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BMO-TYPE SEMINORMS AND SOBOLEV FUNCTIONS *

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Abstract. Following some ideas of a recent paper by Bourgain, Brezis and Mironescu, we give a representation formula of the norm of the gradient of a Sobolev function which does not make use of the distributional derivatives.

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1. Introduction

In this paper we study the relation between certain BMO-type seminorms and the L^p norm of the gradient of a Sobolev function. The starting point of our investigations goes back to the paper [4] by Bourgain, Brezis and Mironescu where a new function space is introduced by means of suitable BMO-type seminorms. Some of the ideas contained in [4] have been later on extended by Ambrosio, Bourgain, Brezis and Figalli in [1] to give a new characterization of sets of finite perimeter.

Here, given a function $f \in L^p(\mathbb{R}^n)$, $p > 1$, for any $\varepsilon > 0$ we consider the following seminorm

$$\kappa_\varepsilon(f;p) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p \mathrm{d}x, \tag{1.1}$$

where the supremum on the right hand side is taken over all families $\mathcal{G}_\varepsilon$ of disjoint open cubes Q' of side length ε and arbitrary orientation.

Our main result, Theorem 2.3, states that a function $f \in L^p(\mathbb{R}^n)$ belongs to $W^{1,p}(\mathbb{R}^n)$ if and only if the above seminorms have finite limit as ε tends to 0. Moreover, the following formula holds true

$$\lim_{\varepsilon \to 0^+} \kappa_\varepsilon(f;p) = \gamma(n,p) \int_{\mathbb{R}^n} |\nabla f|^p \mathrm{d}x, \tag{1.2}$$

where γ is the constant given in (2.2).

Keywords and phrases. Sobolev functions, Nikol'skij spaces, BMO-type seminorms.

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In the special case $p = 1$ and $f = \chi_E$, where χ_E denotes the characteristic function of a measurable set E, it was proved in the aforementioned paper [1] that

$$\lim_{\varepsilon\to 0^+} \kappa_\varepsilon(\chi_E; 1) = \frac{1}{2}P(E),$$

where $P(E)$ is the *perimeter* of E in the sense of De Giorgi. More generally, if f belongs to the space $SBV(\mathbb{R}^n)$ of *special functions of bounded variation*, see the definition in ([2], Sect. 4.1), one has

$$\lim_{\varepsilon\to 0^+} \kappa_\varepsilon(f; 1) = \frac{1}{4}\int_{\mathbb{R}^n} |\nabla f| \,\mathrm{d}x + \frac{1}{2}|D^s f|(\mathbb{R}^n),$$

where ∇f and $D^s f$ are the absolutely continuous part and the singular part, respectively, of the gradient measure Df. The proof of this formula was given in ([6], Thm. 3.3) under an extra technical assumption on f, subsequently removed in ([5], Cor. 6.2).

Note, that for a general BV function no such formula may hold, as shown by a one dimensional example of [6], see Remark 2.5. However in Proposition 2.4 we show that one can still characterize the functions in $BV(\mathbb{R}^n)$ as the functions $f \in L^1(\mathbb{R}^n)$ such that $\limsup_{\varepsilon\to 0} \kappa_\varepsilon(f; 1)$ is finite.

In the case of the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$, with $0 < s < 1$, one could expect that a formula of the type (1.2) should hold using the quantity $\kappa_\varepsilon(f; p, s)$ defined as in (1.1) and replacing the scaling factor ε^{n-p} by ε^{n-sp}. However, it is not so. An almost immediate consequence of the definition yields that

$$\sup_{\varepsilon>0} \kappa_\varepsilon(f; p, s) \le C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y, \tag{1.3}$$

for some constant C depending only on n and $p \ge 1$, while the opposite inequality is false as shown by a simple example, see Remark 3.3. The fact that the two quantities in (1.3) are not equivalent is clarified by the last result of paper. Indeed in Proposition 3.4 we show that for every $p \ge 1$ and $s \in (0, 1]$ the supremum in ε of $\kappa_\varepsilon(f; p, s)$ is equivalent to the Nikol'skij seminorm of f raised to p.

2. The Sobolev space $W^{1,p}(R^n)$

Given a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $p \ge 1$, and $\varepsilon > 0$, we denote by $\mathcal{G}_\varepsilon$ a family of disjoint open cubes Q' of side length ε and arbitrary orientation. We set

$$\kappa_\varepsilon(f; p) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p \mathrm{d}x. \tag{2.1}$$

Let us introduce also the following important constant

$$\gamma(n, p) := \max_{\nu\in\mathbb{S}^{n-1}} \int_Q |x\cdot\nu|^p \,\mathrm{d}x, \tag{2.2}$$

where $Q = (-1/2, 1/2)^n$ and $n \ge 1$.

Remark 2.1. We observe that for $p = 1, 2$ the constant $\gamma(n, p)$ is independent of the dimension. Indeed it can be proved, (see [6], Lem. 3.1), that $\gamma(n, 1) = 1/4$ for all $n \ge 1$. On the other hand, if $p = 2$ we have for any $\nu \in \mathbb{S}^{n-1}$

$$\int_Q |x\cdot\nu|^2 \,\mathrm{d}x = \sum_{i,j=1}^n \nu_i\nu_j \int_Q x_i x_j \,\mathrm{d}x = \sum_{i=1}^n \nu_i^2 \int_Q x_i^2 \,\mathrm{d}x = \frac{1}{12} = \gamma(n, 2).$$

Note however that in general γ may depend on the dimension. In fact, if $p \geq 1$ we have $\gamma(1,p) = 2^{-p}/(p+1)$. On the other hand, setting $T := \{(x,y) \in \mathbb{R}^2 : -1/2 < x < 1/2,\ -x < y < 1/2\}$, we may estimate

$$\gamma(2,p) \geq \int_Q \left|\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right|^p \mathrm{d}x\mathrm{d}y = 2^{1-\frac{p}{2}} \int_T (x+y)^p \,\mathrm{d}x\mathrm{d}y = \frac{2^{1-\frac{p}{2}}}{(p+1)(p+2)}\cdot$$

Therefore $\gamma(2,p) > \gamma(1,p)$ for p large.

