  as they are not built to deal with source terms. In order to make sure that the viscosity is sufficiently big to be resolved by the grid, we use the viscosity coefficient  $\nu = 0.02$ . The test domain is given by  $\Omega = [-0.1; 0.9]$  and the final time is T = 0.05.

Figure 5 shows the combined  $L^1$  error for all three conserved quantities, *i.e.*,  $L^1(\rho) + L^1(m) + L^1(E)$ . We observe the expected convergence stalling for the ST-SIPG and the ST-NIPG method for  $\mathcal{V}^0$ . In addition, the ST-NIPG method converges with only second order for  $\mathcal{V}^2$ . (The convergence orders for the presented grids are (from coarse to fine) 3.05, 2.67, 2.46, dropping to 2.17 for the next level of refinement, which is not shown here.) Otherwise, the size of the errors computed with the ST-SIPG and the ST-NIPG method is very similar to the size of the errors computed with the ST-LDG method. We conclude that the results for this test are consistent with the results for the test involving the linear advection diffusion equation.

#### 6.2.2. Modified Sod test

Our next test is a modified Sod problem, similar to the test in [39]. We consider initial data

$$(\rho, m, E) = \begin{cases} (1.0, 0.0, 2.5) & \text{if} & x < 0, \\ (0.125, 0.0, 0.25) & \text{if} & x > 0, \end{cases}$$

on the domain  $\Omega = [-0.5, 0.5]$ . The viscosity  $\nu = 2.5 \times 10^{-5}$  is fairly small.

Figure 6 shows the result for velocity using entropy stable flux and  $\mathcal{V}^2$  with the final time T = 0.2. We first consider the case of a coarse grid width  $h = 2.0 \times 10^{-2}$ . In this case, the small viscosity  $\nu = 2.5 \times 10^{-5}$  cannot be resolved and the solution shows oscillations around the shock for all methods if no streamline diffusion and shock capturing terms are used. Using artificial viscosity terms with  $\alpha = 0.9$  for the ST-IP methods only slightly changes the solution. For this test case, these artificial viscosity terms seem not to be strong enough. When using



FIGURE 6. Modified Sod test: Comparison for the velocity component for the ST-IP method and the ST-LDG method for  $\mathcal{V}^2$ . (As the results for the ST-SIPG and the ST-NIPG method are very similar, we show them together and refer to the method as 'ST-IP' method.) The plots show the solution for using the methods both without  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms ("w/o") and with  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms ("w/o").

the ST-IP method with artificial viscosity terms with  $\alpha = 0$  or the LDG method with artificial viscosity terms, the oscillations are almost gone.

For a fine mesh width  $h = 5.0 \times 10^{-4}$ , the artificial diffusion terms are not necessary: The diffusion from the physical viscosity and heat conduction term is sufficient for reducing the oscillations around the shock. But the presence of the streamline diffusion and shock capturing terms also does not deteriorate the solution. Overall, we observe fairly similar behavior for all three methods (ST-SIPG, ST-NIPG, ST-LDG) for this test.

Finally, we examine the criterion for entropy stability for the ST-SIPG method for this test. At each time step  $t^n$ , we evaluate the quotient  $\max_x \lambda_2(\mathbf{V}_{n,-}^h(x))/\lambda_1(\mathbf{V}_{n,-}^h(x))$  with  $\lambda_{1,2}$  given by (3.5) and observe values between 7 and 14 for the computations with artificial diffusion terms and values between 8 and 16 for the computations without artificial diffusion terms, respectively. We also evaluate  $\Lambda^h/\lambda^h$  with  $\Lambda^h$  and  $\lambda^h$  given by (3.7). This results in values between 9 and 19 for the former case, and values between 9 and 22 for the latter case, *i.e.*, in fairly similar bounds. Based on the condition  $\sigma > \frac{c_{\text{inv}}\Lambda}{\lambda}$  given by (4.7), this would require to choose  $\sigma \approx 200$  for our tests involving polynomials of degree 2, which is a fairly high value. We repeat the test for the ST-SIPG method with this value for  $\sigma$  but did not observe a change in numerical results.



