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A SURVEY ON TRANSITIVITY IN DISCRETE TIME DYNAMICAL SYSTEMS. APPLICATION TO SYMBOLIC SYSTEMS AND RELATED LANGUAGES *

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Abstract. The main goal of this paper is the investigation of a relevant property which appears in the various definition of deterministic topological chaos for discrete time dynamical system: transitivity. Starting from the standard Devaney's notion of topological chaos based on regularity, transitivity, and sensitivity to the initial conditions, the critique formulated by Knudsen is taken into account in order to exclude periodic chaos from this definition. Transitivity (or some stronger versions of it) turns out to be the relevant condition of chaos and its role is discussed by a survey of some important results about it with the presentation of some new results. In particular, we study topological mixing, strong transitivity, and full transitivity. Their applications to symbolic dynamics are investigated with respect to the relationships with the associated languages.

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1. Introduction

Discrete Time Dynamical Systems (DTDS) are mathematical objects studied in a large number of disciplines with different purposes showing some behaviors which are not always recovered by the continuous case. Formally a DTDS is a pair $\langle X, g \rangle$, where the *state space* X is a nonempty set equipped with a metric d

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and the *next state function* $g : X \mapsto X$ is a transformation of X continuous with respect to the metric d. The next state function induces deterministic dynamics by its iterated application starting from a given initial state. A DTDS $\langle X, g\rangle$ is said to be *reversible or homeomorphic* iff the next state map $g : X \mapsto X$ is bijective (one-to-one and onto) and bicontinuous (both $g : X \mapsto X$ and its inverse $g^{-1} : X \mapsto X$ must be continuous). For a general introduction to DTDS with the involded notions see [17, 22, 27].

Several theoretical properties (*e.g.*, transitivity, ergodicity, sensitivity to the initial conditions, expansivity, denseness of periodic orbits, ...) which describe some behaviors of DTDS have been investigated. The more intriguing of these properties, at least at a level of popular divulgement, is sensitive dependence to the initial conditions which is recognized as a central notion in chaos theory since it captures the feature that in chaotic systems small errors in experimental readings lead to large scale divergence, *i.e.*, the system is *unpredictable*. For instance in [18] it is claimed that: "sensitive dependence on initial conditions is also expressed by saying that the system is *chaotic*". Its formal definition is the following one:

Definition 1.1 (Sensitivity). A DTDS $\langle X, g\rangle$ is sensitive to the initial conditions iff

$$\exists \epsilon > 0\ \forall x \in X\ \forall \delta > 0\ \exists y \in X\ \exists n \in \mathbb{N}: \quad d(x, y) < \delta \text{ and } d(g^n(x), g^n(y)) \geq \epsilon.$$

The constant ϵ is called the *sensitivity constant.* Note that if a DTDS is sensitive then trivially it is *perfect* (that is, without isolated points) and also *infinite*, moreover, the state y which is involved in this definition is necessarily $y \neq x$.

The sensitivity only, notwithstanding its intuitive glamor in defining a chaotic DTDS, has the withdraw that, once taken as the unique condition characterizing chaos, it is not sufficient or so intuitive as it seems at a first glance. These considerations induce to investigate definitions of deterministic chaos in which, besides sensitivity, some other properties characterizing chaos and avoiding this kind of non–intuitive behaviors must be involved.

In the definition of *topological chaos* for DTDS introduced by Devaney in [17], the essential components which are involved are transitivity, denseness of periodic orbits (also called regularity), and sensitivity. Several studies treat the relationships among these properties. An important result from the Devaney point of view can be found in [5], where it is proved that for spaces of infinite cardinality the two conditions of transitivity and regularity imply sensitivity. "That is, despite its popular appeal, sensitive dependence is mathematically redundant–so that in fact, [Devaney's] chaos is a property relying only on the topological, and not on the metric, properties of the space." [15].

Knudsen in [24] proved the following results. Let $\langle X, g\rangle$ be a DTDS and $\langle Y, g\rangle$ be any sub-DTDS dense in X [*i.e.*, $g(Y) \subseteq Y$ and $\overline{Y} = X$]. Then: (1) $\langle X, g\rangle$ is unpredictable iff $\langle Y, g\rangle$ is unpredictable; and (2) $\langle X, g\rangle$ is transitive iff $\langle Y, g\rangle$ is transitive. Quoting Knudsen [24]:

"Let us describe some consequences of the two [results]. Take a dynamical system [$\langle X, g\rangle$] that is chaotic according to Devaney's definition. Consider the restriction of the dynamics to the set of periodic points which is clearly invariant. (Above (1) and (2)) imply that the restricted dynamical system (of all periodic points) is topological transitive, and that the system exhibits dependence to initial conditions. Together with the trivial fulfilled condition of denseness of periodic points, this implies that the system is chaotic. Due to the lack of nonperiodicity this is not the kind of system most people would consider labelling chaotic. It might, of course, be argued that since for any periodic point there is another periodic point close by with arbitrary high period, and therefore this is a kind of chaos. Nevertheless, insisting on the distinction between a truly nonperiodic orbit, say, a dense orbit, and a periodic orbit of even arbitrary high period, such a system should not be termed chaotic."

As a consequence of these results, Knudsen proposed "the following definition of chaos which excludes chaos without nonperiodicity" [24] assuming that a DTDS is *Knudsen chaotic* (or *K-chaotic*) iff it has a dense orbit and is sensitive.

Example 1.2. Let us consider the logistic DTDS $\langle [0,1], g\rangle$ where g is defined as $\forall x \in [0,1], g(x) = 4x(1-x)$. It is well known that this system is Devaney chaotic and possesses a dense orbit. By removing from $[0,1]$ all the periodic points and their pre-images, we obtain a DTDS which is transitive, sensitive and inherits the dense orbit. Thus the obtained DTDS is Knudsen but not Devaney chaotic.

Let us recall that sensitivity implies perfectness and as shown below the existence of a dense orbit *plus* perfectness imply transitivity (see Prop. 2.9, Sect. 2); hence Knudsen chaos implies a DTDS which is transitive and sensitive. In other words, in this definition of chaos regularity is omitted.

However, we observe that the only condition obtained considering transitivity and sensitivity together may not be sufficient to consider a DTDS as chaotic. Indeed we illustrate a concrete example of transitive and sensitive DTDS in which there exists a unique equilibrium point which attracts every orbit, this latter being a quite non chaotic behavior. Before showing such a system, we recall that the one-sided (resp., two-sided) full shift on the finite alphabet $\mathcal{A}$ is the DTDS $\left\langle \mathcal{A}^{\mathbb{K}}, \sigma \right\rangle$, with $\mathbb{K} = \mathbb{N}$ (resp., $\mathbb{K} = \mathbb{Z}$), where σ is the left shift mapping on the space $\mathcal{A}^{\mathbb{K}}$ defined as: $\forall \underline{x} \in \mathcal{A}^{\mathbb{K}}, \forall i \in \mathbb{K}, [\sigma(\underline{x})]_i = x_{i+1}$. The set $\mathcal{A}^{\mathbb{K}}$ is endowed with the Tychonoff metric $d_T(\underline{x}, \underline{y}) = \sum_{i\in\mathbb{K}} \frac{1}{4^{|i|}} h(x_i, y_i)$, with h the Hamming distance on $\mathcal{A}$. Let us note that with respect to this metric, the left shift on $\mathcal{A}^{\mathbb{Z}}$ is a uniformly continuous map. Moreover it is a bijection and thus it defines a reversible DTDS.