Finally, observe that if $\bar\nu \in \mathbb{S}^{n-1}$ is a vector maximizing the integral in (2.2), $x_0 \in \mathbb{R}^n$ and $Q_l(x_0)$ is a cube of side length l with center in x_0 then

$$\int_{Q_l(x_0)} |(x-x_0)\cdot\bar\nu|^p \mathrm{d}x = \gamma(n,p) l^{n+p}.$$

Next theorem is the main result of the paper. In its proof and throughout all the paper we shall denote by C a constant whose value may change from line to line.

Theorem 2.2. *Let $p > 1$ and $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$|\nabla f| \in L^p(\mathbb{R}^n) \quad \textit{if and only if} \quad \liminf_{\varepsilon\to 0^+} \kappa_\varepsilon(f;p) < \infty. \tag{2.3}$$

Moreover if $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $p \geq 1$ we have also

$$\lim_{\varepsilon\to 0^+} \kappa_\varepsilon(f;p) = \gamma(n,p) \int_{\mathbb{R}^n} |\nabla f|^p \mathrm{d}x. \tag{2.4}$$

Proof.
Step 1. Let us assume that $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p \geq 1$. We are going to show that

$$\limsup_{\varepsilon\to 0^+} \kappa_\varepsilon(f;p) \leq \gamma(n,p) \int_{\mathbb{R}^n} |\nabla f|^p \mathrm{d}x. \tag{2.5}$$

To this aim we may assume without loss of generality that $|\nabla f| \in L^p(\mathbb{R}^n)$. As a starting point we fix a bounded open set $\Omega \subset \mathbb{R}^n$. Then, given $\sigma > 0$, we take a function $g \in C^\infty_c(\mathbb{R}^n)$ such that $\|f-g\|_{W^{1,p}(\Omega)} < \sigma$ and choose $\varepsilon > 0$ such that

$$|\nabla g(x) - \nabla g(y)| \leq \sigma \quad \text{whenever} \quad x,y \in \mathbb{R}^n \quad \text{and} \quad |x-y| \leq \varepsilon\sqrt{n}/2. \tag{2.6}$$

Given $\delta \in (0,1)$, from the convexity of the function $t \mapsto |t|^p$ we get immediately that for every $a,b \in \mathbb{R}$

$$|a+b|^p = \left|\frac{1}{1+\delta}(1+\delta)a + \frac{\delta}{1+\delta}\frac{(1+\delta)b}{\delta}\right|^p \leq (1+\delta)^p|a|^p + M_\delta|b|^p, \tag{2.7}$$

where we have set

$$M_\delta := \frac{(1+\delta)^p}{\delta^p}\cdot$$

Let us take now a family $\mathcal{G}_\varepsilon$ of disjoint open cubes Q' of side length ε and arbitrary orientation and let us denote by $\mathcal{G}'_\varepsilon$ the subfamily of $\mathcal{G}_\varepsilon$ made by all cubes $Q' \in \mathcal{G}_\varepsilon$ such that $Q' \subset \Omega$. From (2.7) we have that for any $Q' \in \mathcal{G}'_\varepsilon$

$$\begin{aligned}\fint_{Q'} \left|f(x) - \fint_{Q'} f\right|^p \mathrm{d}x &\leq (1+\delta)^p \fint_{Q'} \left|g(x) - \fint_{Q'} g\right|^p \mathrm{d}x + M_\delta \fint_{Q'} \left|(f-g)(x) - \fint_{Q'} (f-g)\right|^p \mathrm{d}x \\ &\leq (1+\delta)^p \fint_{Q'} \left|g(x) - \fint_{Q'} g\right|^p \mathrm{d}x + C_p M_\delta \varepsilon^{p-n} \int_{Q'} |\nabla(f-g)|^p \,\mathrm{d}x.\end{aligned} \tag{2.8}$$

where C_p is the constant in the Poincaré inequality for the cubes. Denote by z the center of the cube $Q' \in \mathcal{G}'_\varepsilon$. For all $x \in Q'$ there exists $\bar{x} \in Q'$ such that

$$g(x) = g(z) + \nabla g(\bar{x}) \cdot (x - z) = g(z) + \nabla g(z) \cdot (x - z) + R(x),$$

where $|R(x)| \leq (\sqrt{n}\sigma\varepsilon)/2$, thanks to (2.6). Therefore, recalling the definition (2.2) of γ and (2.7) again, we have

$$\begin{aligned}
\fint_{Q'} \left| g(x) - \fint_{Q'} g \right|^p \mathrm{d}x &= \fint_{Q'} \left| \nabla g(z) \cdot (x - z) + R(x) - \fint_{Q'} R \right|^p \mathrm{d}x \\
&\leq (1+\delta)^p \fint_{Q'} |\nabla g(z) \cdot (x - z)|^p \mathrm{d}x + 2^p M_\delta \fint_{Q'} |R(x)|^p \,\mathrm{d}x \\
&\leq \gamma(n,p)(1+\delta)^p \varepsilon^p |\nabla g(z)|^p + C M_\delta \sigma^p \varepsilon^p,
\end{aligned}$$

where C depends only on n and p. Another application of (2.6) and (2.7), yields

$$|\nabla g(z)|^p \leq (1+\delta)^p \varepsilon^{-n} \int_{Q'} |\nabla g(x)|^p \mathrm{d}x + C M_\delta \sigma^p.$$

Hence, since $\delta \in (0,1)$,

$$\fint_{Q'} \left| g(x) - \fint_{Q'} g \right|^p \mathrm{d}x \leq (1+\delta)^{2p} \varepsilon^p \gamma(n,p) \fint_{Q'} |\nabla g(x)|^p \mathrm{d}x + C M_\delta \sigma^p \varepsilon^p. \tag{2.9}$$