FIGURE 7. Modified Shu-Osher test: Comparison for the density component for the ST-IP method and the ST-LDG method for  $h = 1.0 \times 10^{-1}$ . (As the results for the ST-SIPG and the ST-NIPG method are fairly similar, we show them together and refer to the method as 'ST-IP' method.) The plots show the solution for using the methods both without  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms ("w/o") and with  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms ("w/").

### 6.2.3. Modified Shu-Osher test

Finally, to test the robustness of our scheme, we use a modified Shu-Osher test. We consider initial data

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.33333) & \text{if} & x < -4.0, \\ (1 + 0.2\sin(5.0x), 0.0, 1.0) & \text{if} & x > -4.0, \end{cases}$$

on the domain  $\Omega = [-5.0, 5.0]$  with final time T = 1.8. The viscosity is chosen as  $\nu = 4.0 \cdot 10^{-3}$  and we use entropy stable flux.

Figure 7 shows the computed solutions for density for  $\mathcal{V}^1$ ,  $\mathcal{V}^2$ , and  $\mathcal{V}^3$  for  $h = 1.0 \cdot 10^{-1}$  for the ST-IP methods and the LDG method. For this grid size, the physical diffusion cannot be resolved. As a consequence, we observe significant overshoot when not using artificial viscosity terms for all methods. When using the ST-LDG method with artificial viscosity terms or the ST-IP methods with artificial viscosity terms and  $\alpha = 0$ , the overshoot is gone. Different to the modified Sod test, the artificial viscosity terms for the ST-IP methods with  $\alpha = 0.9$ also do fairly well for this test case. We observe again that the ST-LDG and the ST-IP methods lead to very comparable results, both when no artificial diffusion terms are used and when artificial diffusion terms with  $\alpha = 0$  are used.

Finally, Figures 8 shows the results for  $\mathcal{V}^1$  using the mesh width  $h = 4.0 \times 10^{-3}$ . In this case, there is no real need for artificial viscosity terms but the numerical results confirm that their presence also does not deteriorate the numerical solution.



FIGURE 8. Modified Shu-Osher test: Comparison for the density component for the ST-IP method and the ST-LDG method for  $h = 4.0 \times 10^{-3}$ . (As the results for the ST-SIPG and the ST-NIPG method are very similar, we show them together and refer to the method as 'ST-IP' method.) The plots show the solution for using the methods both without  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms ("w/o") and with  $\mathcal{B}_{SD}^{(\cdot)}$  and  $\mathcal{B}_{SC}^{(\cdot)}$  terms ("w/").

	ST-NIPG	ST-SIPG	ST-LDG
Entropy stability:			
• Ass. on matrix $\mathbf{A}$ in (3.1)	Ass. 3.1	Ass. 3.2	Ass. 3.1
• applies to compr. NS eqns	$\checkmark$	(3.7) & (4.7)	$\checkmark$
Num. results for $\mathcal{V}^k, k \geq 1$ :			
• smooth flow			
$\circ \mathrm{~w/o}~\mathcal{B}_\mathrm{SD}^{(\cdot)}~\&~\mathcal{B}_\mathrm{SC}^{(\cdot)}$	$O(h^k) / O(h^{k+1})$	$O(h^{k+1})$	$O(h^{k+1})$
$\circ \text{ w} / \mathcal{B}_{SD}^{(\cdot)} \& \mathcal{B}_{SC}^{(\cdot)}, \alpha = 0 \text{ for ST-IP}$	$O(h^k)$	$O(h^k)$	$O(h^{k+1})$
$\circ \text{ w} / \mathcal{B}_{SD}^{(\cdot)} \& \mathcal{B}_{SC}^{(\cdot)}, \alpha = 0.9 \text{ for ST-IP}$	$O(h^k) / O(h^{k+1})$	$O(h^{k+1})$	$O(h^{k+1})$
• shock problem			
$\circ \mathrm{~w/o} ~ \mathcal{B}_{\mathrm{SD}}^{(\cdot)} ~ \& ~ \mathcal{B}_{\mathrm{SC}}^{(\cdot)}$	—— very comparable ——		
$\circ \le \mathcal{B}_{SD}^{(\cdot)} \& \mathcal{B}_{SC}^{(\cdot)}, \alpha = 0 \text{ for ST-IP}$	—— very comparable ——		
$\circ \text{ w} / \mathcal{B}_{SD}^{(\cdot)} \& \mathcal{B}_{SC}^{(\cdot)}, \alpha = 0.9 \text{ for ST-IP}$	solution for ST-IP has more oscillations than for ST-LDG		

TABLE 1. Comparison of the ST-NIPG, the ST-SIPG, and the ST-LDG method.

