# HOMOGENEITY RESULTS FOR INVARIANT DISTRIBUTIONS OF A REDUCTIVE $p$-ADIC GROUP * 

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#### Abstract

Let $k$ denote a complete nonarchimedean local field with finite residue field. Let $G$ be the group of $k$-rational points of a connected reductive linear algebraic group defined over $k$. Subject to some conditions, we establish a range of validity for the Harish-Chandra-Howe local expansion for characters of admissible irreducible representations of $G$. Subject to some restrictions, we also verify two analogues of this result.


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RÉSUMÉ. - Soit $k$ un corps local non archimédien de corps résiduel fini. Soit $G$ le groupe des points $k$-rationnels d'un groupe algébrique linéaire réductif connexe défini sur $k$. Sous certaines conditions, nous établissons le domaine de validité pour le développement local de Harish-Chandra-Howe pour les caractères des représentations irréductibles admissibles de $G$. Nous vérifions également deux analogues de ce résultat.
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## Introduction

Let $k$ be a field with nontrivial discrete valuation and residue field $\mathfrak{f}$. We suppose that $k$ is complete and $\mathfrak{f}$ is finite. Suppose that $G$ is the group of $k$-rational points of a reductive connected linear algebraic group defined over $k$. Let $\mathfrak{g}$ denote the Lie algebra of $G$.

In this paper we prove homogeneity results for $G$-invariant distributions on $G$ and $\mathfrak{g}$. All of these results have their origins in a conjecture of Thomas Hales, Allen Moy, and Gopal Prasad [21,§1]; we now discuss this conjecture and the results of this paper.

## Homogeneity for characters

Suppose that $(\pi, V)$ is an irreducible admissible representation of $G$. We first recall some facts about the character of $(\pi, V)$.

Let $C_{c}^{\infty}(G)$ denote the space of complex-valued, locally constant, compactly supported functions on $G$. Let $d g$ denote a Haar measure on $G$. For $f \in C_{c}^{\infty}(G)$ we define the finite-rank operator $\pi(f): V \rightarrow V$ by

$$
\pi(f) v=\int_{G} f(g) \cdot \pi(g) v d g
$$

[^0]for $v \in V$. In [14] Harish-Chandra shows that the character distribution
$$
\Theta_{\pi}: C_{c}^{\infty}(G) \rightarrow \mathbb{C}
$$
which sends $f \in C_{c}^{\infty}(G)$ to $\operatorname{tr}(\pi(f))$ is represented on $G^{\text {reg }}$, the set of regular semisimple elements in $G$, by a function which is locally constant on $G^{\text {reg }}$. We abuse notation and denote by $\Theta_{\pi}$ both the character distribution and the function which represents it. Thus we have, for all $f \in C_{c}^{\infty}\left(G^{\mathrm{reg}}\right)$,
$$
\Theta_{\pi}(f)=\int_{G} f(g) \cdot \Theta_{\pi}(g) d g
$$

Suppose that either $G=\mathbf{G} \mathbf{L}_{n}(k)$ or $k$ has characteristic zero. Suppose we have a reasonable map $\varphi$ from $\mathfrak{g}$ to $G$ in a neighborhood of zero. (If $k$ has characteristic zero, then the exponential map will do.) Howe [15] and Harish-Chandra [13] showed that in a sufficiently small neighborhood of zero we have a local expansion for the character $\Theta_{\pi}$. That is, for all regular semisimple $X$ in $\mathfrak{g}$ sufficiently close to zero, we have the asymptotic expansion

$$
\Theta_{\pi}(\varphi(X))=\sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu_{\mathcal{O}}}(X) .
$$

This expression is referred to as the Harish-Chandra-Howe local expansion. Here $\mathcal{O}(0)$ is the set of nilpotent orbits, the $c_{\mathcal{O}}(\pi)$ are complex-valued constants which depend on our orbit $\mathcal{O}$ and our representation $(\pi, V)$, and the $\widehat{\mu_{\mathcal{O}}}$ s are functions on the Lie algebra which are determined by $\mathcal{O}$. In other words, sufficiently near the identity, a character is given by a linear combination of functions on $\mathfrak{g}$, and these functions are independent of our representation.

Let $G$ again denote the group of $k$-rational points of an arbitrary connected reductive group defined over $k$. The conjecture of Hales, Moy, and Prasad defines a region on which the Harish-Chandra-Howe local expansion ought to be valid. This predicted region depends on a rational number which Moy and Prasad [21] call $\rho(\pi)$, the depth of $(\pi, V)$. For example, if $(\pi, V)$ has a non-trivial Iwahori-fixed vector, then $\rho(\pi)=0$.

Conjecture 1 (Hales-Moy-Prasad). - If $(\pi, V)$ is an irreducible, admissible representation of $G$, then the Harish-Chandra-Howe local character expansion of $(\pi, V)$ is valid on $G_{\rho(\pi)^{+}} \cap G^{\mathrm{reg}}$.

The set $G_{\rho(\pi)^{+}}$is a $G$-invariant, open, closed subset of $G$. For example, if $\rho(\pi)=0$, then $G_{\rho(\pi)^{+}}$is often referred to as the set of topologically unipotent elements in $G$.

In two remarkable papers [25,28], J.-L. Waldspurger proves this conjecture for integral depth representations of "classical unramified groups" (with some hypotheses on $\mathbf{G}$ and $k$ ); many of the techniques I use were inspired by (or borrowed from) Waldspurger's work. Additionally, Conjecture 1 is known to be true for $\mathbf{G L}_{2}(k)$ [18], $\mathbf{G L}_{3}(k)$ [10], and the groups $\mathbf{S L}_{2}(k)$ [23], $\mathbf{G S p} \mathbf{p}_{4}(k)$ [8], and $\mathbf{S p}_{4}(k)$ [8] when $k$ has odd residual characteristic.

One difficulty with Conjecture 1 is that it assumes the existence of a reasonable $G$-equivariant map from $\mathfrak{g}^{\text {reg }}$, the set of regular semisimple elements in $\mathfrak{g}$, to $G^{\text {reg }}$ on a rather large domain; such a map is not known to exist in general. (See Hypothesis 3.2.1 and the discussion following it.) Another difficulty with this conjecture is that it assumes that the functions $\widehat{\mu_{\mathcal{O}}}$ can be defined. In Theorem 3.5.2 we verify Conjecture 1 subject to the existence of a reasonable map and some additional conditions.

## Two analogues of Conjecture 1

In order to state these analogues, we must first recall some facts about the Moy-Prasad filtrations.

Let $\mathcal{B}(G)$ denote the reduced Bruhat-Tits building of $G$. To each $x \in \mathcal{B}(G)$ and each $r \in \mathbb{R}$, Moy and Prasad [19,21] associate a lattice $\mathfrak{g}_{x, r}$ in $\mathfrak{g}$. Similarly, if $r \in \mathbb{R}_{\geqslant 0}$, then Moy and Prasad define an open compact subgroup $G_{x, r}$ of (the stabilizer of $x$ in) $G$. Without further comment, we will assume that $r \in \mathbb{R}$ (respectively, $r \in \mathbb{R}_{\geqslant 0}$ ) when we discuss objects in $\mathfrak{g}$ (respectively, objects in $G$ ).

Fix $r \in \mathbb{R}$. We define

$$
\mathfrak{g}_{r}:=\bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x, r} \quad \text { and } \quad G_{r}:=\bigcup_{x \in \mathcal{B}(G)} G_{x, r}
$$

These objects have been studied in [2,10,12]. We recall that if $\mathcal{N}$ denotes the set of nilpotent elements in $\mathfrak{g}$ and $\mathcal{U}$ denotes the set of unipotent elements in $G$, then we have [2]

$$
\mathfrak{g}_{r}=\bigcap_{x \in \mathcal{B}(G)} \mathfrak{g}_{x, r}+\mathcal{N} \quad \text { and } \quad G_{r}=\bigcap_{x \in \mathcal{B}(G)} G_{x, r} \cdot \mathcal{U}
$$

Consequently, $\mathfrak{g}_{r}$ (respectively, $G_{r}$ ) is a $G$-domain; that is, it is a $G$-invariant, open, and closed subset of $\mathfrak{g}$ (respectively, $G$ ).

We define the subspace $\mathcal{H}_{r}$ of $C_{c}^{\infty}(G)$ by

$$
\mathcal{H}_{r}:=\sum_{x \in \mathcal{B}(G)} C_{c}\left(G / G_{x, r}\right)
$$

We interpret the sum on the right in the following way. If $f \in \mathcal{H}_{r}$, then we can write $f$ as a finite sum $f=\sum_{i} f_{i}$ where each $f_{i}$ is a complex-valued, compactly supported function which is invariant under right translation by $G_{y_{i}, r}$ for some $y_{i} \in \mathcal{B}(G)$.

We let $C_{c}^{\infty}(\mathfrak{g})$ denote the space of complex-valued, compactly supported, locally constant functions on $\mathfrak{g}$. We define the subspace $\mathcal{D}_{r}$ of $C_{c}^{\infty}(\mathfrak{g})$ by

$$
\mathcal{D}_{r}:=\sum_{x \in \mathcal{B}(G)} C_{c}\left(\mathfrak{g} / \mathfrak{g}_{x, r}\right)
$$

We interpret the sum on the right as above; if $f \in \mathcal{D}_{r}$, then we can write $f$ as a finite sum $f=\sum_{i} f_{i}$ where each $f_{i}$ is a complex-valued, compactly supported function which is invariant under translation by $\mathfrak{g}_{y_{i}, r}$ for some $y_{i} \in \mathcal{B}(G)$.

A distribution on $G$ is a complex-valued, linear function on $C_{c}^{\infty}(G)$. For $g, h \in G$, we let ${ }^{g} h=g h g^{-1}$. If $f \in C_{c}^{\infty}(G)$ and $g \in G$, we define $f^{g} \in C_{c}^{\infty}(G)$ by

$$
f^{g}(h)=f\left({ }^{g} h\right)
$$

for $h \in G$. If $T$ is a distribution on $G$ and $g \in G$, then the distribution ${ }^{g} T$ is defined by

$$
{ }^{g} T(f)=T\left(f^{g}\right)
$$

for $f \in C_{c}^{\infty}(G)$. The distribution $T$ is said to be $G$-invariant if ${ }^{g} T=T$ for all $g \in G$. We let $J(G)$ denote the set of $G$-invariant distributions on $G$.

For $\mathfrak{g}$ we define $J(\mathfrak{g})$, the space of $G$-invariant distributions on $\mathfrak{g}$, in an analogous fashion.
Let $J\left(\mathfrak{g}_{r}\right)$ denote the space of $G$-invariant distributions on $\mathfrak{g}$ with support in $\mathfrak{g}_{r}$. Let $J(\mathcal{N})$ denote the space of $G$-invariant distributions on $\mathfrak{g}$ with support in $\mathcal{N}$. If $T$ is a distribution on $\mathfrak{g}$, then we let $\operatorname{res}_{\mathcal{D}_{r}} T$ denote the restriction of $T$ to $\mathcal{D}_{r}$.

If $r \in \mathbb{Z}$, then a variant of Conjecture 2 (which is our first analogue of Conjecture 1) has been stated by Waldspurger [28, §I.2].