Set $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\sqrt{n}\}$ and observe that $\#(\mathcal{G}'_\varepsilon) \leq \varepsilon^{-n}|\Omega|$. From (2.9), using the Poincaré inequality and recalling (2.8), we get

$$\begin{aligned}
&\varepsilon^{n-p} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p \mathrm{d}x \leq \varepsilon^{n-p} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p \mathrm{d}x + C \sum_{Q' \in \mathcal{G}_\varepsilon \setminus \mathcal{G}'_\varepsilon} \int_{Q'} |\nabla f|^p \mathrm{d}x \\
&\leq (1+\delta)^p \varepsilon^{n-p} \sum_{Q' \in \mathcal{G}'_\varepsilon} \fint_{Q'} \left| g(x) - \fint_{Q'} g \right|^p \mathrm{d}x + C M_\delta \int_\Omega |\nabla(f-g)|^p \mathrm{d}x + C \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} |\nabla f|^p \mathrm{d}x \\
&\leq (1+\delta)^{3p} \gamma(n,p) \sum_{Q' \in \mathcal{G}'_\varepsilon} \int_{Q'} |\nabla g|^p \mathrm{d}x + C M_\delta \sigma^p |\Omega| + C M_\delta \sigma^p + C \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} |\nabla f|^p \mathrm{d}x, \\
&\leq (1+\delta)^{3p} \gamma(n,p) \int_\Omega |\nabla f|^p \mathrm{d}x + C M_\delta \sigma^p + C \int_{\mathbb{R}^n \setminus \Omega_\varepsilon} |\nabla f|^p \mathrm{d}x,
\end{aligned}$$

for some constant C depending only on n, p and $|\Omega|$. Then, taking the supremum over all the families of cubes $\mathcal{G}_\varepsilon$, and then letting first $\varepsilon \to 0^+$, $\sigma \to 0$, $\delta \to 0$ and $\Omega \uparrow \mathbb{R}^n$ we obtain (2.5).

Step 2. Assume again that $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $p \geq 1$. As before we fix a bounded open set Ω. We fix also $\eta \in (0,1)$, $\sigma > 0$ and a function $g \in C^1_c(\mathbb{R}^n)$ such that $\|f - g\|_{W^{1,p}(\Omega)} < \sigma$. We set

$$U_\eta := \{x \in \Omega : |\nabla g(x)| > \eta\}.$$

Then, see the proof of Proposition 3.6 in [6], it is always possible to find k pairwise disjoint open sets $S_j \subset \mathbb{S}^{n-1}$ such that

$$\bigcup_{j=1}^k \overline{S}_j = \mathbb{S}^{n-1}, \qquad \mathrm{diam}\, S_j < \eta \quad \text{for all } j = 1, \dots, k \tag{2.10}$$

$$\left| \bigcup_{j=1}^k \left\{ x \in U_\eta : \frac{\nabla g(x)}{|\nabla g(x)|} \in \partial S_j \right\} \right| = 0, \tag{2.11}$$

where, when $X \subset \mathbb{S}^{n-1}$, the symbol ∂X denotes the relative boundary of X on $\mathbb{S}^{n-1}$. For all $j = 1, \dots, k$ we choose $\mu_j \in S_j$ and set

$$A_j := \left\{ x \in U_\eta : \frac{\nabla g(x)}{|\nabla g(x)|} \in S_j \right\}.$$

By construction the sets A_j are all open and by (2.11)

$$\left| U_\eta \setminus \bigcup_{j=1}^{k} A_j \right| = 0. \tag{2.12}$$

For $\varepsilon > 0$ we consider the family $\mathcal{F}_\varepsilon$ of all open cubes with faces parallel to the coordinate planes, side length ε, centered at all points of the form εv, with $v \in \mathbb{Z}^n$. Then for all $j = 1, \dots, k$ we denote by $\mathcal{R}_j \in SO(n)$ a rotation that takes $\bar{\nu}$ into $\mu_j \in S_j$, where $\bar{\nu}$ is the unit vector maximizing the integral in (2.2). Note that in this way, denoting by x' the center of the cube $Q' \in \mathcal{F}_\varepsilon$, we have, see Remark 2.1,

$$\int_{\mathcal{R}_j(Q')} |(x - x') \cdot \mu_j|^p \mathrm{d}x = \gamma(n,p)\varepsilon^{n+p}.$$

Set now

$$\mathcal{G}_\varepsilon = \bigcup_{j=1}^{k} \bigcup \{\mathcal{R}_j(Q') : \, Q' \in \mathcal{F}_\varepsilon \text{ and } \mathcal{R}_j(Q') \subset A_j\}.$$

For all $j = 1, \dots, k$ we denote by $\mathcal{R}_j(Q'_{h,j})$, $Q'_{h,j} \in \mathcal{F}_\varepsilon$, $h = 1, \dots, m_j$, the elements of $\mathcal{G}_\varepsilon$ contained in A_j. By (2.12) there exists $\varepsilon(\sigma, \eta) < 1$ such that if $\varepsilon < \varepsilon(\sigma, \eta)$ then

$$\left| U_\eta \setminus \bigcup_{j=1}^{k} \bigcup_{h=1}^{m_j} \mathcal{R}_j(Q'_{h,j}) \right| \le \eta^p \quad \text{and (2.6) holds.} \tag{2.13}$$

Denote now by $z_{h,j}$ the centers of the cubes $\mathcal{R}_j(Q'_{h,j})$ and argue as in the proof of (2.9), indicating by $R_{h,j}(x)$ the remainder term. Then, using (2.7) and recalling (2.6) and the second inequality in (2.10), we have

$$\begin{aligned}
\fint_{\mathcal{R}_j(Q'_{h,j})} \left| g(x) - \fint_{\mathcal{R}_j(Q'_{h,j})} g \right|^p \mathrm{d}x &= \fint_{\mathcal{R}_j(Q'_{h,j})} \left| \nabla g(z_{h,j}) \cdot (x - z_{h,j}) + R_{h,j}(x) - \fint_{\mathcal{R}_j(Q'_{h,j})} R_{h,j} \right|^p \mathrm{d}x \\
&\ge \frac{1}{(1+\delta)^p} \fint_{\mathcal{R}_j(Q'_{h,j})} |\nabla g(z_{h,j}) \cdot (x - z_{h,j})|^p \mathrm{d}x - \frac{2^p}{\delta^p} \fint_{\mathcal{R}_j(Q'_{h,j})} |R_{h,j}(x)|^p \mathrm{d}x \\
&\ge \frac{1}{(1+\delta)^{2p}} \fint_{\mathcal{R}_j(Q'_{h,j})} |\nabla g(z_{h,j})|^p |\mu_j \cdot (x - z_{h,j})|^p \mathrm{d}x \\
&\quad - \frac{1}{\delta^p} \fint_{\mathcal{R}_j(Q'_{h,j})} |\nabla g(z_{h,j})|^p \left| \left(\frac{\nabla g(z_{h,j})}{|\nabla g(z_{h,j})|} - \mu_j \right) \cdot (x - z_{h,j}) \right|^p \mathrm{d}x - \frac{C\sigma^p \varepsilon^p}{\delta^p} \\
&\ge \frac{\varepsilon^p \gamma(n,p) |\nabla g(z_{h,j})|^p}{(1+\delta)^{2p}} - \frac{C\eta^p \varepsilon^p}{\delta^p} \|\nabla g\|_{L^\infty}^p - \frac{C\sigma^p \varepsilon^p}{\delta^p}.
\end{aligned}$$