This completes the presentation of our numerical results. We conclude this work with a direct comparison of the ST-IP methods and the ST-LDG method.

### 7. COMPARISON OF THE ST-NIPG, THE ST-SIPG, AND THE ST-LDG METHOD

We summarize our main findings in Table 1. Based on our results, all methods have their pros and cons.

The ST-LDG method is optimal in the sense that it is entropy stable, it shows optimal rates of convergence in smooth flow, and it is robust if the artificial viscosity terms are used. However, it is also the most expensive method due to the introduction of the auxiliary variables  $\mathbf{P}$ . Furthermore, a suitable matrix  $\mathbf{B}$  needs to be found for the method to work.

Also the comparison between ST-SIPG and ST-NIPG does not clearly favor one of them. The ST-NIPG is automatically entropy stable and avoids having to choose a suitable value for the penalty parameter  $\sigma$ , which is necessary for the ST-SIPG method and can lead to high values for  $\sigma$ . On the other hand, the ST-SIPG method

has a better order of convergence for even polynomial degrees. The ST-SIPG method also has the additional advantage of being symmetric, which is important for adjoint consistency.

One open problem for the ST-IP methods is the development of better streamline diffusion and shock capturing terms. We presented two versions, one using  $\alpha = 0$  and one using  $\alpha = 0.9$ . The former one is more robust in the vicinity of shocks (compare the modified Sod test) but causes suboptimal convergence rates in smooth flow; the latter one has optimal convergence rates and adds sufficient diffusion for some tests (compare the modified Shu-Osher test), but not for all. In order to find a suitable transition between these two versions, thorough testing, especially on physically relevant problems in two dimensions, will be necessary.

Concerning other aspects of the extension to two and three dimensions: The proofs of entropy stability of the ST-NIPG and the ST-LDG methods should extend in a fairly straight-forward way to higher dimensions. For the proof of entropy stability of the ST-SIPG method additional work will be necessary as the physical diffusion matrix  $\mathbf{A} \in \mathbb{R}^{8\times8}$  in two dimensions only has rank 5. But exploiting the structure of  $\mathbf{A}$ , we are very positive that it will be possible to transfer the proofs from one dimension to two. In terms of the practical aspects of the methods, the implementation of the ST-NIPG and the ST-SIPG will be fairly straight-forward. However, we expect the development of a suitable preconditioner to cause problems. The implementation for the ST-LDG method will be fairly complex: besides solving the challenging task of finding a suitable matrix  $\mathbf{B}$ , it will be essential to find a way of eliminating the auxiliary variables as otherwise the nonlinear solves will turn out to be too expensive.

### Appendix A. Proof of Lemma 4.5

*Proof.* The proof of (4.8) can be transferred from the proof of the entropy stability of the original scheme. Therefore, we do not give all details here. The full proof can be found in ([26], Thm 3.1). **Step 1.** Define the spatial part

$$\begin{aligned} \mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{\Phi}^h) &= -\sum_{n,i} \int_{I_n} \int_{K_i} \mathbf{F}(\mathbf{V}^h) \cdot \mathbf{\Phi}^h_x \, \mathrm{d}x \, \mathrm{d}t \\ &+ \sum_{n,i} \int_{I_n} \left( \mathbb{F}(\mathbf{V}^h_{i+1/2,L}, \mathbf{V}^h_{i+1/2,R}) \cdot \mathbf{\Phi}^h_{i+1/2,L} - \mathbb{F}(\mathbf{V}^h_{i-1/2,L}, \mathbf{V}^h_{i-1/2,R}) \cdot \mathbf{\Phi}^h_{i-1/2,R} \right) \, \mathrm{d}t. \end{aligned}$$