It is well known that any full shift is Devaney and Knudsen chaotic.

Example 1.3. The left shift on the state space $\Sigma \subset \mathcal{A}^{\mathbb{N}}$ of all definitively null one-sided sequences on a given alphabet $\mathcal{A}$ containing a zero element 0 has the null sequence $\underline{0} = (0, 0, 0, \ldots)$ as the unique equilibrium point, which is a global (finite steps) attractor. But this DTDS is transitive and sensitive to the initial conditions due to the Knudsen results since Σ is dense in $\mathcal{A}^{\mathbb{N}}$.

We remark (see Cor. 2.12, Sect. 2) that if a compact DTDS $\langle X, g\rangle$ is chaotic in the sense of Devaney, then it is chaotic in the sense of Knudsen. Another possibility is to define as chaotic a *perfect* DTDS which has a dense orbit and is regular. Let us note that if a DTDS is only perfect, then the existence of a dense orbit implies transitivity (see Cor. 2.9, Sect. 2), having that also in this case the new definition of chaos implies Devaney chaos. If one considers a compact and perfect (or alternatively, compact and surjective) DTDS then the two definitions of chaos coincide since in this case the existence of a dense orbit is equivalent to transitivity (see Cor. 2.13, Sect. 2).

In the case of DTDS induced by Cellular Automata (CA) (for an introduction to CA as DTDS and their realtionships to chaos see for instance [13]), the simple condition of transitivity implies sensitivity [14] and it is conjectured (and proved only for some classes [8, 9, 12]) that surjective CA, and then transitive CA, are regular. For infinite Subshifts of Finite Type (SFT) and for DTDS on intervals of the real line, transitivity is equivalent to Devaney chaos [11, 35]. Moreover, topological transitivity is an important property of DTDS which is connected to various other disciplines. Several studies deal with the relationships between transitivity and ergodicity [7, 16, 22, 36, 37].

In this paper we focus our attention to the notion of *transitivity* as an important (but not *unique*) component of chaos since, as shown by the above brief discussion, there exists alternative definitions of chaos centered in a modification of transitivity, leaving invariant the sensitivity condition. As to this approach, in literature one can found several definitions of "transitivity" and we make an investigation, with suitable results and counter-examples, about this notions. Once renamed as positive transitivity the Devaney notion, we consider the notion of full transitivity, which in the particular case of compact and homeomorphic DTDS can be found in [16, 36]. We study the relationships among these two notions and other topological properties. In particular we show that these two notions coincide in the following cases: the perfect DTDS and the DTDS whose states are all non-wandering points (extending the same result already proved in [36] for compact and homeomorphic DTDS). As a consequence, since regularity implies that all states are non-wandering, we have that in the Devaney definition of chaos the notion of transitivity can be substituted by the weaker notion of full transitivity.

We also consider some conditions stronger than positive transitivity, in particular the two notions of *mixing* and of *strong transitivity*. Similarly to a result obtained by Kurka in [27] relatively to mixing, we show that a strongly transitive DTDS with at least two disjoint *periodic* orbits is sensitive. In this way one obtains two *Knudsen–like* definitions of chaos, *i.e.*, without regularity, in which giving as fixed the sensitivity condition some property which implies positive transitivity is assumed to hold.

There are strongly transitive DTDS, as the one-sided full shift, which are both Devaney and Knudsen chaotic, but there are also other DTDS, as the irrational rotations of the circle (Rem. 2.21, Sect. 2), which are strongly transitive, not regular and not sensitive (and so neither Devaney nor Knudsen chaotic). In effects in the setting of homeomorphic and compact DTDS, strong transitivity reduces to the classical notion of minimality (see for instance [21, 36]). As consequence of this fact we have that no strongly transitive DTDS can be Devaney chaotic.

Finally, we study the above properties in the case of Subshifts on a finite alphabet $\mathcal{A}$. They are DTDS whose state space is constituted by a suitable set of sequences of $\mathcal{A}^{\mathbb{K}}$ (either $\mathbb{K} = \mathbb{Z}$ or $\mathbb{K} = \mathbb{N}$) which is endowed with the Tychonoff metric. We recall that with respect to this metric $\mathcal{A}^{\mathbb{K}}$ turns out to be a *Cantor space* (*i.e.*, a compact, perfect and totally disconnected space, [38], p. 99). The topology induced from the Hamming distance on the alphabet is the *discrete* one, in which any collection of letters from the alphabet is a clopen subset. The topology induced by the Tychonoff distance on $\mathcal{A}^{\mathbb{K}}$ is just the product topology induced from the discrete topology of $\mathcal{A}$. Let us recall that this topology is the coarsest one with respect to the pointwise convergence of sequences.

The importance of Subshifts is due to the fact that to many DTDS it is possible to associate a suitable Subshift, encoding the states of the system as symbolic sequences. In this way one can understand some dynamical aspects of the original system by investigating the behavior of the associated Susbhift (see for instance [22, 27]). Subshifts have also found significant application in different disciplines, *e.g.*, data storage and transmission, coding and linear algebra [29], and they are studied in the field of language theory [6,11,29]. Indeed a formal language is canonically associated to any subshift. An important subclass of these symbolic systems is constituted by Subshifts of Finite Type (SFT) [31], that is, by those subshifts which can be described by a finite set of words and represented (and then investigated) by a directed graph and a matrix [29]. In [10] the SFT behavior of CA has been investigated and it has been illustrated how to associate to any SFT a CA which contains it. Moreover it has been proved that the class of the SFT turns out to coincide with the class of SFT contained in CA.

We conclude this section with a remark based on Example 1.3 where an unpredictable system with a unique globally attracting equilibrium point has been described. In order to avoid this kind of pathological behavior strictly linked to the requirement that the sensitivity is obtained after a finite number of time steps n making no requirement about the behavior in the successive instants, we also propose a modified, and stronger notion of sensitivity

Definition 1.4 (Strong Sensitivity). A DTDS $\langle X, g\rangle$ is strongly sensitive to the initial conditions iff

$$\exists \epsilon > 0\ \forall x \in X\ \forall \delta > 0\ \exists y \in X\ \exists n_0 \in \mathbb{N}\colon d(x,y) < \delta \text{ and } \forall n \geq n_0,\ d(g^n(x), g^n(y)) \geq \epsilon$$

where the main difference with respect to the definition 1.1 consists in the requirement that the distance between the two orbits must remain greater that ϵ for all time steps after the first instant n_0. It is easy to check that any full shift $\langle \mathcal{A}^{\mathbb{K}}, \sigma \rangle$ is strongly sensitive.