Observing again that $\#(\mathcal{G}'_\varepsilon) \le \varepsilon^{-n}|\Omega|$, from the previous inequality, adding up over j and h we get, arguing as in the proof of (2.9) and recalling (2.13),

$$\begin{aligned}\varepsilon^{n-p}\sum_{Q'\in\mathcal{G}'_\varepsilon}\fint_{Q'}\left|g(x)-\fint_{Q'}g\right|^p\mathrm{d}x &\ge \sum_{j=1}^{k}\sum_{h=1}^{m_j}\frac{\varepsilon^n\gamma(n,p)|\nabla g(z_{h,j})|^p}{(1+\delta)^{2p}}-\frac{C\eta^p}{\delta^p}\|\nabla g\|^p_{L^\infty}-\frac{C\sigma^p}{\delta^p}\\ &\ge \frac{\gamma(n,p)}{(1+\delta)^{3p}}\sum_{j=1}^{k}\sum_{h=1}^{m_j}\int_{Q'_{h,j}}|\nabla g|^p\,\mathrm{d}x-\frac{C\eta^p}{\delta^p}\|\nabla g\|^p_{L^\infty}-\frac{C\sigma^p}{\delta^p}\\ &\ge \frac{\gamma(n,p)}{(1+\delta)^{3p}}\int_\Omega|\nabla g|^p\,\mathrm{d}x-\frac{C\eta^p}{\delta^p}(1+\|\nabla g\|^p_{L^\infty})-\frac{C\sigma^p}{\delta^p}\\ &\ge \frac{\gamma(n,p)}{(1+\delta)^{4p}}\int_\Omega|\nabla f|^p\,\mathrm{d}x-\frac{C\eta^p}{\delta^p}(1+\|\nabla g\|^p_{L^\infty})-\frac{C\sigma^p}{\delta^p},\end{aligned}$$

for some positive constant C now depending on p, n and also on $|\Omega|$. Therefore, choosing η sufficiently small, and thus also ε small enough, we conclude, arguing as in Step 1,

$$\begin{aligned}\sum_{Q'\in\mathcal{G}_\varepsilon}\varepsilon^{n-p}\fint_{Q'}\left|f(x)-\fint_{Q'}f\right|^p\mathrm{d}x &\ge \frac{1}{(1+\delta)^p}\sum_{Q'\in\mathcal{G}_\varepsilon}\varepsilon^{n-p}\fint_{Q'}\left|g(x)-\fint_{Q'}g\right|^p\mathrm{d}x-\frac{C}{\delta^p}\int_\Omega|\nabla f-\nabla g|^p\mathrm{d}x\\ &\ge \frac{\gamma(n,p)}{(1+\delta)^{5p}}\int_\Omega|\nabla f|^p\mathrm{d}x-\frac{C\sigma^p}{\delta^p},\end{aligned}$$

for some constant depending as before only on p, n and $|\Omega|$. Then, taking the supremum over all the families $\mathcal{G}_\varepsilon$ and letting first $\varepsilon\to0^+$, $\sigma\to0^+$, $\delta\to0^+$ and $\Omega\uparrow\mathbb{R}^n$, we get

$$\liminf_{\varepsilon\to0^+}\kappa_\varepsilon(f,p)\ge\gamma(n,p)\int_{\mathbb{R}^n}|\nabla f|^p\mathrm{d}x.$$

This inequality, together with (2.5) concludes the proof of (2.4) when $f\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

Step 3. Now let $p>1$ and let us assume that $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. In order to prove (2.3) we fix $\sigma>0$ and set $f_\sigma := \varrho_\sigma * f$, where ϱ is a standard mollifier and $\varrho_\sigma(x)=\sigma^{-n}\varrho(x/\sigma)$. Then, given any family $\mathcal{G}_\varepsilon$ of disjoint open cubes Q' with side length ε and arbitrary orientation we have, using the definition of f_σ, Jensen inequality and Fubini's theorem,

$$\begin{aligned}\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\left|f_\sigma(x)-\fint_{Q'}f_\sigma\right|^p\mathrm{d}x &= \sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\left|\int_{\mathbb{R}^n}\varrho(y)f_\sigma(x-\sigma y)\,\mathrm{d}y-\fint_{Q'}\int_{\mathbb{R}^n}\varrho(y)f_\sigma(z-\sigma y)\,\mathrm{d}y\mathrm{d}z\right|^p\mathrm{d}x\\ &\le \int_{\mathbb{R}^n}\varrho(y)\left(\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\left|f(x-\sigma y)-\fint_{Q'}f(z-\sigma y)\,\mathrm{d}z\right|^p\mathrm{d}x\right)\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\varrho(y)\left(\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'-\sigma y}\left|f(x)-\fint_{Q'-\sigma y}f\right|^p\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

Therefore, since $\int\varrho=1$, we immediately deduce

$$\varepsilon^{n-p}\sum_{Q'\in\mathcal{G}_\varepsilon}\fint_{Q'}\left|f_\sigma(x)-\fint_{Q'}f_\sigma\right|\mathrm{d}x\le\kappa_\varepsilon(f;p)$$

and thus, since $f_\sigma\in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, from Step 2 we have

$$\gamma(n,p)\int_{\mathbb{R}^n}|\nabla f_\sigma|^p\mathrm{d}x\le\liminf_{\varepsilon\to0^+}\kappa_\varepsilon(f_\sigma;p)\le\liminf_{\varepsilon\to0^+}\kappa_\varepsilon(f;p).$$