Conjecture 2. - If $r \in \mathbb{R}$, then

$$
\operatorname{res}_{\mathcal{D}_{r}} J\left(\mathfrak{g}_{r}\right)=\operatorname{res}_{\mathcal{D}_{r}} J(\mathcal{N}) .
$$

In the stunning paper [28], Waldspurger proved this conjecture for "classical unramified groups" with $r \in \mathbb{Z}$ (and some hypotheses on $\mathbf{G}$ and $k$ ). For the group $\mathbf{G L}_{n}(k)$ and $r \in \mathbb{Z}+\frac{1}{n}$, it is verified in [9]. Additionally, it is known to be true for the groups $\mathbf{S L}_{2}(k), \mathbf{G S p} \mathbf{p}_{4}(k)$, and $\mathbf{S p}_{4}(k)$ when the residual characteristic of $k$ is odd [8]. In Theorem 2.1 .5 we prove Conjecture 2 subject to some conditions. In fact, as in $[13,15,28]$, we prove an apparently stronger statement.

Let $J\left(G_{r}\right)$ denote the space of $G$-invariant distributions on $G$ with support in $G_{r}$. Let $J(\mathcal{U})$ denote the space of $G$-invariant distributions on $G$ with support in $\mathcal{U}$. If $T$ is a distribution on $G$, then we let $\operatorname{res}_{\mathcal{H}_{r}} T$ denote the restriction of $T$ to $\mathcal{H}_{r}$. We now state the second analogue of Conjecture 1.

Conjecture 3. - If $r \in \mathbb{R}_{\geqslant 0}$, then

$$
\operatorname{res}_{\mathcal{H}_{r}} J\left(G_{r}\right)=\operatorname{res}_{\mathcal{H}_{r}} J(\mathcal{U}) .
$$

In the case when $r=0$, Waldspurger [27] has a conjecture describing the dimension of $\operatorname{res}_{\mathcal{H}_{0}} J(\mathcal{U})$. In Section 4.2, we show that there is a dual basis for $\operatorname{res}_{\mathcal{H}_{r}} J\left(G_{r}\right)$ consisting of functions having support in $G_{0}$. If $r>0$, then in Theorem 4.1.4 we prove Conjecture 3 subject to some conditions.

Remark. - There is no difficulty in extending the main results of this paper to the case where $\mathbf{G}$ is disconnected; we refer the reader to [7] and [11] for the appropriate definitions in this setting.

## 1. Notation

In addition to the notation introduced in the introduction, we will require the following.

### 1.1. Basic notation

We let $\nu$ denote our discrete valuation on $k$, and we suppose that $\mathfrak{f}$ has characteristic $p$. Denote the ring of integers of $k$ by $R$ and the prime ideal by $\wp$. Fix a uniformizing element $\varpi$. Let $\Lambda$ be a fixed complex-valued additive character on $k^{+}$which is nontrivial on $R$ and trivial on $\wp$.

Let $K$ be a fixed maximal unramified extension of $k$.
Let $\mathbf{G}$ be a connected, reductive, linear algebraic group defined over $k$. We let $G=\mathbf{G}(k)$, the group of $k$-rational points of $\mathbf{G}$. We denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$. We let $\mathfrak{g}=\mathfrak{g}(k)$, the vector space of $k$-rational points of $\mathfrak{g}$. Let [, ] denote the Lie bracket operation for $\mathfrak{g}$.

Let $L$ be the minimal Galois extension of $K$ such that $\mathbf{G}$ is $L$-split. As in [19] we define $\ell=[L: K]$. We also denote by $\nu$ the unique extension of $\nu$ to any algebraic extension of $k$. As in [19], we normalize $\nu$ by requiring $\nu\left(L^{\times}\right)=\mathbb{Z}$.

If $g, h \in G$, then ${ }^{g} h=g h g^{-1}$. If $S \subset G$, then we let ${ }^{G} S$ denote the set $\left\{{ }^{g}{ }_{s} \mid g \in G\right.$ and $\left.s \in S\right\}$. If $h \in G$, then we write ${ }^{G} h$ for ${ }^{G}\{h\}$, the $G$-orbit of $h$. If $g \in G$ and $X \in \mathfrak{g}$, then ${ }^{g} X=\operatorname{Ad}(g) X$.

If $S \subset \mathfrak{g}$, then we let ${ }^{G} S$ denote the set $\left\{{ }^{g} X \mid g \in G\right.$ and $\left.X \in S\right\}$. If $X \in \mathfrak{g}$, then we write ${ }^{G} X$ for ${ }^{G}\{X\}$, the $G$-orbit of $X$.

An element $X \in \mathfrak{g}$ is called nilpotent provided that there exists $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{G})$ such that $\lim _{t \rightarrow 0}{ }^{\lambda(t)} X=0$. Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathfrak{g}$ and let $\mathcal{O}(0)$ denote the set of nilpotent $G$-orbits in $\mathfrak{g}$. We say that $h \in G$ is unipotent provided that there exists $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{G})$ such that $\lim _{t \rightarrow 0}{ }^{\lambda(t)} h=1$. Let $\mathcal{U}$ denote the set of unipotent elements in $G$.

Let $n$ denote the rank of $\mathbf{G}$. We say that an element $g \in G$ is regular semisimple if the coefficient of $t^{n}$ in $\operatorname{det}(t-1+\operatorname{Ad}(g))$ is nonzero. We let $G^{\text {reg }}$ denote the set of regular semisimple elements in $G$. Similarly, we say that an element $X \in \mathfrak{g}$ is regular semisimple if the coefficient of $t^{n}$ in $\operatorname{det}(t-\operatorname{ad}(X))$ is nonzero. We let $\mathfrak{g}^{\text {reg }}$ denote the set of regular semisimple elements in $\mathfrak{g}$.

If a group $H$ acts on a set $S$, then $S^{H}$ denotes the set of $H$-fixed points of $S$.
If $S$ is a set, then we let $|S|$ denote the cardinality of $S$.
For a subset $S$ of $\mathfrak{g}$ (respectively, $G$ ) we let $[S]$ denote the characteristic function of $S$ on $\mathfrak{g}$ (respectively, $G$ ).

### 1.2. Apartments, buildings, and associated notation

Let $\mathcal{B}(G)=\mathcal{B}(\mathbf{G}, k)$ denote the reduced Bruhat-Tits building of $G$.
We let dist: $\mathcal{B}(G) \times \mathcal{B}(G) \rightarrow \mathbb{R}_{\geqslant 0}$ denote a (nontrivial) $G$-invariant distance function as discussed in $[24, \S 2.3]$. For $x, y \in \mathcal{B}(G)$, let $[x, y]$ denote the geodesic in $\mathcal{B}(G)$ from $x$ to $y$ and let $(x, y]$ denote $[x, y] \backslash\{x\}$. We define $(x, y)$ and $[x, y)$ similarly.

For $\Omega \subset \mathcal{B}(G)$, we let $\operatorname{stab}_{G}(\Omega)$ denote the stabilizer of $\Omega$ in $G$.
Given a maximal $k$-split torus $\mathbf{S}$ we have the torus $S=\mathbf{S}(k)$ in $G$ and the corresponding apartment $\mathcal{A}(S)=\mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(G)$.

We let $\Phi(S)=\Phi(\mathbf{S}, k)=\Phi(\mathcal{A})$ denote the set of roots of $\mathbf{G}$ with respect to $k$ and $\mathbf{S}$; we denote by $\Psi(S)=\Psi(\mathcal{A})=\Psi(\mathbf{S}, k)=\Psi(\mathbf{S}, k, \nu)$ the set of affine roots of $\mathbf{G}$ with respect to $k, \mathbf{S}$, and $\nu$. If $\psi \in \Psi(\mathcal{A})$, then $\dot{\psi} \in \Phi(\mathcal{A})$ denotes the gradient of $\psi$.

For $\psi \in \Psi(\mathcal{A})$, let $U_{\psi}$ and $U_{\psi}^{+}:=U_{\psi^{+}}$denote the corresponding subgroups of the root group $U_{\dot{\psi}}$ (see [21, §2.4 and §3.1]).

### 1.3. The Moy-Prasad filtrations of $\mathfrak{g}$

We will require a basic understanding of the "root decomposition" of the lattices $\mathfrak{g}_{x, r}$; however, we will not repeat the definition of the $\mathfrak{g}_{x, r}$ (see, $[19,21]$ ).

Suppose that $\mathbf{S}$ is a maximal $k$-split torus. For $\psi \in \Psi(\mathcal{A}(\mathbf{S}, k))$, we can define a lattice $\mathfrak{g}_{\psi}$ in the root space $\mathfrak{g}_{\dot{\psi}}$ of $\mathfrak{g}$. Let $\mathfrak{m}$ denote the Lie algebra of the $k$-Levi subgroup $C_{\mathbf{G}}(\mathbf{S})$. Let $\mathfrak{m}=\mathfrak{m}(k)$. Fix $r \in \mathbb{R}$. For $x \in \mathcal{A}(\mathbf{S}, k)$, let $\mathfrak{m}_{r}=\mathfrak{m} \cap \mathfrak{g}_{x, r}$. The lattice $\mathfrak{m}_{r} \subset \mathfrak{m}$ is independent of the choice of $x \in \mathcal{A}(\mathbf{S}, k)$. If $x \in \mathcal{A}(\mathbf{S}, k)$, then

$$
\mathfrak{g}_{x, r}=\mathfrak{m}_{r}+\sum_{\psi \in \Psi(\mathbf{S}, k) ; \psi(x) \geqslant r} \mathfrak{g}_{\psi}
$$

We define $\mathfrak{g}_{x, r^{+}}:=\bigcup_{s>r} \mathfrak{g}_{x, s}$. For $X \in \mathfrak{g}$ and $x \in \mathcal{B}(G)$ we let $\mathrm{d}_{x}(X)$ denote the depth of $X$ in the $x$-filtration, that is, $\mathrm{d}_{x}(X)=t$ where $t \in \mathbb{R}$ is the unique real number such that $X \in \mathfrak{g}_{x, t} \backslash \mathfrak{g}_{x, t^{+}}$.

Remark 1.3.1. - The function $(x, X) \rightarrow \mathrm{d}_{x}(X)$ is continuous in the variable $x$ and locally constant in the variable $X$. In fact, it follows from the root decomposition of the Moy-Prasad filtration lattices that the function $(x, X) \rightarrow \mathrm{d}_{x}(X)$ satisfies a very strong version of uniform
continuity in the variable $x$. Namely, for all $\varepsilon>0$, there exists a $\delta>0$ such that for all $x, y \in \mathcal{B}(G)$ and all $X \in \mathfrak{g}$ we have

$$
\text { if } \operatorname{dist}(x, y)<\delta, \quad \text { then }\left|\mathrm{d}_{x}(X)-\mathrm{d}_{y}(X)\right|<\varepsilon
$$

We also define $\mathfrak{g}_{r+}:=\bigcup_{s>r} \mathfrak{g}_{s}$. We have

$$
\mathfrak{g}_{r^{+}}=\bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x, r^{+}} .
$$

### 1.4. The Moy-Prasad filtrations of $G$

For $x \in \mathcal{B}(G)$, we will denote the parahoric subgroup attached to $x$ by $G_{x}\left(=G_{x, 0}\right)$, and we denote its pro-unipotent radical by $G_{x}^{+}\left(=G_{x, 0^{+}}\right)$. Note that both $G_{x}$ and $G_{x}^{+}$depend only on the facet of $\mathcal{B}(G)$ to which $x$ belongs. If $F$ is a facet in $\mathcal{B}(G)$ and $x \in F$, then we define $G_{F}=G_{x}$ and $G_{F}^{+}=G_{x}^{+}$.
For $x \in \mathcal{B}(G)$ the quotient $G_{x} / G_{x}^{+}$is the group of $\mathfrak{f}$-rational points of a connected reductive group $\mathrm{G}_{x}$ defined over $\mathfrak{f}$. We let $Z_{x}$ denote the $\mathfrak{f}$-split torus in the center of $\mathrm{G}_{x}$ corresponding to the maximal $k$-split torus in the center of $\mathbf{G}$.