As a final conclusion of this introduction, we are aware that the Devaney definition of chaos is from different points of view unsatisfactory. For instance we have above discussed the Knudsen criticisms about full periodic chaos, but of course it is not the unique criticism. In our opinion, the two essential notions of chaos are transitivity and sensitivity or some variations of their. This is particularly true in the wide class of DTDS generated by (one-dimensional, two-sided) CA. Indeed, notwithstanding the simplicity of the local rule, the induced global dynamics show a great variety of dynamical behaviors from the simpler ones (unique global attracting equilibrium point) to the more and more complicated (shift and subshift, fractal-like space-time diagram , see for instance [13]). In the CA global dynamics transitivity implies sensitivity [14], and so transitivity is equivalent to K-chaos. In the particular case of elementary (binary of radius 1, see [13]) CA we have the stronger result that transitivity is equivalent to both K and D-chaos. So, at least in this wide class of dynamical systems transitivity captures two of the possible definitions of chaos.

These are the reasons which induce us to investigate in this paper transitivity, with some of its variations such as full transitivity, topological mixing and strong transitivity, with a particular regard to the context of symbolic dynamics. Of course in literature one can recently find a discussion about variations of sensitivity [2, 25]. A particular interesting new definition of chaos can be found in [33] with respect to a notion called of *extreme sensitive dependence on initial condition*; it is interesting to note that one-sided full shift is both extremely sensitive and strongly sensitive, since these conditions are based on the existence of some suitable state. The relationships among some of these variations will be the argument of a forthcoming paper of ours.

2. Positive transitivity

In what follows, we denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{N}_+$ the set of strictly positive natural number. The notion of transitivity given by Devaney [17] is renamed by us according to the following definition.

Definition 2.1 (Positive Transitivity). A DTDS $\langle X, g\rangle$ is (topologically) *positively transitive* if and only if for any pair A and B of non empty open subsets of X there exists an integer $n \in \mathbb{N}_+$ such that $g^n(A) \cap B \neq \emptyset$.

Intuitively, a positively transitive map g has points which under forward iteration of g eventually move from one arbitrarily small neighborhood to any other. An example of a positively transitive (and regular and strongly sensitive) DTDS is the full shift.

Remark 2.2. If a DTDS is positively transitive, then for any pair A, B of nonempty open subsets of X both the following conditions must hold:

PT1) $\exists n_1 \in \mathbb{N}_+ : g^{n_1}(A) \cap B \neq \emptyset$ PT2) $\exists n_2 \in \mathbb{N}_+ : g^{n_2}(B) \cap A \neq \emptyset$.

Example 2.3. *A non positively transitive DTDS.*

Let $\langle \mathbb{N}, g_s\rangle$ be the DTDS where the state space $\mathbb{N}$ is equipped with the trivial metric $d_{tr} : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}_+$, where $d_{tr}(x, y) = 1$ if $x \neq y$ and $d_{tr}(x, y) = 0$ otherwise, and $g_s : \mathbb{N} \mapsto \mathbb{N}$ is the successor mapping defined as follows: $\forall x \in \mathbb{N},\ g_s(x) = x+1$. The pair of open sets $A = \{3\}$ and $B = \{5\}$ verifies PT1) but not PT2) and thus it is not a positively transitive DTDS.

Let us observe that in literature there are alternative ways by which the notion of transitivity is defined. In some works the intersection condition $g^n(A) \cap B \neq \emptyset$ of the 2.1 is replaced by $A \cap g^{-n}(B) \neq \emptyset$ (see for instance [2,7] in the context of surjective DTDS on compact spaces) giving rise to an equivalent notion of transitivity. In other works (see for instance [3,4,11,21]) there are definitions of transitivity in which the integer n involved in Definition 2.1 is allowed to assume also the value $n = 0$. It is easy to prove they are equivalent to the positive transitivity. For further conditions equivalent to positive transitivity see for instance [1,26,36].

We recall that a subset $A \subseteq X$ is said to be *g positively (or forward) invariant* iff $\forall a \in A, g(a) \in A$ (*i.e.*, $g(A) \subseteq A$) and a pair $\langle A, g\rangle$ is a *dynamical sub-system* of $\langle X, g\rangle$ if A is a topologically closed and positively invariant subset of X. In the investigation of topological dynamics, if possible, it is useful to divide the space in "basic" blocks which constitute a partition of the initial system in sub-dynamical systems, and then to study the properties of each block. Positive transitivity is a topological property which avoid such a partition. To clarify this aspect we introduce the following

Definition 2.4 (Indecomposability). A DTDS $\langle X, g\rangle$ is *indecomposable* iff X is not the union of two nonempty open, disjoint, and positively invariant subset.

Of course, if two open subsets decompose X, then they must be also closed. This means that for an indecomposable DTDS the state space X cannot be split into two (nontrivial) clopen sub-dynamical systems. Indecomposability is in a certain sense an irreducibility condition [32]. Note that in [28] this property is also called condition of invariant connection and X is said to be invariantly connected. It is easy to check that a positively transitive DTDS is indecomposable.

In the sequel we denote by $B_\delta(x)$ the open ball centered in $x \in X$ and of radius $\delta > 0$ and by $\gamma_x : \mathbb{N} \mapsto X$ the orbit (or positive motion) of initial state $x \in X$ defined as: $\forall t \in \mathbb{N}, \gamma_x(t) = g^t(x)$. We give now the following

Definition 2.5 (Topologically Transitive Point). Let $\langle X, g\rangle$ be a DTDS. A state $x_0 \in X$ is said to be *topologically transitive* iff the positive motion γ_{x_0} of initial state x_0 is dense in X, *i.e.*, $\overline{\{g^t(x_0)\}_{t\in\mathbb{N}}} = X$.

Obviously, topological transitive points can occur only if the topological space underlying the involved DTDS is separable (*i.e.*, it admits a dense countable subset). Let us note that in some works in which DTDS on compact spaces are considered(see for instance [19,22,36]) the notion of transitivity is given in terms of the existence of a topologically transitive point. It is easy to prove that a DTDS which admits a topologically transitive point is indecomposable. It is important to stress that the existence of topologically transitive point is not a sufficient condition to guarantee the positive transitivity of a DTDS.

Example 2.6. *A DTDS which possesses a dense orbit but which is not positively transitive.*

Let $\langle \mathbb{N}, g_s\rangle$ be the DTDS of Example 2.3. The orbit of initial state $0 \in \mathbb{N}$ coincides with the whole state space $\mathbb{N}$ and then it is a dense orbit.

This behavior holds also in the case of compact DTDS.

Example 2.7. *A compact and non positively transitive DTDS with a dense orbit.* Let us consider the state space $X = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N}\}$ equipped with the metric induced by the usual metric of $\mathbb{R}$. This space is compact (and non perfect). Let $g : X \mapsto X$ be the continuous mapping defined as: $\forall x \in X$, $g(x) = \frac{1}{2}x$. A possible positive orbit dense in X is the sequence of initial state $x = 1$. But the dynamical system is not positively transitive. Indeed, if we consider the two nonempty open sets $A = \{1/2\}$ and $B = \{1\}$, we have that $\forall n \in \mathbb{N}$: $g^n(A) \cap B = \emptyset$.

Example 2.8. *A positively transitive DTDS which does not possess any dense orbit.*

Let $\langle Per(\sigma), \sigma\rangle$ be the dynamical system constituted by all periodic points of the two-sided full shift $\langle \mathcal{A}^{\mathbb{Z}}, \sigma\rangle$. It is easy to show that this system is positively transitive but it does not possess any dense orbit.