Since $f_\sigma \to f$ in $L^p_{\rm loc}(R^n)$, we conclude that if $\liminf_{\varepsilon\to 0^+}\kappa_\varepsilon(f;p)<\infty$ then f is weakly differentiable and $|\nabla f|\in L^p(\mathbb{R}^n)$. The reverse implication is an immediate consequence of Step 1. $\square$

Observe that if $p=2$ the result proved in the previous theorem holds in a stronger form. In fact, as observed in Remark 2.1, in this case we have that the integral

$$\int_Q |x\cdot\nu|^2\,\mathrm{d}x$$

is constant with respect to $\nu\in\mathbb{S}^{n-1}$. Therefore, in the argument used in the Step 2 it is not necessary to rotate the cubes. Thus, if we denote by $\mathcal{I}_\varepsilon$ a family of disjoint open cubes of side length ε with all faces parallel to the coordinate planes and set for a function $f\in L^2_{\rm loc}(\mathbb{R}^n)$

$$K_\varepsilon(f;2):=\varepsilon^{n-2}\sup_{\mathcal{I}_\varepsilon}\sum_{Q'\in\mathcal{I}_\varepsilon}\fint_{Q'}\left|f(x)-\fint_{Q'}f\right|^2\mathrm{d}x$$

we have he following result.

Theorem 2.3. *Let $f\in L^2_{\rm loc}(\mathbb{R}^n)$. Then*

$$|\nabla f|\in L^2(\mathbb{R}^n)\quad \textit{if and only if}\quad \liminf_{\varepsilon\to 0^+}K_\varepsilon(f;2)<\infty.$$

Moreover if $f\in W^{1,2}_{\rm loc}(\mathbb{R}^n)$ we have also

$$\lim_{\varepsilon\to 0^+}K_\varepsilon(f;2)=\frac{1}{12}\int_{\mathbb{R}^n}|\nabla f|^2\mathrm{d}x.$$

Using a result proved in [6] we are now going to show a characterization of BV functions similar to the one provided by Theorem 2.2. To this aim, we recall that if $f\in L^1_{\rm loc}(\mathbb{R}^n)$ and $\Omega\subset\mathbb{R}^n$ is an open set the total variation of f in Ω, possibly equal to $+\infty$, is defined by setting

$$|Df|(\Omega):=\sup\left\{\int_\Omega f(x)\,\mathrm{div}\,\varphi(x)\,\mathrm{d}x:\ \varphi\in C^1_c(\Omega),\ \|\varphi\|_\infty\le 1\right\}.$$

We recall also, (see [2], Def. 3.35), that if $E\subset\mathbb{R}^n$ is a measurable set the perimeter of E in an open set Ω is defined as

$$P(E;\Omega):=|D\chi_E|(\Omega).$$

In the following, to denote the perimeter of E in the whole $\mathbb{R}^n$ we shall simply write $P(E)$.

Proposition 2.4. *Let $f\in L^1_{\rm loc}(\mathbb{R}^n)$. Then,*

$$\frac14|Df|(\mathbb{R}^n)\le\liminf_{\varepsilon\to 0^+}\kappa_\varepsilon(f;1)\le\limsup_{\varepsilon\to 0^+}\kappa_\varepsilon(f;1)\le\frac12|Df|(\mathbb{R}^n).\tag{2.14}$$

Proof. Arguing exactly in the Step 3 of the proof of Theorem 2.2 we have that if $f\in L^1_{\rm loc}(\mathbb{R}^n)$, then for all $\sigma>0$

$$\liminf_{\varepsilon\to 0^+}\kappa_\varepsilon(f_\sigma;1)\le\liminf_{\varepsilon\to 0^+}\kappa_\varepsilon(f;1).\tag{2.15}$$

On the other hand, since $\nabla f_\sigma \in L^1_{\rm loc}(\mathbb{R}^n)$ Theorem 3.3 in [6] yields that

$$\frac{1}{4}\int_{\mathbb{R}^n} |\nabla f_\sigma|\,\mathrm{d}x = \lim_{\varepsilon\to 0^+} \kappa_\varepsilon(f_\sigma;1).$$

Then, the first inequality in (2.14) follows immediately by combining the last equation with (2.15).

In order to prove the second inequality in (2.14) we may assume without loss of generality that $|Df|(\mathbb{R}^n) < \infty$. By the coarea formula for BV functions ([2], Thm. 3.40), we have

$$|Df|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} P(\{x\in\mathbb{R}^n:\ f(x)>t\})\,\mathrm{d}t.$$

Thus, for every integer j we may choose $t_{j,N} \in (\frac{j}{N}, \frac{j+1}{N})$ such that

$$P(\{x\in\mathbb{R}^n:\ f(x)>t_{j,N}\}) \le N\int_{\frac{j}{N}}^{\frac{j+1}{N}} P(\{x\in\mathbb{R}^n:\ f(x)>t\})\,\mathrm{d}t.$$

Then, we set for all $j\in\mathbb{Z}$

$$E_{j,N} := \{x\in\mathbb{R}^n:\ f(x)>t_{j,N}\} \quad \text{for } j\ge 0,$$

$$E_{j,N} := \{x\in\mathbb{R}^n:\ f(x)\le t_{j,N}\} \quad \text{for } j< 0,$$

$$h_N := \frac{1}{N}\sum_{j=0}^{+\infty}\chi_{E_{j,N}} - \frac{1}{N}\sum_{j=1}^{+\infty}\chi_{E_{-j,N}}.$$