We define $G_{x, r^{+}}:=\bigcup_{s>r} G_{x, s}$ and $G_{r^{+}}:=\bigcup_{s>r} G_{s}$. We have

$$
G_{r^{+}}=\bigcup_{x \in \mathcal{B}(G)} G_{x, r^{+}}
$$

### 1.5. Generalized $r$-facets

Suppose $r \in \mathbb{R}$. In this subsection we recall the definition of generalized $r$-facets and some of their properties. These were originally developed in $[11, \S 3.2]$ for the Lie algebra.

Definition 1.5.1. - For $x \in \mathcal{B}(G)$, define

$$
F^{*}(x):=\left\{y \in \mathcal{B}(G) \mid \mathfrak{g}_{x, r}=\mathfrak{g}_{y, r} \text { and } \mathfrak{g}_{x, r^{+}}=\mathfrak{g}_{y, r^{+}}\right\} .
$$

Remark 1.5.2. - Alternatively, we can define $F^{*}(x)$ by

$$
F^{*}(x)=\left\{y \in \mathcal{B}(G) \mid G_{x,|r|}=G_{y,|r|} \text { and } G_{x,|r|^{+}}=G_{y,|r|^{+}}\right\} .
$$

DEFInition 1.5.3.-

$$
\mathcal{F}(r):=\left\{F^{*}(x) \mid x \in \mathcal{B}(G)\right\} .
$$

Example 1.5.4. - The set $\mathcal{F}(0)$ is the set of all facets in $\mathcal{B}(G)$.
An element of $\mathcal{F}(r)$ is called a generalized $r$-facet.
Definition 1.5.5. - Suppose $F^{*} \in \mathcal{F}(r)$. Fix $x \in F^{*}$. Define

$$
\begin{aligned}
\mathfrak{g}_{F^{*}} & :=\mathfrak{g}_{x, r} \quad \text { and } \quad \mathfrak{g}_{F^{*}}^{+}:=\mathfrak{g}_{x, r^{+}}, \\
G_{F^{*}} & :=G_{x,|r|} \quad \text { and } \quad G_{F^{*}}^{+}:=G_{x,|r|^{+}}, \\
\mathfrak{g}_{F^{*},-r} & =\mathfrak{g}_{x,-r} \quad \text { and } \quad V_{F^{*}}:=\mathfrak{g}_{x, r} / \mathfrak{g}_{x, r^{+}} .
\end{aligned}
$$

We recall some of the basic properties of generalized $r$-facets.
Remark 1.5.6. - Suppose $F_{1}^{*}, F_{2}^{*} \in \mathcal{F}(r)$.

1. From [11, Lemma 3.2.8] we have

$$
N_{G}\left(G_{F_{1}^{*}}\right) \cap N_{G}\left(G_{F_{1}^{*}}^{+}\right)=\operatorname{stab}_{G}\left(F_{1}^{*}\right)
$$

and

$$
N_{G}\left(\mathfrak{g}_{F_{1}^{*}}\right) \cap N_{G}\left(\mathfrak{g}_{F_{1}^{*}}^{+}\right)=\operatorname{stab}_{G}\left(F_{1}^{*}\right) .
$$

2. If $F_{1}^{*} \cap \overline{F_{2}^{*}} \neq \emptyset$, then from [11, Lemma 3.2.15] we have $F_{1}^{*} \subset \overline{F_{2}^{*}}$.
3. If $F_{1}^{*} \subset \overline{F_{2}^{*}}$, then from [11, Lemma 3.2.17] there exists an $x_{2} \in F_{2}^{*}$ such that

$$
G_{x_{2}} \subset \operatorname{stab}_{G}\left(F_{1}^{*}\right)
$$

4. If $F_{1}^{*} \subset \overline{F_{2}^{*}}$, then from [11, Corollary 3.2.19] we have

$$
\mathfrak{g}_{F_{1}^{*}}^{+} \subset \mathfrak{g}_{F_{2}^{*}}^{+} \subset \mathfrak{g}_{F_{2}^{*}} \subset \mathfrak{g}_{F_{1}^{*}} \quad \text { and } \quad G_{F_{1}^{*}}^{+} \subset G_{F_{2}^{*}}^{+} \subset G_{F_{2}^{*}} \subset G_{F_{1}^{*}}
$$

Suppose $F^{*} \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that $F^{*} \cap \mathcal{A} \neq \emptyset$. Let $A\left(\mathcal{A}, F^{*}\right)$ denote the smallest affine subspace of $\mathcal{A}$ containing $F^{*} \cap \mathcal{A}$. From [11, Corollary 3.2.14] we have that $\operatorname{dim} A\left(\mathcal{A}, F^{*}\right)$ is independent of the apartment $\mathcal{A}$. That is, if $\mathcal{A}^{\prime}$ is another apartment such that $\mathcal{A}^{\prime} \cap F^{*} \neq \emptyset$, then $\operatorname{dim} A\left(\mathcal{A}^{\prime}, F^{*}\right)=\operatorname{dim} A\left(\mathcal{A}, F^{*}\right)$. Consequently, the following definition makes sense.

DEFINITION 1.5.7. - If $F^{*} \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that $\mathcal{A} \cap F^{*} \neq \emptyset$, then

$$
\operatorname{dim} F^{*}:=\operatorname{dim} A\left(\mathcal{A}, F^{*}\right)
$$

Remark 1.5.8. - If $F_{1}^{*}, F_{2}^{*} \in \mathcal{F}(r)$ and $F_{1}^{*} \subset \overline{F_{2}^{*}}$, then $\operatorname{dim} F_{1}^{*} \leqslant \operatorname{dim} F_{2}^{*}$ with equality if and only if $F_{1}^{*}=F_{2}^{*}$.

## 2. A proof of Conjecture 2

Fix $r \in \mathbb{R}$. Subject to some conditions, in this section we prove Conjecture 2.

### 2.1. An extension of Conjecture 2

Before stating the extension of Conjecture 2, we require a few additional definitions.
Definition 2.1.1. - Suppose $x \in \mathcal{B}(G)$ and $s \leqslant r$. Define

$$
\begin{aligned}
\tilde{J}_{x, s, r^{+}}:=\{T \in J(\mathfrak{g}) \mid & \text { for } f \in C\left(\mathfrak{g}_{x, s} / \mathfrak{g}_{x, r^{+}}\right) \text {if } \operatorname{supp}(f) \cap\left(\mathcal{N}+\mathfrak{g}_{x, s^{+}}\right)=\emptyset, \\
& \text { then } T(f)=0\} .
\end{aligned}
$$

Remark 2.1.2. - Suppose $s \leqslant r$. Since

$$
\mathfrak{g}_{r^{+}} \subset \mathfrak{g}_{x, r^{+}}+\mathcal{N} \subset \mathfrak{g}_{x, s^{+}}+\mathcal{N},
$$

it follows that $J\left(\mathfrak{g}_{r^{+}}\right) \subset \tilde{J}_{x, s, r^{+}}$.

DEFINITION 2.1.3.-

$$
\tilde{J}_{r^{+}}:=\bigcap_{x \in \mathcal{B}(G)} \bigcap_{s \leqslant r} \tilde{J}_{x, s, r^{+}}
$$

The sums in the definitions below should be interpreted as in the introduction.
Definition 2.1.4. - Define the space

$$
\mathcal{D}_{r^{+}}:=\sum_{x \in \mathcal{B}(G)} C_{c}\left(\mathfrak{g} / \mathfrak{g}_{x, r^{+}}\right)
$$

and its subspace

$$
\mathcal{D}_{r^{+}}^{r}:=\sum_{x \in \mathcal{B}(G)} C\left(\mathfrak{g}_{x, r} / \mathfrak{g}_{x, r^{+}}\right) .
$$

We can now state an extension of Conjecture 2.
Theorem 2.1.5. - Suppose that all the hypotheses of Section 2.2 hold.

1. If $T \in \tilde{J}_{r^{+}}$, then

$$
\operatorname{res}_{\mathcal{D}_{r+}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{\mathcal{D}_{r^{+}}^{r}} T=0 .
$$

2. We have

$$
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r^{+}}} \tilde{J}_{r^{+}}\right) \leqslant|\mathcal{O}(0)|
$$

3. If $\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r^{+}}} J(\mathcal{N})\right)=|\mathcal{O}(0)|$, then

$$
\operatorname{res}_{\mathcal{D}_{r^{+}}} \tilde{J}_{r^{+}}=\operatorname{res}_{\mathcal{D}_{r^{+}}} J(\mathcal{N})
$$

Remark 2.1.6. - The condition

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r^{+}}} J(\mathcal{N})\right)=|\mathcal{O}(0)| \tag{1}
\end{equation*}
$$

in Theorem 2.1.5 (3) is known to be true if $k$ has characteristic zero. The positive characteristic situation is more delicate (see, for example, Corollary 3.4.6).

Remark 2.1.7. - From Remark 2.1.2 we have $J(\mathcal{N}) \subset J\left(\mathfrak{g}_{r^{+}}\right) \subset \tilde{J}_{r^{+}}$. From [2] for each $r \in \mathbb{R}$ there exists an $s \in \mathbb{R}$ such that $\mathcal{D}_{r}=\mathcal{D}_{s^{+}}$and $\mathfrak{g}_{r}=\mathfrak{g}_{s^{+}}$. Thus, subject to the hypotheses of Section 2.2 and condition (1), Conjecture 2 follows from Theorem 2.1.5 (3).

### 2.2. The hypotheses of [11, §4.2]

The hypotheses listed below place some restrictions on $k$ and $\mathbf{G}$; in particular, they are all valid if $p$ is larger than some constant which can be determined by examining the absolute root datum of $\mathbf{G}$. The reader may consult [11, §4.2] for more information.