In the case of perfect DTDS, we have the following result:

Proposition 2.9 [34]**.** *Let $\langle X, g\rangle$ be a perfect DTDS. If there exists a topologically transitive point, then $\langle X, g\rangle$ is positively transitive.*

We now recall some sufficient conditions involving the properties of the metric space of a DTDS, in order that positive transitivity implies the existence of a dense periodic orbit:

Proposition 2.10 [34]**.** *Let $\langle X, g\rangle$ be a DTDS where the metric space (X, d) is separable and of second category (for this latter see* [23]*). If $\langle X, g\rangle$ is positively transitive, then it admits a topologically transitive point.*

Corollary 2.11. *Let $\langle X, g\rangle$ be a compact DTDS. If it is positively transitive then it admits a topologically transitive point.*

As a consequence of the previous result we have the following

Corollary 2.12. *Let $\langle X, g\rangle$ be a compact DTDS. If $\langle X, g\rangle$ is Devaney chaotic then it is Knudsen chaotic.*

There are two cases in which the positive transitivity is equivalent to the existence of a dense orbit. The former is obtained as combination of the Corollary 2.11 with the Proposition 2.9.

Corollary 2.13. *Let $\langle X, g\rangle$ be a compact and perfect DTDS. Then $\langle X, g\rangle$ is positively transitive if and only if it admits a topologically transitive point.*

Proposition 2.14 [36]**.** *Let $\langle X, g\rangle$ be a compact DTDS with a surjective next state mapping g. Then $\langle X, g\rangle$ is positively transitive if and only if it admits a topologically transitive point.*

Another interesting consequence of the positive transitivity is that $\overline{g(X)} = X$ (global denseness) which is in some cases strictly related to surjectivity of the involved system. In fact, in the setting of compact DTDS, positive transitivity implies that $g(X) = X$. We remark that the existence of a dense orbit is not a sufficient condition to obtain $\overline{g(X)} = X$, as the following example shows.

Example 2.15. Let us consider the DTDS of Example 2.3. The set $g_s(X) = \{1, 2, \ldots\}$ is a closed subset of X, so $\overline{g_s(X)} = g_s(X) \neq X$.

We introduce now some notions which refer to conditions stronger than positive transitivity.

2.1. Mixing condition

Let us introduce the following

Definition 2.16 (Topological Mixing). A DTDS $\langle X, g\rangle$ is *topologically mixing* iff for any pair A and B of non empty open subsets of X there exists an integer $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $g^n(A) \cap B \neq \emptyset$.

In [11] it has been proved that a topologically mixing DTDS with cardinality strictly greater than 2 is necessarily perfect (and so infinite). The following property can be useful in order to study the Knudsen-like chaotic (*i.e.*, without regularity) behavior of a DTDS:

Proposition 2.17 [27]**.** *Any topologically mixing DTDS with at least three states is sensitive to the initial conditions.*

2.2. Strong transitivity and minimality

Another condition stronger than positive transitivity is the following:

Definition 2.18 (Strong Transitivity). A DTDS $\langle X, g\rangle$ is *strongly transitive* iff for any nonempty open set $A \subseteq X$ we have $\bigcup_{n\in\mathbb{N}} g^n(A) = X$.

Thus a strongly transitive mapping g has points which eventually move under iteration of g from one arbitrarily small neighborhood to any other point. Therefore g must be surjective. Let us note that in [21] the notion of strong transitivity is called denseness of all backward orbits and some classes of DTDS where this property is equivalent to positive transitivity are exhibited.

In analogy to the above quoted Kurka result about the relationship between mixing and sensitivity, we can now prove the following relationship between strong transitivity and sensitivity. Also in this case we have a condition which furnishes a Knudesn-like chaotic (*i.e.*, always without regularity) behavior.

Proposition 2.19. *Any strongly transitive DTDS with at least two disjoint periodic orbits is sensitive to the initial conditions.*

Proof. Let 2ϵ be the minimum distance between two disjoint periodic orbits $\mathcal{C}_1$ and $\mathcal{C}_2$. Chosen a state x and a real $\delta > 0$, there exist two states $y_1, y_2 \in B_\delta(x)$ and an integer $n \in \mathbb{N}$ such that $g^n(y_1) \in \mathcal{C}_1$ and $g^n(y_2) \in \mathcal{C}_2$. Since $d(g^n(y_1), g^n(x)) + d(g^n(x), g^n(y_2)) \geq d(g^n(y_1), g^n(y_2)) \geq 2\epsilon$, we obtain that either $d(g^n(y_1), g^n(x)) \geq \epsilon$ or $d(g^n(y_2), g^n(x)) \geq \epsilon$. □

As an example of strongly transitive and regular (which trivially has at least two disjoint periodic orbits) DTDS is the tent map T on $[0,1]$ defined as $\forall x \in [0,1], T(x) = 1 - |2x-1|$.

Let us note that in the case of homeomorphic DTDS on compact spaces Definition 2.18 can be rewritten as:

$$\text{for any open set } A \neq \emptyset \text{ there exists } k \in \mathbb{N} \text{ such that } \bigcup_{n=0}^{k} g^n(A) = X. \tag{1}$$

The previous condition is the one considered in [20] in the case of compact but not necessary homeomorphic DTDS. We now recall the classical condition of minimality.

Definition 2.20 (Minimality). A DTDS $\langle X, g\rangle$ is minimal iff the unique closed and strictly invariant subsets of X are X and $\emptyset$.

Equivalently, a DTDS is minimal iff every positive motion is dense in X. Thus minimality implies positive transitivity. Moreover, it is easy to show that a minimal DTDS either does not possess any periodic point or it is constituted by a unique periodic orbit.

Remark 2.21 (Private communication of François Blanchard)**.** With a slight modification of Theorem 5.1 in [36], it is not hard to prove that for homeomorphic and compact DTDS minimality is equivalent to condition (1). Therefore, we have the following result:

Proposition 2.22. *Let $\langle X, g\rangle$ a homeomorphic and compact DTDS. $\langle X, g\rangle$ is minimal if and only if it is strongly transitive.*

As an immediate consequence of the previous facts, we have that homeomorphic, compact and strongly transitive DTDS cannot be regular (unless they are constituted by a unique cycle), and then they cannot be Devaney chaotic. This fact has also an intuitive explanation: some invertible minimal systems (irrational rotations of the circle, adding machines, see for instance [21, 27]) are considered to be the "most" deterministic or "un-chaotic" dynamical systems, barring periodic points and being not sensitive. Then they are neither Devaney nor Knudsen-like chaotic. In conclusion, on compact spaces, strong transitivity is a property having a completely different meaning in the homeomorphic case with respect to the non-homeomorphic one. Let us recall that the two-sided full shift $\langle \mathcal{A}^{\mathbb{Z}}, \sigma\rangle$ is a homeomorphic and compact DTDS which is Devaney chaotic and mixing but it is not strongly transitive, while the one-sided full shift $\langle \mathcal{A}^{\mathbb{N}}, \sigma\rangle$ is a non-homeomorphic DTDS which is Devaney chaotic, mixing and strongly transitive.