From the definition of h_N, using the coarea formula, we have $|Dh_N|(\mathbb{R}^n) \le |Df|(\mathbb{R}^n)$. Now, let us fix a family $\mathcal{G}_\varepsilon$ of disjoint open cubes of side ε, with arbitrary orientation and recall that if E is a measurable set contained in an open cube Q of side l, then (see [1], Sect. 5 or [7]),

$$|E||Q\setminus E| \le \frac{l^{n+1}}{4}P(E;Q).$$

From this inequality we estimate easily

$$\begin{aligned}
\sum_{Q'\in\mathcal{G}_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|h_N(x) - \fint_{Q'} h_N\right| \mathrm{d}x &\le \frac{1}{N}\sum_{Q'\in\mathcal{G}_\varepsilon} \varepsilon^{n-1}\sum_{j=-\infty}^{\infty}\fint_{Q'}\left|\chi_{E_{j,N}}(x) - \fint_{Q'}\chi_{E_{j,N}}\right|\mathrm{d}x \\
&= \frac{1}{N}\sum_{Q'\in\mathcal{G}_\varepsilon}\frac{1}{\varepsilon^{n+1}}\sum_{j=-\infty}^{\infty} 2|E_{j,N}\cap Q'||Q'\setminus E_{j,N}| \\
&\le \frac{1}{2N}\sum_{Q'\in\mathcal{G}_\varepsilon}\sum_{j=-\infty}^{\infty} P(E_{j,N};Q') \\
&\le \frac{1}{2N}\sum_{j=-\infty}^{\infty} P(E_{j,N}) = \frac{1}{2}|Dh_N|(\mathbb{R}^n) \le \frac{1}{2}|Df|(\mathbb{R}^n). \qquad (2.16)
\end{aligned}$$

Fix now a bounded open set Ω and denote by $\mathcal{G}'_\varepsilon$ the family of all cubes in $\mathcal{G}_\varepsilon$ that are contained in Ω. Note that though the functions f and h_N are not necessarily bounded, by construction we have that $\|f-h_N\|_{L^\infty(\mathbb{R}^n)} \le 2/N$.

Thus, using the Poincaré inequality and the Sobolev imbedding theorem, we have, for some positive constant C depending only on n,

$$\begin{aligned}
&\sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\right| \mathrm{d}x \\
&\quad\le \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \left(\fint_{Q'} \left|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\right|^{\frac{n}{n-1}} \mathrm{d}x\right)^{\frac{n-1}{n}} \\
&\quad\le \frac{C}{N^{1/n}} \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \left(\fint_{Q'} \left|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\right| \mathrm{d}x\right)^{\frac{n-1}{n}} \\
&\quad\le \frac{C}{N^{1/n}} \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\right| \mathrm{d}x + \frac{C}{N^{1/n}} \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \\
&\quad\le \frac{C}{N^{1/n}} |D(f-h_N)|(\mathbb{R}^n) + \frac{C|\Omega|}{\varepsilon N^{1/n}}.
\end{aligned}$$

Then, from this inequality, setting $\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\sqrt{n}\}$, recalling (2.16), using the Poincaré inequality and still denoting by C a constant depending only on n, we get easily that

$$\begin{aligned}
\sum_{Q'\in\mathcal{G}_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|f(x) - \fint_{Q'} f\right| \mathrm{d}x &\le \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|f(x) - \fint_{Q'} f\right| \mathrm{d}x + \sum_{Q'\in\mathcal{G}_\varepsilon\setminus\mathcal{G}'_\varepsilon} C|Df|(Q') \\
&\le \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|h_N(x) - \fint_{Q'} h_N\right| \mathrm{d}x \\
&\quad + \sum_{Q'\in\mathcal{G}'_\varepsilon} \varepsilon^{n-1} \fint_{Q'} \left|(f(x)-h_N(x)) - \fint_{Q'}(f-h_N)\right| \mathrm{d}x + C|Df|(\mathbb{R}^n\setminus\Omega_\varepsilon) \\
&\le \frac{1}{2}|Df|(\mathbb{R}^n) + \frac{C}{N^{1/n}}|Df|(\mathbb{R}^n) + \frac{C|\Omega|}{\varepsilon N^{1/n}} + C|Df|(\mathbb{R}^n\setminus\Omega_\varepsilon).
\end{aligned}$$

The last inequality in (2.14) follows by letting first $N \to \infty$ and then taking the supremum over all $\mathcal{G}_\varepsilon$ and letting $\varepsilon \to 0^+$ and $\Omega \uparrow \mathbb{R}^n$. □

Remark 2.5. As already mentioned in the Introduction, if f belongs to the space $SBV(\mathbb{R}^n)$ of special functions of bounded variation one has

$$\lim_{\varepsilon\to0^+} \kappa_\varepsilon(f;1) = \frac{1}{4}\int_{\mathbb{R}^n} |\nabla f|\,\mathrm{d}x + \frac{1}{2}|Df|(\mathbb{R}^n).$$

If f is the function coinciding with the Cantor-Vitali function in $[0,1]$, $f(x) = 0$ for $x \le 0$, and $f(x) = 1$ for $x \ge 1$, then $\nabla f = 0$ and, (see [6], Exp. 2.2),

$$\liminf_{\varepsilon\to0^+} \kappa_\varepsilon(f;1) < \frac{1}{2}|Df|(\mathbb{R}).$$

We conclude this section by observing that Theorem 2.2 holds also in an open set Ω with the same proof replacing $\kappa_\varepsilon(f;p)$ by the quantity

$$\kappa_\varepsilon(f;p;\Omega) := \varepsilon^{n-p} \sup_{\mathcal{G}_\varepsilon} \sum_{Q'\in\mathcal{G}_\varepsilon} \fint_{Q'} \left|f(x) - \fint_{Q'} f\right|^p \mathrm{d}x,$$

where the supremum on the right hand side is taken over all families $\mathcal{G}_\varepsilon$ of disjoint open cubes Q' of side length ε and arbitrary orientation contained in Ω. A similar local version holds also for Theorem 2.3 and Proposition 2.4.

3. Fractional Sobolev and Nikol'skij spaces

In this section we extend the previous results to the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ and to the Nikol'skij spaces $N^{s,p}(\mathbb{R}^n)$, where $p \geq 1$ and $s \in (0,1)$.

We start by recalling that the *Gagliardo seminorm* of a function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ is defined by setting for $0 < s < 1$

$$[f]_{s,p} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, {\rm d}x{\rm d}y \right)^{\frac{1}{p}}. \tag{3.1}$$

Then the Sobolev fractional space $W^{s,p}(\mathbb{R}^n)$ is the set of all functions $u \in L^p(\mathbb{R}^n)$ such that the above seminorm is finite.