One can show that  $\mathcal{B}^s_{\mathrm{DG}}(\mathbf{V}^h, \mathbf{V}^h) \ge 0$ : Due to the definition of the entropy flux function Q, there holds for the entropy potential  $\psi_x = \mathbf{V}_x \cdot \mathbf{F}$ . This implies

$$\begin{aligned} \mathcal{B}^{s}_{\mathrm{DG}}(\mathbf{V}^{h},\mathbf{V}^{h}) &= -\sum_{n,i} \int_{I_{n}} \int_{K_{i}} \psi(\mathbf{V}^{h})_{x} \,\mathrm{d}x \,\mathrm{d}t \\ &+ \sum_{n,i} \int_{I_{n}} \left( \mathbb{F}(\mathbf{V}^{h}_{i+1/2,L},\mathbf{V}^{h}_{i+1/2,R}) \cdot \mathbf{V}^{h}_{i+1/2,L} - \mathbb{F}(\mathbf{V}^{h}_{i-1/2,L},\mathbf{V}^{h}_{i-1/2,R}) \cdot \mathbf{V}^{h}_{i-1/2,R} \right) \,\mathrm{d}t. \end{aligned}$$

Evaluating the (spatial) integral over  $\psi_x$  and using the definition of the flux F from (2.5) gives

$$\begin{split} \mathcal{B}^{s}_{\mathrm{DG}}(\mathbf{V}^{h},\mathbf{V}^{h}) &= \sum_{n,i} \int_{I_{n}} \left( \mathbb{F}^{*}(\mathbf{V}^{h}_{i+1/2,L},\mathbf{V}^{h}_{i+1/2,R}) \cdot \mathbf{V}^{h}_{i+1/2,L} - \psi(\mathbf{V}^{h}_{i+1/2,L}) \right) \, \mathrm{d}t \\ &- \sum_{n,i} \int_{I_{n}} \left( \mathbb{F}^{*}(\mathbf{V}^{h}_{i-1/2,L},\mathbf{V}^{h}_{i-1/2,R}) \cdot \mathbf{V}^{h}_{i-1/2,R} - \psi(\mathbf{V}^{h}_{i-1/2,R}) \right) \, \mathrm{d}t \\ &- \frac{1}{2} \sum_{n,i} \int_{I_{n}} \mathbf{V}^{h}_{i+1/2,L} \cdot \mathbb{D}(\mathbf{V}^{h}_{i+1/2,R} - \mathbf{V}^{h}_{i+1/2,L}) \, \mathrm{d}t \\ &+ \frac{1}{2} \sum_{n,i} \int_{I_{n}} \mathbf{V}^{h}_{i-1/2,R} \cdot \mathbb{D}(\mathbf{V}^{h}_{i-1/2,R} - \mathbf{V}^{h}_{i-1/2,L}) \, \mathrm{d}t. \end{split}$$

Reordering the sum and exploiting the compact support of the approximate solutions results in

$$\begin{aligned} \mathcal{B}_{\mathrm{DG}}^{s}(\mathbf{V}^{h},\mathbf{V}^{h}) &= -\sum_{n,i} \int_{I_{n}} \left\{ \mathbb{F}^{*}(\mathbf{V}_{i+1/2,L}^{h}\mathbf{V}_{i+1/2,R}^{h}) \cdot \left(\mathbf{V}_{i+1/2,R}^{h} - \mathbf{V}_{i+1/2,L}^{h}\right) - (\psi(\mathbf{V}_{i+1/2,R}^{h}) - \psi(\mathbf{V}_{i+1/2,L}^{h})) \right\} \mathrm{d}t \\ &+ \frac{1}{2} \sum_{n,i} \int_{I_{n}} \left(\mathbf{V}_{i+1/2,R}^{h} - \mathbf{V}_{i+1/2,L}^{h}\right) \cdot \mathbb{D}(\mathbf{V}_{i+1/2,R}^{h} - \mathbf{V}_{i+1/2,L}^{h}) \, \mathrm{d}t. \end{aligned}$$

The terms in the first sum cancel due to (2.4), the terms in the second sum are non-negative due to the definition of the diffusion operator  $\mathbb{D}$ .