3. Full transitivity

Definition 3.1 (Full Transitivity). A DTDS $\langle X, g\rangle$ is (topologically) *full transitive* if and only if for any pair A and B of non empty open subsets of X there exists an integer number $t \in \mathbb{Z}$ such that $A \cap g^{-t}(B) \neq \emptyset$.

Full transitivity is a weaker condition than the positive one. In the particular case of homeomorphic and compact DTDS, this notion is the one considered for instance in [16,36,37], but let us stress that our definition precludes this restrictive constraints. It is obvious that a positively transitive DTDS is full transitive. The converse does not hold as we can see in the following example.

Example 3.2. *A full transitive DTDS which is not positively transitive.*
Let $\langle \mathbb{N}, g_s \rangle$ be the DTDS of Example 2.3. We prove that $\langle \mathbb{N}, g_s \rangle$ is full transitive. Let A and B be two disjoint open subsets of $\mathbb{N}$ and a, b their corresponding minimal elements. Then the integer $t_1 = b - a$ is such that $g_s^{t_1}(A) \cap B \neq \emptyset$.

Note that the compact DTDS of Example 2.7 exhibits the same behavior of the previous one. Trivially the existence of a dense orbit is a sufficient condition to guarantee the full transitivity and it is easy to check that every full transitive DTDS is indecomposable. As regards as global denseness we want to remark that full transitivity is not sufficient condition to obtain $\overline{g(X)} = X$ (see Ex. 2.3).

We now consider the homeomorphic DTDS, with attention to compact spaces. Let us introduce the following notion:

Definition 3.3 (Full Transitive Point). Let $\langle X, g \rangle$ be a homeomorphic DTDS. A state $x_0 \in X$ is said to be *full transitive* iff the set $\{g^t(x_0)\}_{t\in\mathbb{Z}}$ is dense in X.

Obviously a topologically transitive state x_0 of a homeomorphic DTDS is also a full transitive point. The converse does not hold, not even under the condition of a compact metric space (see Ex. 3.5). It is easy to check that the existence of a full transitive point implies the full transitivity (but it is not a sufficient condition to guarantee the positive one – see Ex. 3.4 – not even if X is a compact set – see Ex. 3.5). The converse holds if the system is homeomorphic and compact. This fact is proved in [16].

We ask whether full transitivity coincides to the positive transitivity for homeomorphic DTDS. The answer is no.

Example 3.4. *A homeomorphic and full transitive DTDS which is not positively transitive.*

Let $X = \mathbb{Z}$ be the set of integer numbers equipped with the trivial metric and let $g_p : \mathbb{Z} \mapsto \mathbb{Z}$ be the mapping defined as follows: $\forall x \in \mathbb{Z}, g_p(x) = x - 1$. It is easy to show that the system $\langle \mathbb{Z}, g_p \rangle$ is homeomorphic and full transitive, but it is not positively transitive. Let us note that any state is a full transitive point.

This behavior also holds in the case of compact and homeomorphic DTDS.

Example 3.5. *A homeomorphic, full transitive and compact DTDS which is not positively transitive.*

Let $\langle S, \sigma_S \rangle$ be the dynamical subsystem of the full shift on the binary alphabet described as follows. A configuration $\underline{x} \in \{0,1\}^{\mathbb{Z}}$ belongs to S iff either $\underline{x} = \underline{0} = (\ldots, 0, 0, 0, \ldots)$ or $\underline{x} = \underline{1} = (\ldots, 1, 1, 1, \ldots)$ or $\underline{x}$ is of the kind $(\ldots, 0, 0, 0, 1, 1, 1, \ldots)$ The mapping σ_S is the restriction of the shift mapping σ to the set S. It is easy to check that $\langle S, \sigma_S \rangle$ is a homeomorphic and compact DTDS (in effect it is a SFT, see Ex. 4.9 of Sect. 4) which is non perfect. Let us note that each state $\underline{x}_0$ of $S \setminus \{\underline{0}, \underline{1}\}$

generates a dense full orbit but the subshift does not possess any topologically transitive point, and so the system is full transitive but not positively transitive. Furthermore we observe that the system is not regular.

Let us recall that a state $x \in X$ of a DTDS $\langle X, g\rangle$ is a non-wandering point iff for any neighborhood U of x, there exists an integer $n > 0$ such that $g^{-n}(U) \cap U \neq \emptyset$. We denote by $\Omega(g)$ the set of all the non-wandering points of the system.

The following theorems extends to a general DTDS the result given in [36] for the homeomorphic and compact setting.

Theorem 3.6. *Let $\langle X, g\rangle$ a DTDS. The system is positively transitive iff it is full transitive and $\Omega(g) = X$.*

Proof. It is obvious that a positively transitive is full transitive and non wandering. For the sequel of argument, let us suppose now that there exist two nonempty open sets $U, V \subseteq X$ such that $\forall t > 0$, $g^t(U) \cap V = \emptyset$. Since the system is full transitive, there is an integer $s \geq 0$ such that $g^{-s}(U) \cap V \neq \emptyset$ (equivalently, $g^s(V) \cap U \neq \emptyset$). Let us consider the nonempty open set $A = g^{-s}(U) \cap V$. We now show that there exists an integer $N > s$ such that $g^{-N}(A) \cap A \neq \emptyset$. We suppose the contrary (that is for any $M \geq s$ we have $g^{-M}(A) \cap A = \emptyset$). Let $m \leq s$ be the greatest integer such that $A' = g^{-m}(A) \cap A \neq \emptyset$ (such an integer exists). We have that $g^{-m'}(A') \cap A' \neq \emptyset$, for some $m' > 0$. This fact implies that

$$\begin{aligned}\emptyset \neq g^{-m'}\left(g^{-m}(A) \cap A\right) \cap g^{-m}(A) \cap A &= g^{-m'-m}(A) \cap g^{-m'}(A) \cap g^{-m}(A) \cap A \\ &\subseteq g^{-m'-m}(A) \cap A,\end{aligned}$$

and then there exists an integer $N > s$ such that $g^{-N}(A) \cap A \neq \emptyset$, and that $g^N(A) \cap A \neq \emptyset$. Since $g^s(A) \subseteq g^s(g^{-s}(U)) \cap g^s(V) = U \cap g^s(V) \subseteq U$ we have that $g^N(A) = g^{N-s}(g^s(A)) \subseteq g^{N-s}(U)$. Recalling that $A \subseteq V$, we conclude that $\emptyset \neq g^N(A) \cap A \subseteq g^{N-s}(U) \cap V$, contradiction. □

Example 3.7. Let us consider the system of Example 3.5. One can deduce that the system is not positively transitive also from the fact that the unique non-wandering points are $\underline{0}$ and $\underline{1}$.

As a consequence of the previous theorem, since regularity implies that all states are non-wandering, we have that in the Devaney definition of chaos the notion of transitivity can be substituted by the weaker notion of full transitivity. There is an important case, involving condition of perfectness of the system, where full transitivity implies positive transitivity.