The integral in (3.1) is closely related to the fractional difference quotients. Before proving this fact let us set for any number $h \neq 0$ and $i = 1, \ldots, n$

$$\Delta_{i,h} f(x) := f(x + he_i) - f(x) \qquad\qquad x \in \mathbb{R}^n,$$

where $\{e_1, \ldots, e_n\}$ is the standard base of $\mathbb{R}^n$. The following result is a straightforward variant of a result proved in ([8], Lem. 2.5). For the reader's convenience we give here the proof. But before, we need to introduce some more notation. Given $f \in L^p(\mathbb{R}^n)$, $s \in (0,1]$ and $d > 0$, we set

$$M_d(f;p,s) := \sup_{0<|h|<d} \max_{i=1,\ldots,n} \frac{\|\Delta_{i,h} f\|_{L^p(\mathbb{R}^n)}}{|h|^s}. \tag{3.2}$$

Lemma 3.1. *Let $p \geq 1$. There exists a constant C depending only on n and p such that for every $f \in L^p(\mathbb{R}^n)$, $0 < s' < s \leq 1$ and $d > 0$*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+s'p}} \, {\rm d}x{\rm d}y \leq C \left[\frac{d^{(s-s')p} M_d(f;p,s)^p}{s - s'} + \frac{\|f\|^p_{L^p(\mathbb{R}^n)}}{s' d^{s'p}} \right]. \tag{3.3}$$

Proof. We follow the proof of Lemma 1.50 in [3], setting for any vector $v \in \mathbb{R}^n$

$$v^{(0)} := 0, \qquad v^{(k)} := \sum_{i=1}^{k} (v \cdot e_i) e_i \quad \text{for all } k = 1, \ldots, n.$$

Then, given $x, v \in \mathbb{R}^n$, we decompose the increment of f from x to $x + v$ along the coordinate axes as follows

$$f(x+v) - f(x) = \sum_{k=1}^{n} \Delta_{k,v_k} f\left(x + v^{(k-1)}\right). \tag{3.4}$$

Thus, given $d > 0$, with a change of variable we may estimate

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+s'p}} \, {\rm d}x{\rm d}y &= \int_{|v|<d} {\rm d}v \int_{\mathbb{R}^n} \frac{|f(x{+}v) - f(x)|^p}{|v|^{n+s'p}} \, {\rm d}x + \int_{|v|>d} {\rm d}v \int_{\mathbb{R}^n} \frac{|f(x{+}v) - f(x)|^p}{|v|^{n+s'p}} \, {\rm d}x \\ &\leq C \int_{|v|<d} \frac{{\rm d}v}{|v|^{n+(s'-s)p}} \int_{\mathbb{R}^n} \sum_{k=1}^{n} \frac{\left|\Delta_{k,v_k} f(x + v^{(k-1)})\right|^p}{|v|^{sp}} \, {\rm d}x \\ &\quad + 2^p \|f\|_{L^p(\mathbb{R}^n)} \int_{|v|>d} \frac{{\rm d}v}{|v|^{n+s'p}}, \end{aligned}$$

from which (3.3) follows. ☐

We now need to extend the definition given in (2.1). To this aim, let us fix $p \geq 1$ and $0 < s \leq 1$. Given a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, for any $\varepsilon > 0$, we set

$$\kappa_\varepsilon(f;p,s) := \varepsilon^{n-sp} \sup_{\mathcal{G}_\varepsilon} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p \mathrm{d}x, \tag{3.5}$$

where $\mathcal{G}_\varepsilon$ denotes a family of disjoint open cubes Q' of side length ε and arbitrary orientation.

The next result relates the supremum of the above quantity to the Gagliardo seminorms of an L^p function.

Proposition 3.2. *Let $p \geq 1$, There exists a constant C, depending only on n and p such that for every $f \in L^p(\mathbb{R}^n)$, $0 < s < 1$,*

$$\sup_{\varepsilon>0} \kappa_\varepsilon(f;p,s) \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\mathrm{d}y. \tag{3.6}$$

Moreover, for any $0 < s' < s$ and any $d > 0$

$$\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+s'p}} \,\mathrm{d}x\mathrm{d}y \leq c(n,p) \left[\frac{d^{(s-s')p}}{s-s'} \sup_{0<\varepsilon<d} \kappa_\varepsilon(f;p,s) + \frac{\|f\|^p_{L^p(\mathbb{R}^n)}}{s' d^{s'p}} \right]. \tag{3.7}$$

Remark 3.3. Note that the seminorm $[f]_{s,p}$ is not equivalent to the supremum of the *BMO*-type seminorms $\kappa_\varepsilon(f;p,s)$. To see this take a function f belonging to the Nikol'skij space $N^{s,p}(\mathbb{R}^n)$ and not in $W^{s,p}(\mathbb{R}^n)$ and use Proposition 3.4.

Proof of Proposition 3.2*.* Let $\mathcal{G}_\varepsilon$ be a family of disjoint cubes of side length ε and arbitrary orientation. Then, we have

$$\begin{aligned} \varepsilon^{n-sp} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'} \left| f(x) - \fint_{Q'} f \right|^p \mathrm{d}x &\leq \varepsilon^{n-sp} \sum_{Q' \in \mathcal{G}_\varepsilon} \fint_{Q'}\fint_{Q'} |f(x)-f(y)|^p \mathrm{d}x\mathrm{d}y \\ &= \varepsilon^{-n-sp} \sum_{Q' \in \mathcal{G}_\varepsilon} \int_{Q'}\int_{Q'} \frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}} |x-y|^{n+ps} \mathrm{d}x\mathrm{d}y \\ &\leq C \sum_{Q' \in \mathcal{G}_\varepsilon} \int_{Q'}\int_{Q'} \frac{|f(x)-f(y)|^p}{|x-y|^{n+ps}} \mathrm{d}x\mathrm{d}y \leq C[f]^p_{s,p}, \end{aligned}$$

for some constant C depending only on n and p. From this inequality (3.6) follows at once.