Step 2. Define the temporal part

$$\mathcal{B}_{\mathrm{DG}}^{t}(\mathbf{V}^{h}, \mathbf{\Phi}^{h}) = -\sum_{n,i} \int_{I_{n}} \int_{K_{i}} \mathbf{U}(\mathbf{V}^{h}) \cdot \mathbf{\Phi}_{t}^{h} \, \mathrm{d}x \, \mathrm{d}t + \sum_{n,i} \int_{K_{i}} \left( \mathbf{U}(\mathbf{V}_{n+1,-}^{h}) \cdot \mathbf{\Phi}_{n+1,-}^{h} - \mathbf{U}(\mathbf{V}_{n,-}^{h}) \cdot \mathbf{\Phi}_{n,+}^{h} \right) \, \mathrm{d}x.$$

Set  $\Phi^h = \mathbf{V}^h$  and use integration by parts with respect to time. The boundary terms evaluated at  $t_{n+1,-}$  cancel resulting in

$$\mathcal{B}_{\mathrm{DG}}^{t}(\mathbf{V}^{h},\mathbf{V}^{h}) = \sum_{n,i} \int_{I_{n}} \int_{K_{i}} \mathbf{U}(\mathbf{V}^{h})_{t} \cdot \mathbf{V}^{h} \, \mathrm{d}x \, \mathrm{d}t + \sum_{n,i} \int_{K_{i}} \left( \mathbf{U}(\mathbf{V}_{n,+}^{h}) \cdot \mathbf{V}_{n,+}^{h} - \mathbf{U}(\mathbf{V}_{n,-}^{h}) \cdot \mathbf{V}_{n,+}^{h} \right) \, \mathrm{d}x.$$

By the definition of the entropy function,  $\mathbf{U}(\mathbf{V}^h)_t \cdot \mathbf{V}^h = S(\mathbf{U}(\mathbf{V}^h))_t$ . Evaluating the time integral and adding a zero-sum involving  $S(\mathbf{U}(\mathbf{V}^h_{n,-}))$ , this implies

$$\begin{aligned} \mathcal{B}_{\mathrm{DG}}^{t}(\mathbf{V}^{h},\mathbf{V}^{h}) &= \sum_{n,i} \int_{K_{i}} \left( S(\mathbf{U}(\mathbf{V}_{n+1,-}^{h})) - S(\mathbf{U}(\mathbf{V}_{n,-}^{h})) \right) \, \mathrm{d}x \\ &+ \sum_{n,i} \int_{K_{i}} \left( S(\mathbf{U}(\mathbf{V}_{n,-}^{h})) - S(\mathbf{U}(\mathbf{V}_{n,+}^{h})) \right) \, \mathrm{d}x + \sum_{n,i} \int_{K_{i}} \left( \mathbf{U}(\mathbf{V}_{n,+}^{h}) - \mathbf{U}(\mathbf{V}_{n,-}^{h}) \right) \cdot \mathbf{V}_{n,+}^{h} \, \mathrm{d}x. \end{aligned}$$

The first sum corresponds to a telescoping sum. For the second sum, the change of variables  $\mathbf{V}(\xi) = \mathbf{V}_{n,+}^h + \xi(\mathbf{V}_{n,-}^h - \mathbf{V}_{n,+}^h), \xi \in [0,1]$ , is used, resulting in

$$S(\mathbf{U}(\mathbf{V}_{n,-}^{h})) - S(\mathbf{U}(\mathbf{V}_{n,+}^{h})) = \int_{0}^{1} \frac{\mathrm{d}}{\mathrm{d}\xi} \left( S(\mathbf{U}(\mathbf{V}(\xi))) \right) \mathrm{d}\xi = \int_{0}^{1} S_{\mathbf{U}}(\mathbf{U}(\mathbf{V}(\xi))) \cdot \left(\mathbf{U}_{\mathbf{V}}(\mathbf{V}(\xi))\mathbf{V}_{\xi}(\xi)\right) \mathrm{d}\xi$$
$$= \mathbf{V}_{n,+}^{h} \cdot \left(\mathbf{U}(\mathbf{V}_{n,-}^{h}) - \mathbf{U}(\mathbf{V}_{n,+}^{h})\right) + \int_{0}^{1} \xi \left(\mathbf{V}_{n,-}^{h} - \mathbf{V}_{n,+}^{h}\right) \cdot \left(\mathbf{U}_{\mathbf{V}}(\mathbf{V}(\xi))\left(\mathbf{V}_{n,-}^{h} - \mathbf{V}_{n,+}^{h}\right)\right) \mathrm{d}\xi.$$