Theorem 3.8. *Let $\langle X, g\rangle$ a perfect DTDS. The system is positively transitive iff it is full transitive.*

Proof. We show that if the system is full transitive, then every state is a non-wandering point. Let $x \in X$ be a state and U be a neighborhood of x. We know that there exists an integer $t \in \mathbb{Z}$ such that $U \cap g^{-t}(U) \neq \emptyset$. If $t \neq 0$, we trivially have that x is non-wandering. Otherwise, using the fact that X is perfect there is a state $y \in U$ with $y \neq x$. Now, we are able to find two disjoints open set $A, B \subseteq U$ containing x and y respectively and an integer s such that $A \cap g^{-s}(B) \neq \emptyset$. Necessarily we have $s \neq 0$, so there exists an integer $n > 0$ such that $U \cap g^{-n}(U) \neq \emptyset$. Theorem 3.6 concludes the proof. □

In conclusion full transitivity is a property which has a proper meaning in some settings like DTDS with isolated points. In the next sections we will deal with systems (such as the subshifts) which may be of this kind.

4. Subshifts and related languages

We define the cylinder of block $u \in \mathcal{A}^n$ and position $m \in \mathbb{Z}$ as the set $C_m(u) = \{\underline{x} \in \mathcal{A}^{\mathbb{Z}} \mid x_m \cdots x_{m+n-1} = u\}$. Note that cylinders form a basis of clopen subsets of $\mathcal{A}^{\mathbb{Z}}$ for the topology induced by the Tychonoff metric.

Definition 4.1 (**Subshift**)**.** A (two-sided) *subshift* over the alphabet $\mathcal{A}$ is a DTDS $\langle S, \sigma_S \rangle$, where S is a non empty closed, strictly σ-invariant ($\sigma(S) = S$) subset of $\mathcal{A}^{\mathbb{Z}}$, and σ_S is the restriction of the shift map σ to S.

In the sequel, in the context of a given subshift $\langle S, \sigma_S \rangle$, for the sake of simplicity we will denote by $C_m(u)$ the subset $C_m(u) \cap S$, if there is no confusion. Let us note that in relative topological space (S, d_T), where d_T is the restriction to S of the Tychonoff metric defined on $\mathcal{A}^{\mathbb{Z}}$, the set $C_m(u) \cap S$ is open.

A subshift $\langle S, \sigma_S \rangle$ distinguishes the words or finite blocks constructed over the alphabet $\mathcal{A}$ in two types: *admissible blocks*, *i.e.*, blocks appearing in some configuration of S and blocks which are not admissible, called *forbidden*, *i.e.*, blocks which do not appear in any configuration of S. We will write $w \prec \underline{x}$ to denote that the $\mathcal{A}$–word $w = (w_1, \ldots, w_n) \in \mathcal{A}^*$ appears in the configuration $\underline{x} \in \mathcal{A}^{\mathbb{Z}}$, formally $\exists i \in \mathbb{Z}$ s.t. $x_i = w_1, \ldots, x_{i+n-1} = w_n$ (also indicated by $\underline{x}_{[i,i+n-1]} = w$). We will denote by $w \not\prec \underline{x}$ the fact that w does not appear in the configuration $\underline{x}$. A word $u = u_1 \cdots u_m \in \mathcal{A}^*$ is a sub-block (or sub-word) of $w = w_1 \cdots w_n \in \mathcal{A}^*$, written as $u \sqsubseteq w$, iff $u = w_i \cdots w_j$, for some $1 \leq i \leq j \leq n$. To every subshift we can associate a formal language according to the following:

Definition 4.2 (**Language of a Subshift**)**.** Let $\langle S, \sigma_S \rangle$ be a subshift over the alphabet $\mathcal{A}$. The *language* of S is the collection of all admissible blocks: $\mathcal{L}(S) = \{w \in \mathcal{A}^* : \exists \underline{x} \in S \ s.t. \ w \prec \underline{x}\}$.

A canonical way to generate a subshift consists of fixing a collection of words considered as forbidden blocks. Precisely, let $\mathcal{F}$ be any subset of $\mathcal{A}^*$, and let us construct the set $S(\mathcal{F}) = \{\underline{x} \in \mathcal{A}^{\mathbb{Z}} : \forall w \in \mathcal{F}, w \not\prec \underline{x}\}$. Then $S(\mathcal{F})$ is a subshift, named the subshift generated by $\mathcal{F}$. Let us note that two different families of forbidden blocks may generate the same subshift.

Definition 4.3 (**Subshift of Finite Type**)**.** A subshift is of *finite type* iff it can be generated by a finite set $\mathcal{F}$ of forbidden blocks.

In the case of a SFT, the finite set $\mathcal{F}$ generally is composed by blocks of different length. Nevertheless, starting from $\mathcal{F}$ it is always possible to construct a set of forbidden blocks $\mathcal{F}'$ consisting of the same length and generating the same subshift. We have just to complete in all possible ways each block from $\mathcal{F}$, up to reach the length of the longest forbidden block in $\mathcal{F}$. In the case of a SFT $\langle S, \sigma_s \rangle$ we will denote by $\mathcal{F}_h$ a set of forbidden blocks in which all words $w \in \mathcal{F}_h$ have the same length h and generating the subshift, *i.e.*, $S = S(\mathcal{F}_h)$.

To every SFT we can associate a graph according to the following:

Definition 4.4 (**Graph associated to a SFT).** Let $\langle S, \sigma_S \rangle$ be a SFT generated by a set $\mathcal{F}_h$ of forbidden blocks. The *graph* $G_h(S)$ associated to S is the pair $\langle V(S), E(S) \rangle$ where the vertex set is $V(S) = \mathcal{A}^{h-1}$ and the edge set $E(S)$ contains all the pairs $(a, b) \in \mathcal{A}^{h-1} \times \mathcal{A}^{h-1}$ such that $a_2 = b_1, \ldots, a_{h-1} = b_{h-2}$ and $a_1 a_2 \cdots a_{h-1} b_{h-1} \notin \mathcal{F}_h$. The block $a_1 a_2 \cdots a_{h-1} b_{h-1}$ is called the word generated by the blocks a and b.

Trivially, bi-infinite paths along nodes on the graph $G_h(S)$ correspond to bi-infinite strings of the SFT S. In the general case we can minimize the subgraph $G_h(S)$ removing all the unnecessary nodes and the corresponding outgoing and incoming edges. In this way the finite paths along nodes on the graph correspond to the words of the language $\mathcal{L}(S)$. From now on, we will consider only minimized graphs. We will denote by $A_h(S)$ the adjacency matrix of the graph $G_h(S)$ associated to a SFT $\langle S, \sigma_S \rangle$ generated by a set $\mathcal{F}_h$ of forbidden blocks.

We recall now the following definitions concerning the notion of irreducibility for a language and a square matrix.

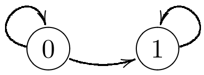
Definition 4.5 (**Irreducible Language).** A language $\mathcal{L} \subseteq \mathcal{A}^*$ is *irreducible* iff for every ordered pair of blocks $u, v \in \mathcal{L}$ there is a block $w \in \mathcal{L}$ such that $uwv \in \mathcal{L}$.

Definition 4.6 (**Irreducible Matrix).** An order k square matrix $M = [m_{i,j}]$ is irreducible iff $\forall i, j \in \{1, \ldots, k\}$, $\exists p = p(i, j) \in \mathbb{N}$, $p > 0$ such that $m_{i,j}^{(p)} \neq 0$, where $m_{i,j}^{(p)}$ is the (i, j)-component of the matrix M^p.