Definition 2.2.1. - For $x \in \mathcal{B}(G)$ and $s \in \mathbb{R}$ define the $\mathfrak{f}$-vector space $V_{x, s}:=\mathfrak{g}_{x, s} / \mathfrak{g}_{x, s^{+}}$.
$V_{x, s}$ is naturally a $G_{x} / G_{x}^{+}$-module.
We recall the definition of the $\mathfrak{f}$-Lie algebra $\overline{\mathfrak{g}}_{x}$. Since we have fixed a uniformizer $\varpi$ for $k$, for $s \in \mathbb{R}$ and $j \in \mathbb{Z}$ we have a natural identification of $V_{x, s}$ with $V_{x, s+j \cdot \ell}$. With respect to this identification, we define

$$
\overline{\mathfrak{g}}_{x}:=\bigoplus_{s \in \mathbb{R} / \ell \cdot \mathbb{Z}} V_{x, s}
$$

Note that $\operatorname{dim}_{\mathfrak{f}} \overline{\mathfrak{g}}_{x}=\operatorname{dim}_{k} \mathfrak{g}$. We define a product operation on $\overline{\mathfrak{g}}_{x}$ in the following manner. If $\bar{X}_{s} \in V_{x, s}$ and $\bar{X}_{t} \in V_{x, t}$, then we define $\left[\bar{X}_{s}, \bar{X}_{t}\right]$ to be the image of $\left[X_{s}, X_{t}\right] \in \mathfrak{g}_{x,(s+t)}$ in $V_{x,(s+t)}$ where $X_{s} \in \mathfrak{g}_{x, s}$ and $X_{t} \in \mathfrak{g}_{x, t}$ are any lifts of $\bar{X}_{t}$ and $\bar{X}_{s}$, respectively. Linearly extend this operation to an operation on $\overline{\mathfrak{g}}_{x}$. With this product $\overline{\mathfrak{g}}_{x}$ is an $\mathfrak{f}$-Lie algebra. For $v \in \overline{\mathfrak{g}}_{x}$, define $\operatorname{ad}(v) \in \operatorname{End}_{\mathfrak{f}}\left(\overline{\mathfrak{g}}_{x}\right)$ by $\operatorname{ad}(v) w=[v, w]$ for all $w \in \overline{\mathfrak{g}}_{x}$.

Hypothesis 2.2.2.- Suppose $x \in \mathcal{B}(G)$. If $X \in \mathcal{N} \cap\left(\mathfrak{g}_{x, r} \backslash \mathfrak{g}_{x, r^{+}}\right)$, then there exist $H \in \mathfrak{g}_{x, 0}$ and $Y \in \mathfrak{g}_{x,-r}$ such that

$$
\begin{aligned}
{[H, X] } & =2 X \bmod \mathfrak{g}_{x, r^{+}}, \\
{[H, Y] } & =-2 Y \bmod \mathfrak{g}_{x,(-r)^{+}}, \\
{[X, Y] } & =H \bmod \mathfrak{g}_{x, 0^{+}} .
\end{aligned}
$$

If $(f, h, e)$ denotes the image of $(Y, H, X)$ in $V_{x,-r} \times V_{x, 0} \times V_{x, r} \subset \overline{\mathfrak{g}}_{x}$, then $(f, h, e)$ is an $\mathfrak{s l}_{2}(\mathfrak{f})$ triple, and $\overline{\mathfrak{g}}_{x}$ decomposes into a direct sum of irreducible ( $f, h, e$ )-modules of highest weight at most $(p-3)$. Moreover, there exists $\bar{\lambda} \in X_{*}^{\dagger}\left(\mathrm{G}_{x}\right)$, uniquely determined up to an element of $\mathbf{X}_{*}\left(Z_{x}\right)$ whose differential is zero, such that the following two conditions hold.

1. The image of $d \bar{\lambda}$ in $\operatorname{Lie}\left(\mathrm{G}_{x}\right)$ coincides with the one dimensional subspace spanned by $h$.
2. Suppose $i \in \mathbb{Z}$. For $v \in \overline{\mathfrak{g}}_{x}$

$$
\text { if }{ }^{\bar{\lambda}(t)} v=t^{i} v, \quad \text { then }|i| \leqslant(p-3) \text { and } \operatorname{ad}(h) v=i v .
$$

Definition 2.2.3. - In the notation of Hypothesis 2.2.2, we say that $\bar{\lambda} \in \mathbf{X}_{*}^{f}\left(\mathrm{G}_{x}\right)$ is adapted to the $\mathfrak{s l}_{2}(\mathfrak{f})$-triple obtained from the image of $(Y, H, X)$ in $V_{x,-r} \times V_{x, 0} \times V_{x, r}$.

Hypothesis 2.2.4. - If $X \in \mathcal{N}$, then there exists $m \leqslant(p-2)$ such that $\operatorname{ad}(X)^{m}=0$.
Hypothesis 2.2.5. - Choose $m \in \mathbb{N}$ such that $\operatorname{ad}(X)^{m}=0$ for all $X \in \mathcal{N}$. Suppose either that $k$ has characteristic zero or that the characteristic of $k$ is greater than $m$. There exists a unique $G$-equivariant map $\exp _{t}: \mathcal{N} \rightarrow \mathcal{U}$ defined over $k$ such that for all $X \in \mathcal{N}$ the adjoint action of $\exp _{t}(X)$ on $\mathfrak{g}$ is given by

$$
\sum_{i=0}^{m} \frac{(\operatorname{ad}(X))^{i}}{i!}
$$

Hypothesis 2.2.6. - Suppose Hypothesis 2.2.5 is valid. If $X \in \mathcal{N}$, then there exists a Lie algebra homomorphism $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ defined over $k$ such that $X=\phi\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right)$. Moreover, for any Lie algebra homomorphism $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ defined over $k$ such that $X=\phi\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right)$ there exists a group homomorphism $\rho_{\phi}: \mathbf{S L}_{2} \rightarrow \mathbf{G}$ defined over $k$ such that for all $t \in k$

1. $\rho_{\phi}\left(\begin{array}{cc}1 & t \\ 0 & 1\end{array}\right)=\exp _{t}\left(\phi\left(\begin{array}{cc}0 & t \\ 0 & 0\end{array}\right)\right)$ and
2. $\rho_{\phi}\left(\begin{array}{cc}1 & 0 \\ t & 1\end{array}\right)=\exp _{t}\left(\phi\left(\begin{array}{cc}0 & 0 \\ t & 0\end{array}\right)\right)$.

Finally, if $\phi^{\prime}: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ is another Lie algebra homomorphism defined over $k$ such that $\phi^{\prime}\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)=X$, then there exists $g \in C_{G}(X)$ such that ${ }^{g} \phi^{\prime}=\phi$.

Definition 2.2.7. - Given $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ as in Hypothesis 2.2.6, we let $\lambda_{\phi} \in \mathbf{X}_{*}^{k}(\mathbf{G})$ denote the one-parameter subgroup defined by $\lambda_{\phi}(t)=\rho_{\phi}\left(\begin{array}{cc}t & 0^{0} \\ 0 & t^{-1}\end{array}\right)$.

Hypothesis 2.2.8. - Suppose $x \in \mathcal{B}(G)$. For all $s \in \mathbb{R}_{>0}$ and for all $t \in \mathbb{R}$ there exists a map $\phi_{x}: \mathfrak{g}_{x, s} \rightarrow G_{x, s}$ such that for $V \in \mathfrak{g}_{x, s}$ and $W \in \mathfrak{g}_{x, t}$ we have

$$
\phi_{x}(V) W=W+[V, W] \bmod \mathfrak{g}_{x,(s+t)+.} .
$$

### 2.3. Descent and recovery

The next result may be thought of as a sharpening of [21, Proposition 6.3].
Lemma 2.3.1.- Suppose that Hypotheses 2.2.2, 2.2.4, and 2.2.8 are valid. Suppose $x \in \mathcal{B}(G)$ and $s<r$. Let $\mathbf{S}$ be a maximal $k$-split torus of $\mathbf{G}$ such that $x \in \mathcal{A}(\mathbf{S}, k)$. If $Z \in\left(\mathcal{N}+\mathfrak{g}_{x, s^{+}}\right) \cap\left(\mathfrak{g}_{x, s} \backslash \mathfrak{g}_{x, s^{+}}\right)$, then there exist $X \in{ }^{G_{x}}\left(Z+\mathfrak{g}_{x, s^{+}}\right)$and $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{S})$ such that for all sufficiently small $\varepsilon>0$ we have

1. $X+\mathfrak{g}_{x, s^{+}} \subset \mathfrak{g}_{x+\varepsilon \cdot \lambda, s^{+}}$, and
2. for all $Z^{\prime} \in \mathfrak{g}_{x, s^{+}}$we have

$$
X+Z^{\prime}+\mathfrak{g}_{x+\varepsilon \cdot \lambda, r^{+}} \subset^{G_{x,(r-s)}}\left(X+Z^{\prime}+\mathfrak{g}_{x, r^{+}}\right)
$$

Proof. - Let $X^{\prime}$ be any element of $\left(Z+\mathfrak{g}_{x, s^{+}}\right) \cap \mathcal{N}$. For $X^{\prime}$ we choose $Y^{\prime} \in \mathfrak{g}_{x,-s}, H^{\prime} \in \mathfrak{g}_{x, 0}$, and $\bar{\lambda}^{\prime} \in \mathbf{X}_{*}^{f}\left(\mathrm{G}_{x}\right)$ as in Hypothesis 2.2.2. There exists an $h \in G_{x}$ such that $\bar{\lambda}:={ }^{\bar{h}} \bar{\lambda}^{\prime}$ lies in S , the maximal $\mathfrak{f}$-split torus in $\mathrm{G}_{x}$ corresponding to $\mathbf{S}$. Let $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{S})$ be the lift of $\bar{\lambda}$. Let $X$ (respectively, $H$, respectively, $Y$ ) denote ${ }^{h} X^{\prime}$ (respectively, ${ }^{h} H^{\prime}$, respectively, ${ }^{h} Y^{\prime}$ ). Let $(\bar{Y}, \bar{H}, \bar{X})$ denote the image of $(Y, H, X)$ in $V_{x,-s} \times V_{x, 0} \times V_{x, s}$. The triple $(\bar{Y}, \bar{H}, \bar{X})$ forms an $\mathfrak{s i}_{2}(\mathfrak{f})$-triple under the Lie algebra product inherited from $\mathfrak{g}$.

Since for sufficiently small $\varepsilon>0$ we have $\mathfrak{g}_{x, s^{+}} \subset \mathfrak{g}_{x+\varepsilon \cdot \lambda, s^{+}}$, in order to establish the first claim of the lemma it is enough to show that $X \in \mathfrak{g}_{x+\varepsilon \cdot \lambda, s^{+}}$for all sufficiently small $\varepsilon>0$.