In order to prove (3.7) it is enough to show that for any $d > 0$

$$M_d(f;p,s)^p \leq c(n,p) \sup_{0<\varepsilon<3d} \kappa_\varepsilon(f;p,s) \tag{3.8}$$

and then to apply Lemma 3.1 with d replaced by $d/3$. To this aim, let us fix $0 < |h| < d$ and $i \in \{1,\ldots,n\}$. Then, let us cover almost all $\mathbb{R}^n$ with the family $\mathcal{F}$ of the open cubes $Q_v = v + (-|h|/2, |h|/2)^n$ with $v \in |h|\mathbb{Z}^n$. Then, let us denote by $\hat{Q}_v$ the cube $v + (-3|h|/2, 3|h|/2)^n$, that is the cube with the same center of Q_v and triple side length. Observe that we may distribute all the cubes Q_v in 3^n distinct subfamilies $\mathcal{F}_k$, $k = 1,\ldots,3^n$, in such a way that if $Q_v, Q_{v'} \in \mathcal{F}_k$ for some k and $v \neq v'$, then $\hat{Q}_v \cap \hat{Q}_{v'} = \emptyset$. In this way we get the following estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{|f(x+he_i)-f(x)|^p}{|h|^{sp}}\,\mathrm{d}x\\
&\le C|h|^{-sp}\sum_{v\in|h|\mathbb{Z}^n}\int_{Q_v}\left(\left|f(x+he_i)-\fint_{Q_v+he_i} f\right|^p+\left|f(x)-\fint_{\hat{Q}_v} f\right|^p+\left|\fint_{Q_v+he_i} f-\fint_{\hat{Q}_v} f\right|^p\right)\mathrm{d}x\\
&\le C|h|^{n-sp}\sum_{k=1}^{3^n}\sum_{Q_v\in\mathcal{F}_k}\left(\fint_{Q_v+he_i}\left|f(x)-\fint_{Q_v+he_i} f\right|^p\mathrm{d}x+2\fint_{\hat{Q}_v}\left|f(x)-\fint_{\hat{Q}_v} f\right|^p\mathrm{d}x\right)\\
&\le C\big(\kappa_{|h|}(f;p,s)+\kappa_{3|h|}(f;p,s)\big),
\end{aligned}$$

where C depends only on n and p. Then, (3.8) follows taking first the supremum with respect to $i\in 1,\dots,n$ and then with respect to $h\in(-d,d)\setminus\{0\}$. □

From the proof of Proposition 3.2 it is clear that (3.6) and (3.7) still hold if we replace $\kappa_\varepsilon(f;p,s)$ by the smaller quantity

$$K_\varepsilon(f;p,s):=\varepsilon^{n-sp}\sup_{\mathcal{I}_\varepsilon}\sum_{Q'\in\mathcal{I}_\varepsilon}\fint_{Q'}\left|f(x)-\fint_{Q'} f\right|^p\mathrm{d}x,$$

where, as in (3.5), $\mathcal{I}_\varepsilon$ denotes a family of disjoint open cubes of side length ε with faces parallel to the coordinate planes.

We conclude this section by a quick discussion on the Nikol'skij space. Recall that if $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, for some $p\ge 1$ and $0<s\le 1$ the *Nikol'skij seminorm* of f is defined by setting

$$[f]_{N^{s,p}}:=\sup_{h\neq 0}\max_{i=1,\dots,n}\frac{\|\varDelta_{i,h}f\|_{L^p(\mathbb{R}^n)}}{|h|^s}.$$

Then the Nikol'skij space $N^{s,p}(\mathbb{R}^n)$ is the space of all functions $f\in L^p(\mathbb{R}^n)$ such that $[f]_{N^{s,p}}$ is finite. Note also that the above seminorm coincides with the supremum with the respect to d of the quantity introduced in (3.2).

Proposition 3.4. *Let $p\ge 1$, $0<s\le 1$. There exists a constant C, depending only on n and p such that for every $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$*

$$\frac{1}{C}\sup_{\varepsilon>0}\kappa_\varepsilon(f;p,s)\le[f]^p_{N^{s,p}}\le C\sup_{\varepsilon>0}\kappa_\varepsilon(f;p,s). \tag{3.9}$$

Proof. The second inequality in (3.9) follows at once by taking the supremum with respect to $d>0$ on both sides of (3.8).

To prove the opposite inequality we fix $\varepsilon>0$ and consider a family $\mathcal{G}_\varepsilon$ of disjoint open cubes Q_i, $i\in I$, of side length ε and arbitrary orientation. Then for every $i\in I$ by a change of variable we have

$$\begin{aligned}
\fint_{Q_i}\left|f(x)-\fint_{Q_i} f\right|^p\mathrm{d}x &\le \fint_{Q_i}\mathrm{d}x\fint_{Q_i}|f(x)-f(y)|^p\,\mathrm{d}y=\fint_{Q_i}\mathrm{d}x\fint_{x-Q_i}|f(x)-f(x+w)|^p\,\mathrm{d}w\\
&\le \varepsilon^{-2n}\int_{B_{\varepsilon\sqrt{n}}}\mathrm{d}w\int_{Q_i}|f(x+w)-f(x)|^p\,\mathrm{d}x.
\end{aligned}$$

Decomposing the difference $f(x+w)-f(x)$ as in (3.4) and summing up we then get

$$
\begin{aligned}
\varepsilon^{n-sp}\sum_{i\in I}\fint_{Q_i}\left|f(x)-\fint_{Q_i}f\right|^p\mathrm{d}x &\le \varepsilon^{-n-sp}\int_{B_{\varepsilon\sqrt{n}}}\mathrm{d}w\int_{\mathbb{R}^n}|f(x+w)-f(x)|^p\,\mathrm{d}x\\
&\le C\varepsilon^{-n-sp}\int_{B_{\varepsilon\sqrt{n}}}\mathrm{d}w\int_{\mathbb{R}^n}\sum_{k=1}^n|\Delta_{k,w_k}f(x+w^{(k-1)})|^p\,\mathrm{d}x\\
&\le C\varepsilon^{-n-sp}\sum_{k=1}^n\int_{B_{\varepsilon\sqrt{n}}}|w^{(k-1)}|^{sp}\,\mathrm{d}w\int_{\mathbb{R}^n}\frac{|\Delta_{k,w_k}f(x+w^{(k-1)})|^p}{|w^{(k-1)}|^{sp}}\,\mathrm{d}x\\
&\le C\varepsilon^{-n-sp}[f]^p_{N^{s,p}}\sum_{k=1}^n\int_{B_{\varepsilon\sqrt{n}}}|w^{(k-1)}|^{sp}\,\mathrm{d}w\le C[f]^p_{N^{s,p}}.
\end{aligned}
$$

Then, the first inequality in (3.9) follows taking the supremum over all families $\mathcal{G}_\varepsilon$ and $\varepsilon>0$. □

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