We now want to study some topological properties of subshifts as DTDS, characterizing the corresponding languages, and for SFT, the associated matrixes and graphs too.

In [11] three different techniques are explained to investigate if a SFT is positively transitive. These techniques involve the language, the graph, and the matrix associated to the SFT.

Theorem 4.7 (see [29], with some improvements in [11])**.** *Let $\langle S, \sigma_S \rangle$ be a SFT generated by a set of forbidden blocks $\mathcal{F}_h$. Then the following statements are equivalent: i) $\langle S, \sigma_S \rangle$ is positively transitive; ii) $\mathcal{L}(S)$ is irreducible; iii) $G_h(S)$ is strongly connected; iv) $A_h(S)$ is irreducible. Moreover, if one of the above equivalent conditions holds, then $\langle S, \sigma_S \rangle$ is regular. Lastly, if $\langle S, \sigma_S \rangle$ has infinite cardinality, the previous statements are equivalent to the following condition: v) $\langle S, \sigma_S \rangle$ is chaotic according to Devaney.*



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

FIGURE 1. Graph and adjacency matrix of Example 4.9.

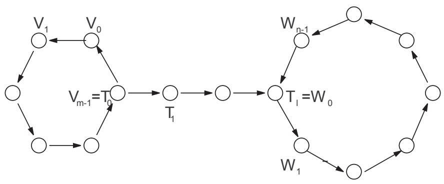


FIGURE 2. Bicyclic graph.

Let us stress that the equivalence between $i)$ and $ii)$ holds even if the subshift is not a SFT. Furthermore, as a consequence of Corollary 2.12, positively transitive and infinite SFT are also Knudsen chaotic. As to the characterization of topologically mixing and strongly transitive subshifts see for instance [11, 29]. We now introduce some new definitions concerning languages, graphs, and matrices which help us to establish if a subshift is full transitive.

Definition 4.8 (**Weakly Irreducible Language).** A language $\mathcal{L} \subseteq \mathcal{A}^*$ is *weakly irreducible* iff for any pair of blocks $u, v \in \mathcal{L}$ there is a block $w \in \mathcal{L}$, s.t. $u, v \sqsubseteq w$.

Example 4.9. *A weakly irreducible language which is not irreducible.*

Let $\langle S, \sigma_S \rangle$ be the SFT over the alphabet $\mathcal{A} = \{0, 1\}$ generated by the set of forbidden blocks $\mathcal{F}_2 = \{10\}$ (see Fig. 1). The language $\mathcal{L}(S)$ is weakly irreducible. However it is not irreducible. Indeed, if we consider the words $u = 1 \in \mathcal{L}(S)$ and $v = 0 \in \mathcal{L}(S)$ we are not able to find any block $w \in \mathcal{L}(S)$ such that $uwv \in \mathcal{L}(S)$.

In the sequel we will refer to a connected component of a directed graph G as a subgraph which is a connected component of the underlying undirected graph of G obtained suppressing the orientations of all the edges of G. We now introduce the following:

Definition 4.10 (**Full Transitive Graph).** A graph G is *full transitive* iff either it is strongly connected or it is *bicyclic*, *i.e.*, a graph of the kind $\langle V, E \rangle$ where the vertex set is $V = \{U_0, \ldots, U_{m-1} = T_0, \ldots, T_i, \ldots, T_l = W_0, \ldots, W_{n-1}\}$ $(m, n, l \in \mathbb{N}_+)$ and the edges are the pairs $(U_i, U_{(i+1) \mod m})$, for $i = 0, \ldots, m-1$, the pairs (T_i, T_{i+1}), for $i = 0, \ldots, l-1$ and the pairs $(W_i, W_{(i+1) \mod n})$, for $i = 0, \ldots, n-1$.

In other words, a full transitive graph is a unique connected component which is either strongly connected or it is constituted by two disjoint cycles and by a unique path which connects them (see Fig. 2).

Definition 4.11 (Full Transitive Matrix). Let $M = [m_{i,j}]$ be an order k matrix. Then, the matrix M is *full transitive* iff either it is irreducible or it is *bicyclic*, *i.e.*, a matrix of the kind

$$M = \begin{bmatrix} l_{1,1} & \cdots & l_{1,m} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ l_{m,1} & \cdots & l_{m,m} & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & r_{1,1} & \cdots & r_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & r_{n,1} & \cdots & r_{n,n} \end{bmatrix} \tag{2}$$

where $L = [l_{i,j}]$ and $R = [r_{i,j}]$ are permutation matrices of order m and n respectively, which have the element 1 in the positions $(m, 1), (i, i+1)$ for $i = 1, \ldots, m-1$ and $(n, 1), (j, j + 1)$ for $j = 1, \ldots, n - 1$ respectively, or there exist a permutation matrix P such that $P^{-1}MP$ has the the structure expressed in 2.

Theorem 4.12. *Let $\langle S, \sigma_S\rangle$ be a subshift. Then the following statements are equivalent: i) $\langle S, \sigma_S\rangle$ is full transitive; ii) $\mathcal{L}(S)$ is weakly irreducible. In the case of a SFT generated by a set $\mathcal{F}_h$ of forbidden blocks, the previous statements are equivalent to the further conditions: iii) $G_h(S)$ is full transitive; iv) $A_h(S)$ is full transitive.*

Proof. *i*) $\Rightarrow$ *ii*) Chosen arbitrarily $u, v \in \mathcal{L}(S)$, there exist a configuration $\underline{z} \in C_0(u)$ and an integer $t \in \mathbb{Z}$ such that $\sigma_S^t(\underline{z}) \in C_n(v)$, where $n = |u|$. Since $\sigma_S^t(\underline{z}) \in C_{-t}(u)$, we have that the words u and v are sub-blocks of the word $w = \sigma_S^t(\underline{z})_{[\min\{-t,n\},\max\{-t+n-1,n+|v|-1\}]} \in \mathcal{L}(S)$.

ii) $\Rightarrow$ *i*) Chosen arbitrarily $u, v \in \mathcal{L}(S)$ and $m, n \in \mathbb{Z}$, let $w \in \mathcal{L}(S)$ be the block such that $u, v \sqsubseteq w$ with $u = w_i \cdots w_j$ and $v = w_k \cdots w_l$, for some $1 \le i \le j \le |w|$ and some $1 \le k \le l \le |w|$. Let us consider a configuration $\underline{z} \in C_{m-i+1}(w)$, we have that $\underline{z} \in C_m(u)$ and then $\sigma_S^t(\underline{z}) \in C_m(v)$, with $t = m - n + k - i \in \mathbb{Z}$.

The equivalence between *i*) and *iv*) can be found in [30]. Moreover the quivalence between *iii*) and *iv*) directly follows from the structure of $G_h(S)$ and $A_h(S)$. □

Example 4.13. *A full transitive subshift which is not positively transitive.*

Let us consider the subshift of Example 4.9. Since the graph is not strongly connected the subshift is not positively transitive. However it is full transitive.