Modulo $\mathfrak{g}_{x, s^{+}}$we can write

$$
X=\sum_{\psi} X_{\psi}
$$

where the sum is over $\psi \in \Psi(\mathbf{S}, k)$ such that $\psi(x)=s$ and $X_{\psi} \in \mathfrak{g}_{\psi} \backslash \mathfrak{g}_{\psi^{+}}$. Since $\bar{\lambda}$ is adapted to the $\mathfrak{s l}_{2}(\mathfrak{f})$-triple $(\bar{Y}, \bar{H}, \bar{X})$, from Hypothesis 2.2 .2 we have $\langle\dot{\psi}, \lambda\rangle=2$ for all $\psi$ occurring in this sum. Thus

$$
\psi(x+\varepsilon \cdot \lambda)=\psi(x)+\varepsilon \cdot\langle\dot{\psi}, \lambda\rangle=s+2 \varepsilon>s
$$

and so $X \in \mathfrak{g}_{x+\varepsilon \cdot \lambda, s^{+}}$.
Since $Z^{\prime} \in \mathfrak{g}_{x, s^{+}}$, we have ${ }^{G_{x,(r-s)}} Z^{\prime} \subset Z^{\prime}+\mathfrak{g}_{x, r^{+}}$. Thus, from Hypothesis 2.2 .8 , in order to establish the second claim of the lemma it is sufficient to show that

$$
X+\mathfrak{g}_{x+\varepsilon \cdot \lambda, r^{+}} \subset X+\operatorname{ad}(X)\left(\mathfrak{g}_{x, r-s}\right)+\mathfrak{g}_{x, r^{+}}
$$

for all sufficiently small $\varepsilon>0$. That is, we need to show that for all $\psi \in \Psi(\mathbf{S}, k)$ such that $\psi(x)=r$ and $\psi(x+\varepsilon \cdot \lambda)>r$, we have $X+\mathfrak{g}_{\psi} \subset X+\operatorname{ad}(X)\left(\mathfrak{g}_{x, r-s}\right)$ modulo $\mathfrak{g}_{x, r^{+}}$.

Fix such a $\psi$. Let $i=\langle\dot{\psi}, \lambda\rangle \in \mathbb{Z}$. Since $\varepsilon \cdot\langle\dot{\psi}, \lambda\rangle>0$, we have $i>0$. From Hypothesis 2.2.2 we have $i \leqslant(p-3)$.

For $j \in \mathbb{Z}$ and $b \in \mathbb{R}$ define

$$
V_{x, b}(j):=\left\{\left.v \in V_{x, b}\right|^{\bar{\lambda}(t)} v=t^{j} \cdot v\right\} .
$$

Because $x \in \mathcal{A}(\mathbf{S}(k))$ and $\bar{\lambda} \in \mathbf{X}_{*}^{\mathfrak{f}}(\mathrm{S})$, we have

$$
V_{x, b}=\sum_{j} V_{x, b}(j)
$$

Define the finite dimensional $(\bar{Y}, \bar{H}, \bar{X})$-module

$$
V(r):=\sum_{j \in \mathbb{Z}} V_{x, r+s j}(i+2 j)
$$

From Hypothesis 2.2.4 and [6, Theorem 5.4.8] the space $V(r)$ is a direct sum of irreducible $(\bar{Y}, \bar{H}, \bar{X})$-modules. Consequently, since $i>0$, the map from $V_{x,(r-s)}(i-2)$ to $V_{x, r}(i)$ which sends $\bar{U} \in V_{x,(r-s)}(i-2)$ to $\overline{[X, U]} \in V_{x, r}(i)$ is surjective. (Here $U \in \mathfrak{g}_{x,(r-s)}$ is any lift of $\bar{U}$ and $\overline{[X, U]}$ denotes the image of $[X, U]$ in $V_{x, r .}$.)

Suppose $W \in \mathfrak{g}_{\psi} \backslash \mathfrak{g}_{\psi^{+}}$. From the above paragraph there exists $U \in \mathfrak{g}_{x,(r-s)}$ such that $[X, U]=W$ modulo $\mathfrak{g}_{x, r^{+}}$.

### 2.4. A proof of Theorem 2.1.5 (1)

In this subsection we begin our proof of Theorem 2.1.5.
A proof of Theorem 2.1.5 (1). - Fix $T \in \tilde{J}_{r^{+}}$.
$" \Rightarrow$ ": Since $\mathcal{D}_{r^{+}}^{r}$ is a subspace of $\mathcal{D}_{r^{+}}$, this implication follows.
$" \Leftarrow "$ : Fix $f \in \mathcal{D}_{r^{+}}$. We will show that $T(f)$ depends only on $\operatorname{res}_{\mathcal{D}_{r+}^{r}} T$. Since $T$ is linear, we may assume that $f \in C_{c}\left(\mathfrak{g} / \mathfrak{g}_{F^{*}}^{+}\right)$for some generalized $r$-facet $F^{*}$ in $\mathcal{F}(r)$. Write $f$ as a finite sum of characteristic functions of cosets in $\mathfrak{g} / \mathfrak{g}_{F^{*}}^{+}$. Again using the linearity of $T$, we may assume that $f=\left[Z+\mathfrak{g}_{F^{*}}^{+}\right]$for some $Z \in \mathfrak{g}$. Without loss of generality, we assume $Z \notin \mathfrak{g}_{F^{*}}$.

Define $\mathrm{m}_{Z}: \mathcal{B}(G) \rightarrow \mathbb{R}$ in the following manner. For $w \in \mathcal{B}(G)$, we let $\mathrm{m}_{Z}(w)=t$ where $Z \in \mathfrak{g}_{w, t} \backslash \mathfrak{g}_{w, t^{+}}$. The function $\mathrm{m}_{Z}$ is continuous. Since $\overline{F^{*}}$ is compact, $\mathrm{m}_{Z}$ attains its maximum on $\overline{F^{*}}$. Thus, there exist $s \in \mathbb{R}$ and $y \in \overline{F^{*}}$ such that

$$
s=\mathrm{m}_{Z}(y) \geqslant \mathrm{m}_{Z}(w)
$$

for all $w \in \overline{F^{*}}$.
Suppose first that $s=r$. Since $y \in \overline{F^{*}}$, from Remark 1.5.6 we have $\mathfrak{g}_{y, r} \subset \mathfrak{g}_{F^{*}}^{+} \subset \mathfrak{g}_{F^{*}} \subset \mathfrak{g}_{y, r}$. Therefore

$$
\begin{equation*}
T\left(\left[Z+\mathfrak{g}_{F^{*}}^{+}\right]\right)=\sum_{\bar{\alpha} \in \mathfrak{g}_{F^{*}}^{+} / \mathfrak{g}_{y, r^{+}}} T\left(\left[Z+\alpha+\mathfrak{g}_{y, r^{+}}\right]\right) \tag{2}
\end{equation*}
$$

Since for all $\alpha \in \mathfrak{g}_{F^{*}}^{+}$we have $\left[Z+\alpha+\mathfrak{g}_{y, r^{+}}\right] \in C\left(\mathfrak{g}_{y, r} / \mathfrak{g}_{y, r^{+}}\right) \subset \mathcal{D}_{r^{+}}^{r}$, the assertion is proved in this case.

Now suppose that $s<r$. Let $F_{y}^{*}$ denote the generalized $r$-facet containing $y$. Since $F_{y}^{*} \subset \overline{F^{*}}$, we have $\mathfrak{g}_{F^{*}}^{+} \subset \mathfrak{g}_{F_{y}^{*}} \subset \mathfrak{g}_{z, r} \subset \mathfrak{g}_{z, s^{+}}$for all $z \in \overline{F_{y}^{*}}$. Thus, we have $\mathrm{m}_{Z+Z^{\prime}}(z)=\mathrm{m}_{Z}(z)$ for all $z \in \overline{F_{y}^{*}}$ and for all $Z^{\prime} \in \mathfrak{g}_{F^{*}}^{+}$. By using a reduction as in equation (2) and the fact that $\mathfrak{g}_{F^{*}}^{+} \subset \mathfrak{g}_{y, r} \subset \tilde{\mathfrak{j}}_{y, s^{+}}$, we may assume that $y \in F^{*}$.

Since $T \in \tilde{J}_{r^{+}} \subset \tilde{J}_{y, s, r^{+}}$, we have that $T\left(\left[Z+\mathfrak{g}_{F^{*}}^{+}\right]\right)=0$ unless $Z \in \mathcal{N}+\mathfrak{g}_{y, s^{+}}$. We may therefore assume that $Z \in\left(\mathcal{N}+\mathfrak{g}_{y, s^{+}}\right) \cap\left(\mathfrak{g}_{y, s} \backslash \mathfrak{g}_{y, s^{+}}\right)$. Let $\mathbf{S}$ be a maximal $k$-split torus such that $y \in \mathcal{A}(\mathbf{S}, k)$. From Lemma 2.3.1 there exist $X \in G_{y}\left(Z+\mathfrak{g}_{y, s^{+}}\right)$and $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{S})$ such that for all sufficiently small $\varepsilon>0$ we have

$$
\begin{equation*}
X+\mathfrak{g}_{y, s^{+}} \subset \mathfrak{g}_{y+\varepsilon \cdot \lambda, s^{+}} \tag{3}
\end{equation*}
$$

and for all $Z^{\prime} \in \mathfrak{g}_{y, s^{+}}$we have

$$
\begin{equation*}
X+Z^{\prime}+\mathfrak{g}_{y+\varepsilon \cdot \lambda, r^{+}} \subset \subset^{G_{y,(r-s)}}\left(X+Z^{\prime}+\mathfrak{g}_{y, r^{+}}\right) . \tag{4}
\end{equation*}
$$

Fix $\varepsilon>0$ so that (3) and (4) are valid. Note that as a consequence of (3) we have $\mathfrak{g}_{y, s^{+}} \subset \mathfrak{g}_{y+\varepsilon \cdot \lambda, s^{+}}$. Let $x=y+\varepsilon \cdot \lambda \in \mathcal{A}(\mathbf{S}, k)$. Fix $h \in G_{y}$ and $Z^{\prime} \in \mathfrak{g}_{y, s^{+}}$such that

$$
X={ }^{h} Z-Z^{\prime} .
$$

Let

$$
\mathfrak{a}:=\left\{{ }^{g} X-X \mid \bar{g} \in G_{y,(r-s)} / G_{y,(r-s)^{+}}\right\}+\mathfrak{g}_{y, r^{+}}
$$

We have $\mathfrak{a} \subset \mathfrak{g}_{y, r} \subset \mathfrak{g}_{y, s^{+}}$, and from (4) we have $\mathfrak{g}_{x, r^{+}} \subset \mathfrak{a}$. The $G$-invariance of $T$ implies that

$$
\begin{aligned}
T\left(\left[Z+\mathfrak{g}_{F^{*}}^{+}\right]\right) & =T\left(\left[X+Z^{\prime}+\mathfrak{g}_{y, r^{+}}\right]\right) \\
& =\text {const } \cdot \sum_{\bar{g} \in G_{y,(r-s)} / G_{y,(r-s)+}} T\left(\left[{ }^{g} X+Z^{\prime}+\mathfrak{g}_{y, r^{+}}\right]\right) \\
& =\text {const } \cdot T\left(\left[X+Z^{\prime}+\mathfrak{a}\right]\right) \\
& =\text { const } \cdot \sum_{\bar{\alpha} \in \mathfrak{a} / \mathfrak{g}_{x, r^{+}}} T\left(\left[X+Z^{\prime}+\alpha+\mathfrak{g}_{x, r^{+}}\right]\right) .
\end{aligned}
$$