Here, we used  $S_{\mathbf{U}}(\mathbf{U}(\mathbf{V}(\xi))) = \mathbf{V}(\xi)$ . Compare ([25], p. 26) for more details. Putting everything together leads to

$$\mathcal{B}_{\mathrm{DG}}^{t}(\mathbf{V}^{h},\mathbf{V}^{h}) = \int_{\Omega} \left( S(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))) - S(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))) \right) \,\mathrm{d}x \\ + \sum_{n,i} \int_{K_{i}} \int_{0}^{1} \xi \,\left(\mathbf{V}_{n,-}^{h} - \mathbf{V}_{n,+}^{h}\right) \cdot \left(\mathbf{U}_{\mathbf{V}}(\mathbf{V}(\xi))(\mathbf{V}_{n,-}^{h} - \mathbf{V}_{n,+}^{h})\right) \,\mathrm{d}\xi \,\mathrm{d}x.$$

Due to S being strictly convex, the terms in the second line are positive, implying

$$\mathcal{B}_{\mathrm{DG}}^{t}(\mathbf{V}^{h},\mathbf{V}^{h}) \geq \int_{\Omega} S\left(\mathbf{U}(\mathbf{V}_{N,-}^{h}(x))\right) \,\mathrm{d}x - \int_{\Omega} S\left(\mathbf{U}(\mathbf{V}_{0,-}^{h}(x))\right) \,\mathrm{d}x.$$

This concludes the proof.

## Appendix B. Using the ST-LDG method for solving the compressible Navier–Stokes equations

The ST-LDG method is based on the existence of a positive semi-definite matrix  $\mathbf{B}(\mathbf{V})$  such that  $\mathbf{B}^2 = \mathbf{A}$  with  $\mathbf{A}$  given by (3.1). In the following we describe the matrix  $\mathbf{B}$  that we use in our numerical tests for the compressible Navier–Stokes equations.

Instead of decomposing the matrix  $\mathbf{A}$  we decompose the reduced matrix  $\mathbf{\hat{A}}$  given by (3.4). We can write  $\mathbf{\tilde{A}} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}^{-1}$  with the matrix  $\mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2)$  containing the positive eigenvalues of  $\mathbf{\tilde{A}}$  and the columns of  $\mathbf{C}$  containing the corresponding eigenvectors. We then define  $\mathbf{\tilde{B}} = \mathbf{C}\mathbf{\Lambda}^{1/2}\mathbf{C}^{-1}$ . In our tests we use

$$\mathbf{C} = \begin{pmatrix} \frac{1}{N_1} \left( \lambda_1 + \nu \frac{v_2^2}{v_3^3} - \frac{\kappa}{R} \frac{1}{v_3^2} \right) & \frac{1}{N_2} \cdot \nu \frac{v_2}{v_3^2} \\ \frac{1}{N_1} \cdot \nu \frac{v_2}{v_3^2} & \frac{1}{N_2} \left( \lambda_2 + \frac{\nu}{v_3} \right) \end{pmatrix}$$

with  $N_1$  and  $N_2$  representing the appropriate normalization factors given by

$$N_1 = \sqrt{\left(\lambda_1 + \nu \frac{v_2^2}{v_3^3} - \frac{\kappa}{R} \frac{1}{v_3^2}\right)^2 + \left(\nu \frac{v_2}{v_3^2}\right)^2} \quad \text{and} \quad N_2 = \sqrt{\left(\nu \frac{v_2}{v_3^2}\right)^2 + \left(\lambda_2 + \frac{\nu}{v_3}\right)^2}$$

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