It is easy to prove that a SFT generated by a set $\mathcal{F}_h$ of forbidden blocks is regular iff every connected component of the graph $G_h(S)$ is a SCC. This fact can be used to a direct proof that in the setting of SFT full transitivity and regularity together imply positive transitivity. We now illustrate some conditions under which a subshift is sensitive to the initial conditions. For this goal, let us introduce a suitable notion of language.

Definition 4.14 (**Right (resp., left) 2-ways Extendible Language).** A language $\mathcal{L} \subseteq \mathcal{A}^*$ is *right (resp., left) 2-ways extendible* iff for any block $u \in \mathcal{L}$ there exist two words $w, w' \in \mathcal{L}$, with $w \neq w'$ and $|w| = |w'|$, such that $uw, uw' \in \mathcal{L}$ (resp., $wu, w'u \in \mathcal{L}$).

Definition 4.15 (**Right (resp., left) 2-ways Extendible Path).** A path $\pi = V_1,$
$\ldots, V_m$ $(m > 1)$ on a graph G is *right (resp., left) 2-ways extendible* iff there exists two paths $\pi' = V_1, \ldots, V_m, R'_1, \ldots, R'_n$ and $\pi'' = V_1, \ldots, V_m, R''_1, \ldots, R''_n$ $(n \geq 1)$ (resp., $\pi'_1 = L'_1, \ldots, L'_n, V_1, \ldots, V_m$ and $\pi''_1 = L''_1, \ldots, L''_n, V_1, \ldots, V_m$) such that $R'_i \neq R''_i$ for some i (resp., $L'_i \neq L''_i$ for some i).

We recall that a sensitive DTDS must be perfect, and with respect to perfectness of subshift, the following result holds:

Proposition 4.16 [10]**.** *A subshift is perfect iff its language is either left or right 2-ways extendible. Moreover a SFT generated by a set $\mathcal{F}_h$ of forbidden blocks is perfect iff every path on the graph $G_h(S)$ is either left or right 2-ways extendible.*

Theorem 4.17 [10]**.** *Let $\langle S, \sigma_S \rangle$ be a subshift. Then the following statements are equivalent: i) $\langle S, \sigma_S \rangle$ is sensitive to the initial conditions with sensitivity constant $\epsilon = 1$; ii) $\mathcal{L}(S)$ is right 2-ways extendible. Moreover, in the case of a SFT generated by a set $\mathcal{F}_h$ of forbidden blocks, the previous statements are equivalent to the following condition: iii) every path on the graph $G_h(S)$ is right 2-ways extendible.*

In other words a SFT is perfect iff every finite path on the corresponding graph is extendible in at least two different ways either at the first or at the last vertex. It is sensitive to the initial conditions iff this fact holds at the last vertex of every path. As a consequence of Theorems 4.12 and 4.17, we can state that a full but non positively transitive SFT is not sensitive to the initial conditions.

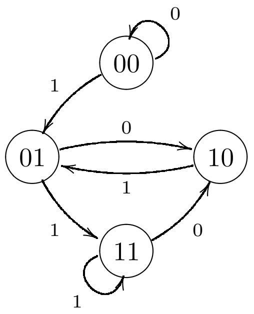
Example 4.18. *A non perfect subshift.*
Let us consider the subshift of Example 4.9. It is not perfect since the path (0)(1) is not extendible in two different ways.

Example 4.19. *A sensitive subshift.*
Let $\langle S, \sigma_S \rangle$ be the SFT over the alphabet $\mathcal{A} = \{0, 1\}$ generated by the set of forbidden blocks $\mathcal{F}_3 = \{100\}$ (see Fig. 3). Every finite path on $G_3(S)$ is right 2-ways extendible. Thus $\langle S, \sigma_S \rangle$ is sensitive to the initial condition (and perfect).

Example 4.20. *A perfect subshift which is not sensitive to the initial condition.*
Let $\langle S, \sigma_S \rangle$ be the SFT over the alphabet $\mathcal{A} = \{0, 1\}$ generated by the set of forbidden blocks $\mathcal{F}_3 = \{110\}$ (see Fig. 4). Every finite path on the graph $G_3(S)$ is left 2-ways extendible but the path (01)(11) is not right 2-ways extendible.

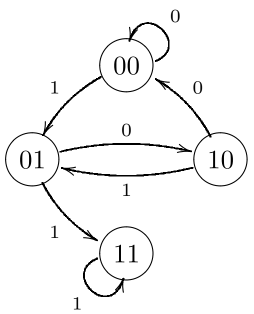
4.1. One-Sided Subshifts

Definition 4.21 (**One-sided Subshift).** A one-sided (two-sided) subshift over the alphabet $\mathcal{A}$ is a DTDS $\langle S, \sigma_S \rangle$, where S is a non empty closed, σ-invariant $(\sigma(S) \subseteq S)$ subset of $\mathcal{A}^{\mathbb{N}}$, and σ_S is the restriction of the shift map σ to S.



$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

FIGURE 3. Graph and adjacency matrix of Example 4.19.



$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

FIGURE 4. Graph and adjacency matrix of Example 4.20.

In contrast to the sided-sided case, $\langle S, \sigma_S \rangle$ is not a homeomorphic system. The notions of language, SFT, graph, and matrix are similar to the ones of two-sides subshifts. As to the results given in this section for two-sided subshifts and involving some topological properties, like positive transitivity, full transitivity, topological mixing, and sensitivity, we have that they hold also in the one-sided case. As to strong transitivity we have that for SFT it is a condition equivalent to positive transitivity [21]. Let us introduce the following definition.

Definition 4.22 (Expansivity and Positive Expansivity). A homeomorphic DTDS (resp., a DTDS) is *expansive* (resp., *positively expansive*) iff there exists a constant $\epsilon > 0$ such that for every pair $x, y \in X$ with $x \neq y$, there is an integer $t \in \mathbb{Z}$ (resp., $t \in \mathbb{N}$) such that $d(g^t(x), g^t(y)) \geq \epsilon$.

In the case of perfect DTDS, positive expansivity is a condition stronger than sensitivity to the initial conditions. It is easy to prove that two-sided subshifts are expansive, while one-sided subshifts are positively expansive.

5. Conclusion

We have studied the notion of transitivity as an important but not unique component of chaos and we made an investigation about the several definitions of transitivity found in literature. We have considered the notions of positive and full transitivity and we have studied the relationships among these two notion and other properties. We have shown that these two notions coincide in the case of perfect and non-wandering DTDS, having this latter as a consequence that in the Devaney definition of chaos the notion of positive transitivity can be substituted by full transitivity. We have also considered some conditions stronger than positive transitivity, in particular the two notions of mixing and of strong transitivity. Similarly to an existing result related to mixing, we have shown that a strongly transitive DTDS with at least two disjoint periodic orbits is sensitive. In this way one obtains two Knudsen-like definitions of chaos, *i.e.*, without regularity, in which giving as fixed the sensitivity condition some property which implies positive transitivity is assumed to hold. We have remarked that in the setting of homeomorphic and compact DTDS, strong transitivity reduces to the classical notion of minimality, having as a consequence that no strongly transitive DTDS can be Devaney chaotic. Finally, we have studied the above properties in the case of Subshifts. We have characterized the languages associated to full transitive and sensitive subshifts (and also the graphs for the SFT of this kind).

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