For all $\alpha \in \mathfrak{a}$ we have $\left[X+Z^{\prime}+\alpha+\mathfrak{g}_{x, r^{+}}\right] \in C\left(\mathfrak{g}_{x, s^{+}} / \mathfrak{g}_{x, r^{+}}\right)$.
If $\mathfrak{g}_{x, s^{+}}=\mathfrak{g}_{x, r}$, then we are finished. Otherwise, we claim that we can iteratively apply the above process to write

$$
T\left(\left[Z+\mathfrak{g}_{F^{*}}^{+}\right]\right)=\sum_{m \in \mathcal{M}} c(m) \cdot T\left(\left[m+\mathfrak{g}_{y_{m}, r^{+}}\right]\right)
$$

with $\mathcal{M}$ a finite subset of $\mathfrak{g}, c(m) \in \mathbb{Q}$ for all $m \in \mathcal{M}, y_{m} \in \mathcal{B}(G)$ for all $m \in \mathcal{M}$, and $\left[m+\mathfrak{g}_{y_{m}, r^{+}}\right] \in C\left(\mathfrak{g}_{y_{m}, r} / \mathfrak{g}_{y_{m}, r^{+}}\right)$for all $m \in \mathcal{M}$. Suppose we cannot do this. In this case there exists an infinite sequence of quadruples $\left(m_{i}, F_{i}^{*}, y_{i}, s_{i}\right) \in \mathfrak{g} \times \mathcal{F}(r) \times \mathcal{B}(G) \times \mathbb{R}$ with $s_{1}<s_{2}<\cdots<r, y_{i} \in F_{i}^{*}, m_{i} \in \mathfrak{g}_{y_{i}, s_{i}} \cap\left(\mathcal{N}+\mathfrak{g}_{y_{i}, s_{i}^{+}}\right)$, and $m_{i} \notin \mathfrak{g}_{w, s_{i}^{+}}$for all $w \in \overline{F_{i}^{*}}$. By choosing a subsequence and using the action of $G$ on $\mathfrak{g}, \mathcal{B}(G)$, and $\mathcal{F}(r)$, we may assume that $F_{i}^{*}=F^{*}$ for all $i$ and some $F^{*} \in \mathcal{F}(r)$. Since $\overline{F^{*}}$ is compact, by choosing a subsequence of $\left\{\left(m_{i}, F^{*}, y_{i}, s_{i}\right)\right\}$ we may assume that the $y_{i}$ converge to $y \in \overline{F^{*}}$. For each $i \in \mathbb{N}$, define $t_{i} \in \mathbb{R}$ by $m_{i} \in \mathfrak{g}_{y, t_{i}} \backslash \mathfrak{g}_{y, t_{i}^{+}}$. The set $\left\{t_{i}\right\}$ is finite. Thus, by choosing a subsequence of $\left\{\left(m_{i}, F^{*}, y_{i}, s_{i}\right)\right\}$, we may assume that there exists a $t \in \mathbb{R}$ such that $m_{i} \in \mathfrak{g}_{y, t} \backslash \mathfrak{g}_{y, t^{+}}$for all $i \in \mathbb{N}$. Since $y_{i} \rightarrow y$, it follows from Remark 1.3.1 that for all $c>0$, there exists an $I_{c} \in \mathbb{N}$ such that for all $i \geqslant I_{c}$

$$
\left|s_{i}-t\right|<c
$$

Since $s_{i} \geqslant t$ for all $i \in \mathbb{N}$, this contradicts the fact that $\left\{s_{i}\right\}$ is a strictly increasing sequence.

### 2.5. Some comments on nilpotent orbits

We suppose that all the hypotheses of Section 2.2 hold.
Suppose $\mathcal{O}_{1}, \mathcal{O}_{2} \in \mathcal{O}(0)$. We will write $\mathcal{O}_{1} \leqslant \mathcal{O}_{2}$ provided that $\mathcal{O}_{1} \subset \overline{\mathcal{O}_{2}}$. We will write $\mathcal{O}_{1}<\mathcal{O}_{2}$ provided that $\mathcal{O}_{1} \leqslant \mathcal{O}_{2}$ and $\mathcal{O}_{1} \neq \mathcal{O}_{2}$. This defines a partial order on $\mathcal{O}(0)$.

DEFINITION 2.5.1.-

$$
I_{r}:=\left\{\left(F^{*}, v\right) \mid F^{*} \in \mathcal{F}(r) \text { and } v \in V_{F^{*}}\right\}
$$

Suppose $\left(F^{*}, v\right) \in I_{r}$. If $X \in \mathfrak{g}_{F^{*}}$ has image $v \in V_{F^{*}}$, then we will write $\left[\left(F^{*}, v\right)\right]$ for the characteristic function of the coset $X+\mathfrak{g}_{F^{*}}^{+}$.

We now introduce a relation on $I_{r}$. Recall that for $F^{*} \in \mathcal{F}(r)$ and $\mathcal{A}$ an apartment in $\mathcal{B}(G)$ such that $F^{*} \cap \mathcal{A} \neq \emptyset$, we let $A\left(\mathcal{A}, F^{*}\right)$ denote the smallest affine subspace of $\mathcal{A}$ which contains $F^{*} \cap \mathcal{A}$. Suppose $F_{1}^{*}, F_{2}^{*} \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that

$$
\emptyset \neq A\left(\mathcal{A}, F_{1}^{*}\right)=A\left(\mathcal{A}, F_{2}^{*}\right)
$$

Then $\mathfrak{g}_{F_{1}^{*}} \cap \mathfrak{g}_{F_{2}^{*}}$ maps onto $V_{F_{i}^{*}}$ with kernel $\mathfrak{g}_{F_{1}^{*}}^{+} \cap \mathfrak{g}_{F_{2}^{*}}^{+}$. This gives us a natural identification of $V_{F_{1}^{*}}$ with $V_{F_{2}^{*}}$ which we denote by $V_{F_{1}^{*}} \stackrel{i}{=} V_{F_{2}^{*}}$.

DEFINITION 2.5.2. - For $\left(F_{1}^{*}, v_{1}\right)$ and $\left(F_{2}^{*}, v_{2}\right)$ in $I_{r}$ we write $\left(F_{1}^{*}, v_{1}\right) \sim\left(F_{2}^{*}, v_{2}\right)$ if and only if there exist a $g \in G$ and an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that

1. $\emptyset \neq A\left(\mathcal{A}, F_{1}^{*}\right)=A\left(\mathcal{A}, g F_{2}^{*}\right)$, and
2. ${ }^{g} v_{2} \stackrel{i}{=} v_{1}$ in $V_{g F_{2}^{*}} \stackrel{i}{=} V_{F_{1}^{*}}$.

Here $g F_{2}^{*}$ is the image of $F_{2}^{*}$ under the action of $g$ on $\mathcal{B}(G)$, and ${ }^{g} v_{2}$ is the image of ${ }^{g} Z$ in $V_{g F_{2}^{*}}$ where $Z \in \mathfrak{g}_{F_{2}^{*}}$ is any lift of $v_{2}$.

The set $I_{r}$ is too large. Suppose $F^{*} \in \mathcal{F}(r)$. Recall that an element $e \in V_{F^{*}}$ is degenerate if and only if there exists a lift $E \in \mathfrak{g}_{F^{*}}$ of $e$ such that $E \in \mathcal{N}$.

DEFINITION 2.5.3.-

$$
I_{r}^{n}:=\left\{\left(F^{*}, v\right) \in I_{r} \mid v \text { is a degenerate element of } V_{F^{*}}\right\}
$$

Remark 2.5.4. - Suppose that all of the hypotheses of Section 2.2 are valid. From [11, §5.3] we associate to each $\left(F^{*}, e\right) \in I_{r}^{n}$ a nilpotent orbit $\mathcal{O}\left(F^{*}, e\right)$ with the following property: there exists $X \in \mathfrak{g}_{F^{*}} \cap \mathcal{O}\left(F^{*}, e\right)$ such that

1. the image of $X$ in $V_{F^{*}}$ is $e$,
2. for all $\mathcal{O} \in \mathcal{O}(0)$ such that $\mathcal{O} \cap\left(X+\mathfrak{g}_{F^{*}}^{+}\right) \neq \emptyset$ we have ${ }^{G} X=\mathcal{O}\left(F^{*}, e\right) \leqslant \mathcal{O}$, and
3. for all $x \in F^{*}$ we have [11, Corollary 5.2.3] $\mathcal{O}\left(F^{*}, e\right) \cap\left(X+\mathfrak{g}_{F^{*}}^{+}\right)=G_{x}^{+} X$.

Example 2.5.5. - Suppose $\left(F^{*}, e\right) \in I_{r}^{n}$ and $e$ is trivial. In this case we have $\mathcal{O}\left(F^{*}, e\right)=\{0\}$.
Remark 2.5.6. - Suppose that all of the hypotheses of Section 2.2 are valid. If $\left(F_{1}^{*}, e_{1}\right)$, $\left(F_{2}^{*}, e_{2}\right) \in I_{r}^{n}$ and $\left(F_{1}^{*}, e_{1}\right) \sim\left(F_{2}^{*}, e_{2}\right)$, then $\mathcal{O}\left(F_{1}^{*}, e_{1}\right)=\mathcal{O}\left(F_{2}^{*}, e_{2}\right)$ (see [11, Lemma 5.4.1]).

For nilpotent elements, we now define certain subsets of $\mathcal{B}(G)$.
DEFINITION 2.5.7. - If $Z \in \mathcal{N}$ and $s \in \mathbb{R}$, then define $\mathcal{B}(Z, s):=\left\{z \in \mathcal{B}(G) \mid Z \in \mathfrak{g}_{z, s}\right\}$.
The set $\mathcal{B}(Z, s)$ is a nonempty and convex subset of $\mathcal{B}(G)$. Moreover, it is the union of generalized $s$-facets.

Fix $X \in \mathcal{N}$ ( $X$ may be trivial). From Hypothesis 2.2.6 there exists a (possibly trivial) Lie algebra homomorphism $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ such that $X=\phi\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$. Let $Y=\phi\left(\begin{array}{ll}0 & 0 \\ 1 & 0\end{array}\right)$ and $H=\phi\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$. Then $(Y, H, X)$ is an $\mathfrak{s l}_{2}(k)$-triple.

DEFINITION 2.5.8. - We denote by $\mathcal{B}(\phi, r)$ or $\mathcal{B}(Y, H, X)$ the set $\mathcal{B}(X, r) \cap \mathcal{B}(Y,-r)$.
Example 2.5.9. - If $\phi$ is trivial, then $\mathcal{B}(\phi, r)=\mathcal{B}(G)$.

The set $\mathcal{B}(\phi, r)$ is a nonempty, closed, and convex subset of $\mathcal{B}(G)$. Moreover, it is also the union of generalized $r$-facets. Following [11, §5.5] we now define $I_{r}^{d}$, the set of "distinguished" elements in $I_{r}^{n}$.

DEFINITION 2.5.10. - We define $I_{r}^{d} \subset I_{r}^{n}$ to be those pairs $\left(F^{*}, e\right) \in I_{r}^{n}$ for which there exists a Lie algebra homomorphism $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ defined over $k$ such that $F^{*}$ is a maximal generalized $r$-facet in $\mathcal{B}(\phi, r)$ and the image of $\phi\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right)$ in $V_{F^{*}}$ is $e$.

Remark 2.5.11. - Suppose that all of the hypotheses of Section 2.2 are valid. From [11, Theorem 5.6.1] there is a bijective correspondence between $I_{r}^{d} / \sim$ and $\mathcal{O}(0)$ given by the map sending $\left(F^{*}, e\right) \in I_{r}^{d}$ to $\mathcal{O}\left(F^{*}, e\right)$.

For induction purposes, the following definitions will be useful.
Definition 2.5.12. - For $\mathcal{O} \in \mathcal{O}(0)$ we define

$$
I_{r}^{n}\left(\mathcal{O}^{+}\right):=\left\{\left(F^{*}, e\right) \in I_{r}^{n} \mid \mathcal{O}<\mathcal{O}\left(F^{*}, e\right)\right\}
$$

and

$$
I_{r}^{d}\left(\mathcal{O}^{+}\right):=I_{r}^{n}\left(\mathcal{O}^{+}\right) \cap I_{r}^{d}
$$

### 2.6. A proof of Theorem 2.1.5 (2) and (3)

Suppose $T \in J(\mathfrak{g})$. We denote by $\operatorname{res}_{I_{r}} T$ the restriction of $T$ to the linear span of $\left\{\left[\left(F^{*}, v\right)\right] \mid\right.$ $\left.\left(F^{*}, v\right) \in I_{r}\right\}$. We define $\operatorname{res}_{I_{r}^{n}} T$ and $\operatorname{res}_{I_{r}^{d}} T$ similarly. If $\mathcal{O} \in \mathcal{O}(0)$, we define $\operatorname{res}_{I_{r}^{n}\left(\mathcal{O}^{+}\right)} T$ and $\operatorname{res}_{I_{r}^{d}\left(\mathcal{O}^{+}\right)} T$ analogously.

Our first result follows from the definition of $\tilde{J}_{r^{+}}$.
Lemma 2.6.1. - Fix $T \in \tilde{J}_{r^{+}}$. We have

$$
\operatorname{res}_{I_{r}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{I_{r}} T=0 .
$$

Lemma 2.6.2. - Suppose that all of the hypotheses of Section 2.2 are valid. Fix $T \in \underline{\tilde{J}_{r+}}$. Suppose $\left(F^{*}, e\right) \in I_{r}^{n}$. Let $X \in \mathfrak{g}_{F^{*}} \cap \mathcal{O}\left(F^{*}, e\right)$ represent e. Suppose $H^{*} \in \mathcal{F}(r)$ and $H^{*} \subset \overline{F^{*}}$. Let $e^{\prime}$ denote the image of $X$ in $V_{H^{*}}$. If $\operatorname{res}_{I_{r}^{n}\left(\mathcal{O}\left(F^{*}, e\right)+\right)} T=0$, then there exists $c \in \mathbb{N}$ such that $T\left(\left[\left(F^{*}, e\right)\right]\right)=c \cdot T\left(\left[\left(H^{*}, e^{\prime}\right)\right]\right)$.

Proof. - From Remark 1.5.6 we have

$$
\mathfrak{g}_{H^{*}}^{+} \subset \mathfrak{g}_{F^{*}}^{+} \subset \mathfrak{g}_{F^{*}} \subset \mathfrak{g}_{H^{*}}
$$

Thus we can write

$$
T\left(\left[\left(F^{*}, e\right)\right]\right)=T\left(\left[X+\mathfrak{g}_{F^{*}}^{+}\right]\right)=\sum_{\bar{\alpha} \in \mathfrak{g}_{F^{*}}^{+} / \mathfrak{g}_{H^{*}}^{+}} T\left(\left[X+\alpha+\mathfrak{g}_{H^{*}}^{+}\right]\right) .
$$

Fix $\bar{\alpha} \in \mathfrak{g}_{F^{*}}^{+} / \mathfrak{g}_{H^{*}}^{+}$. Let $v$ denote the image of $X+\alpha$ in $V_{H^{*}}$. (Note that if $\bar{\alpha}$ is trivial, then $\left(H^{*}, v\right)=\left(H^{*}, e^{\prime}\right)$.) It will be enough to show that $T\left(\left[\left(H^{*}, v\right)\right]\right)$ is either zero or equal to $T\left(\left[\left(H^{*}, e^{\prime}\right)\right]\right)$.

Since $T \in \tilde{J}_{r^{+}}$, if $\left(H^{*}, v\right) \notin I_{r}^{n}$, then $T\left(\left[\left(H^{*}, v\right)\right]\right)$ is zero. Thus we may suppose that $\left(H^{*}, v\right) \in I_{r}^{n}$.

Choose $Y \in \mathfrak{g}_{H^{*}}^{+}$such that $X+\alpha+Y \in \mathcal{O}\left(H^{*}, v\right)$. Since $X+\alpha+Y \in X+\mathfrak{g}_{F^{*}}^{+}$, from Remark 2.5.4 we have $\mathcal{O}\left(F^{*}, e\right) \leqslant \mathcal{O}\left(H^{*}, v\right)$. If $\mathcal{O}\left(F^{*}, e\right)<\mathcal{O}\left(H^{*}, v\right)$, then $T\left(\left[\left(H^{*}, v\right)\right]\right)=0$ by hypothesis.

Suppose $\mathcal{O}\left(F^{*}, e\right)=\mathcal{O}\left(H^{*}, v\right)$. From Remark 1.5 .6 there exists an $x \in F^{*}$ so that $G_{x} \subset \operatorname{stab}_{G}\left(H^{*}\right)$. From Remark 2.5.4, there exists an $h \in G_{x}^{+}$such that $^{h}(X+\alpha+Y)=X$. Since $T$ is $G$-invariant we have

$$
T\left(\left[\left(H^{*}, v\right)\right]\right)=T\left(\left[X+\alpha+Y+\mathfrak{g}_{H^{*}}^{+}\right]\right)=T\left(\left[X+\mathfrak{g}_{H^{*}}^{+}\right]\right)=T\left(\left[\left(H^{*}, e^{\prime}\right)\right]\right) .
$$

Lemma 2.6.3. - Suppose that all of the hypotheses of Section 2.2 are valid. Fix $T \in \tilde{J}_{r^{+}}$. If $\mathcal{O} \in \mathcal{O}(0)$, then

$$
\operatorname{res}_{I_{r}^{n}\left(\mathcal{O}^{+}\right)} T=0 \quad \text { if and only if } \quad \operatorname{res}_{I_{r}^{d}\left(\mathcal{O}^{+}\right)} T=0 .
$$

Proof. - Fix $\mathcal{O} \in \mathcal{O}(0)$.
" $\Rightarrow "$ : Since $I_{r}^{d} \subset I_{r}^{n}$, this implication follows.
$" \Leftarrow "$ : We will prove this by induction (with respect to the partial order on $\mathcal{O}(0)$ ).
Suppose that $\left(F^{*}, e\right) \in I_{r}^{n}\left(\mathcal{O}^{+}\right)$. Note that $e$ is not trivial. From Hypothesis 2.2.2 there exists an $\mathfrak{s l}_{2}(\mathfrak{f})$-triple $(f, h, e) \in V_{x,-r} \times V_{x, 0} \times V_{x, r}$ completing $e$. From [11, Corollary 4.3.2] we can choose an $\mathfrak{s l}_{2}(k)$-triple $(Y, H, X)$ which lifts $(f, h, e)$. If $F^{*}$ is a maximal generalized $r$-facet in $\mathcal{B}(Y, H, X)$, then $\left(F^{*}, e\right) \in I_{r}^{d}$.

Otherwise, there exists a maximal generalized $r$-facet $H^{*} \subset \mathcal{B}(Y, H, X)$ such that $F^{*} \subset \overline{H^{*}}$. Let $e^{\prime}$ denote the image of $X$ in $V_{H^{*}}$. We have $\left(H^{*}, e^{\prime}\right) \in I_{r}^{d}$. From [11, Lemma 5.3.3] we have $\mathcal{O}\left(H^{*}, e^{\prime}\right)={ }^{G} X$ which is $\mathcal{O}\left(F^{*}, e\right)$. Thus $\left(H^{*}, e^{\prime}\right) \in I_{r}^{d}\left(\mathcal{O}^{+}\right)$. We have

$$
I_{r}^{n}\left(\mathcal{O}\left(H^{*}, e^{\prime}\right)^{+}\right)=I_{r}^{n}\left(\mathcal{O}\left(F^{*}, e\right)^{+}\right)
$$

and, since $\left(H^{*}, e^{\prime}\right) \in I_{r}^{d}\left(\mathcal{O}^{+}\right)$, we have $T\left(\left[\left(H^{*}, e^{\prime}\right)\right]\right)=0$.
If $\mathcal{O}\left(F^{*}, e\right)$ is maximal with respect to our ordering on $\mathcal{O}(0)$, then $I_{r}^{n}\left(\mathcal{O}\left(H^{*}, e^{\prime}\right)^{+}\right)$is empty and from Lemma 2.6.2 there exists $c \in \mathbb{N}$ such that $c \cdot T\left(\left[\left(F^{*}, e\right)\right]\right)=T\left(\left[\left(H^{*}, e^{\prime}\right)\right]\right)=0$.

Suppose now that $\mathcal{O}\left(F^{*}, e\right)$ is not maximal with respect to our ordering on $\mathcal{O}(0)$. By induction we can assume that $\operatorname{res}_{I_{r}^{n}\left(\mathcal{O}\left(H^{*}, e^{\prime}\right)^{+}\right)} T=0$. Thus from Lemma 2.6.2 there exists $c \in \mathbb{N}$ such that $c \cdot T\left(\left[\left(F^{*}, e\right)\right]\right)=T\left(\left[\left(H^{*}, e^{\prime}\right)\right]\right)=0$.

Corollary 2.6.4. - Suppose that all of the hypotheses of Section 2.2 are valid. Fix $T \in \tilde{J}_{r^{+}}$. We have

$$
\operatorname{res}_{I_{r}^{n}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{I_{r}^{d}} T=0 .
$$

Proof. - " $\Rightarrow "$ : Since $I_{r}^{d} \subset I_{r}^{n}$, this implication follows.
" $\Leftarrow$ ": Let 0 denote the trivial nilpotent orbit. From Lemma 2.6 .3 we have $\operatorname{res}_{I_{r}^{n}\left(0^{+}\right)} T=0$. Thus it is sufficient to show that if $\left(F^{*}, e\right) \in I_{r}^{n}$ and $\mathcal{O}\left(F^{*}, e\right)=0$ (that is, $e=0$ ), then $T\left(\left[\left(F^{*}, e\right)\right]\right)=0$. Let $H^{*}$ be an open generalized $r$-facet such that $F^{*} \subset \overline{H^{*}}$. From Lemma 2.6.2 there exists $c \in \mathbb{N}$ such that

$$
c \cdot T\left(\left[\left(F^{*}, e\right)\right]\right)=c \cdot T\left(\left[\mathfrak{g}_{F^{*}}^{+}\right]\right)=T\left(\left[\mathfrak{g}_{H^{*}}^{+}\right]\right)=T\left(\left[\left(H^{*}, 0\right)\right]\right) .
$$

Since $\left(H^{*}, 0\right) \in I_{r}^{d}$, we have $T\left(\left[\left(H^{*}, 0\right)\right]\right)=0$.
Lemma 2.6.5. - Suppose that all of the hypotheses of Section 2.2 are valid. Fix $T \in \tilde{J}_{r^{+}}$. Suppose $\left(F_{1}^{*}, e_{1}\right),\left(F_{2}^{*}, e_{2}\right) \in I_{r}^{d}$ and $\left(F_{1}^{*}, e_{1}\right) \sim\left(F_{2}^{*}, e_{2}\right)$. If $\operatorname{res}_{I_{r}^{d}\left(\mathcal{O}\left(F_{1}^{*}, e_{1}\right)^{+}\right)} T=0$, then

$$
T\left(\left[\left(F_{1}^{*}, e_{1}\right)\right]\right)=0 \quad \text { if and only if } \quad T\left(\left[\left(F_{2}^{*}, e_{2}\right)\right]\right)=0 .
$$

Remark 2.6.6. - The apparent asymmetry in the statement of Lemma 2.6.5 is explained by Remark 2.5.6.

Proof. - From Lemma 2.6.3 we know that $\operatorname{res}_{I_{r}^{n}\left(\mathcal{O}\left(F_{1}^{*}, e_{1}\right)^{+}\right)} T=0$. Since $T$ is $G$-invariant, we have that $T\left(\left[\left(g F^{*},{ }^{g} e\right)\right]\right)=T\left(\left[\left(F^{*}, e\right)\right]\right)$ for all $g \in G$ and $\left(F^{*}, e\right) \in I_{r}^{n}$.

From [11, Corollary 5.6.2] there exist $g \in G$ and an $\mathfrak{s l}_{2}(k)$-triple $(Y, H, X)$ in $\mathfrak{g}$ such that

1. $X \in \mathfrak{g}_{F_{1}^{*}} \cap \mathfrak{g}_{g F_{2}^{*}}$,
2. $X$ has image $e_{1}$ in $V_{F_{1}^{*}}$ and image ${ }^{g} e_{2}$ in $V_{g F_{2}^{*}}$, and
3. $F_{1}^{*}$ and $g F_{2}^{*}$ are maximal generalized $r$-facets in $\mathcal{B}(Y, H, X)$.

Without loss of generality, $g=1$.
From [11, Lemma 5.1.4] there exists an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ for which

$$
A\left(\mathcal{A}, F_{1}^{*}\right)=A\left(\mathcal{A}, F_{2}^{*}\right)
$$

and $F_{i}^{*} \cap \mathcal{A}$ is open in $A\left(\mathcal{A}, F_{2}^{*}\right)$. Since $\mathcal{B}(Y, H, X)$ is convex, we can produce a finite sequence $\left.\underline{\{H}_{i}^{*}\right\}_{\underline{i=1}}^{n}$ of maximal generalized $r$-facets in $\mathcal{B}(Y, H, X)$ such that $H_{1}^{*}=F_{1}^{*}, H_{n}^{*}=F_{2}^{*}$, and $\overline{H_{i}^{*}} \cap \overline{H_{i+1}^{*}} \neq \emptyset$ for $1 \leqslant i \leqslant(n-1)$. Let $v_{i}$ denote the image of $X$ in $V_{H_{i}^{*}}$. We have $\left(H_{i}^{*}, v_{i}\right) \in I_{r}^{d}$ and so we may assume that $n=2$. That is, we assume that $\overline{F_{1}^{*}} \cap \overline{F_{2}^{*}} \neq \emptyset$. Let $H^{*} \in \mathcal{F}(r)$ lie in $\overline{F_{1}^{*}} \cap \overline{F_{2}^{*}}$. From Lemma 2.6.2 and our hypothesis there exist $c_{1}, c_{2} \in \mathbb{N}$ such that

$$
\frac{1}{c_{1}} \cdot T\left(\left[\left(F_{1}^{*}, e_{1}\right)\right]\right)=T\left(\left[X+\mathfrak{g}_{H^{*}}^{+}\right]\right)=\frac{1}{c_{2}} \cdot T\left(\left[\left(F_{2}^{*}, e_{2}\right)\right]\right)
$$

We now present the proof of Theorem 2.1.5 (2).
Proof of Theorem 2.1.5 (2). - From Theorem 2.1.5 (1) (which was proved in Section 2.4) we have

$$
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r^{+}}} \tilde{J}_{r^{+}}\right)=\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r^{+}}^{r}} \tilde{J}_{r^{+}}\right)
$$

Since $\mathcal{D}_{r^{+}}^{r}$ is the linear span of $\left\{\left[\left(F^{*}, e\right)\right] \mid\left(F^{*}, e\right) \in I_{r}\right\}$, from Lemma 2.6.1, Corollary 2.6.4, and the above equality we have

$$
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r^{+}}} \tilde{J}_{r^{+}}\right)=\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{I_{r}^{d}} \tilde{J}_{r^{+}}\right)
$$

From Remark 2.5.11 we can choose representatives $\left(F_{\mathcal{O}}, e_{\mathcal{O}}\right) \in I_{r}^{d}$, indexed by $\mathcal{O} \in \mathcal{O}(0)$, for $I_{r}^{d} / \sim$. An induction argument applied to Lemma 2.6 .5 shows that for $T \in \tilde{J}_{r^{+}}$we have

$$
\operatorname{res}_{I_{r}^{d}} T=0 \quad \text { if and only if } \quad T\left(\left[\left(F_{\mathcal{O}}^{*}, e_{\mathcal{O}}\right)\right]\right)=0 \quad \text { for all } \mathcal{O} \in \mathcal{O}(0)
$$

Thus

$$
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{I_{r}^{d}} \tilde{J}_{r^{+}}\right) \leqslant|\mathcal{O}(0)|
$$

Finally, we can finish the proof of Theorem 2.1.5.
Proof of Theorem 2.1.5 (3). - By hypothesis we have

$$
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r}+} J(\mathcal{N})\right)=|\mathcal{O}(0)|
$$

Since $J(\mathcal{N}) \subset \tilde{J}_{r+}$ and $|\mathcal{O}(0)|=\left|I_{r}^{d} / \sim\right|<\infty$, Theorem 2.1.5 (2) says

$$
|\mathcal{O}(0)|=\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r}+} J(\mathcal{N})\right) \leqslant \operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{r}+} \tilde{J}_{r^{+}}\right) \leqslant|\mathcal{O}(0)|
$$

Thus

$$
\operatorname{res}_{\mathcal{D}_{r^{+}}} J(\mathcal{N})=\operatorname{res}_{\mathcal{D}_{r^{+}}} \tilde{J}_{r^{+}} .
$$

## 3. A proof of Conjecture 1

Eventually, we will be required to assume that there exists a "nice" $G$-invariant, bilinear, symmetric, nondegenerate form on $\mathfrak{g}$ (see Hypothesis 3.4.1). However, it is notationally simpler to avoid identifying $\mathfrak{g}$ with its dual for the first few pages of this section.

### 3.1. The Fourier transform

Let $\mathfrak{g}^{*}$ denote the dual of $\mathfrak{g}$.
Let $d X$ be a Haar measure on $\mathfrak{g}$. For any $f \in C_{c}^{\infty}(\mathfrak{g})$, we define the Fourier transform $\hat{f} \in C_{c}^{\infty}\left(\mathfrak{g}^{*}\right)$ of $f$ by

$$
\hat{f}(\chi)=\int_{\mathfrak{g}} f(X) \cdot \Lambda(\chi(X)) d X
$$

for $\chi \in \mathfrak{g}^{*}$. Let $d \chi$ be a Haar measure on $\mathfrak{g}^{*}$. For $f \in C_{c}^{\infty}\left(\mathfrak{g}^{*}\right)$ we use the natural identification of $\mathfrak{g}^{* *}$ with $\mathfrak{g}$ and define the Fourier transform $\hat{f} \in C_{c}^{\infty}(\mathfrak{g})$ by

$$
\hat{f}(X)=\int_{\mathfrak{g}^{*}} f(\chi) \cdot \Lambda(\chi(X)) d \chi
$$

for $X \in \mathfrak{g}$. We normalize our measures $d X$ and $d \chi$ so that for $X \in \mathfrak{g}$ and $f \in C_{c}^{\infty}(\mathfrak{g})$

$$
\hat{\hat{f}}(X)=f(-X)
$$

DEFINITION 3.1.1. - For $x \in \mathcal{B}(G)$ and $r \in \mathbb{R}$, the Moy-Prasad lattices in $\mathfrak{g}^{*}$ are defined by

$$
\mathfrak{g}_{x, r}^{*}:=\left\{\chi \in \mathfrak{g}^{*} \mid \chi(X) \in \wp \text { for all } X \in \mathfrak{g}_{x,(-r)^{+}}\right\} .
$$

We note that $f \in C\left(\mathfrak{g}_{x, r} / \mathfrak{g}_{x, s}\right)$ if and only if $\hat{f} \in C\left(\mathfrak{g}_{x,(-s)^{+}}^{*} / \mathfrak{g}_{x,(-r)^{+}}^{*}\right)$.

### 3.2. The map hypothesis

HYPOTHESIS 3.2.1. - Suppose $r>0$. There exists a bijective map $\varphi: \mathfrak{g}_{r} \rightarrow G_{r}$ such that 1. for all pairs $(x, s) \in \mathcal{B}(G) \times \mathbb{R} \geqslant r$ we have
(a) $\varphi\left(\mathfrak{g}_{x, s}\right)=G_{x, s}$,
(b) for all $X \in \mathfrak{g}_{x, r}$ and for all $Y \in \mathfrak{g}_{x, s}$ we have $\varphi(X) \cdot \varphi(Y)=\varphi(X+Y)$ modulo $G_{x, s^{+}}$, and
(c) $\varphi$ induces a group isomorphism of $\mathfrak{g}_{x, s} / \mathfrak{g}_{x, s^{+}}$with $G_{x, s} / G_{x, s^{+}}$;
2. for all $g \in G$ we have $\operatorname{Int}(g) \circ \varphi=\varphi \circ \operatorname{Ad}(g)$;
3. and $\varphi$ carries $d X$ into $d g$.

Remark 3.2.2. - A map satisfying Hypothesis 3.2 .1 often exists. For example, for $\mathbf{G} \mathbf{L}_{n}(k)$ realized in the usual way the map $X \mapsto(1+X)$ works for all $r>0$; for a split classical group in odd residual characteristic the Cayley transform works for all $r>0$; if $k$ has characteristic zero, then the exponential map works for $r$ sufficiently large.

Remark 3.2.3. - A map satisfying the requirements of Hypothesis 3.2.1 will bijectively map $\mathfrak{g}_{r} \cap \mathfrak{g}^{\text {reg }}$ to $G_{r} \cap G^{\mathrm{reg}}$.

### 3.3. A Kirillov result

Fix an irreducible admissible representation $(\pi, V)$ of $G$. Choose $r \in \mathbb{R}_{>0}$ such that $\mathfrak{g}_{r}=\mathfrak{g}_{\rho(\pi)^{+}}$(see [2]). Assume that $\varphi: \mathfrak{g}_{r} \rightarrow G_{r}$ satisfies Hypothesis 3.2.1.

DEFINITION 3.3.1. - If $f \in C_{c}^{\infty}\left(\mathfrak{g}_{r}\right)$, then we define the function $f \circ \varphi^{-1} \in C_{c}^{\infty}(G)$ by

$$
f \circ \varphi^{-1}(g):= \begin{cases}f(X) & \text { if } X \in \mathfrak{g}_{r} \text { and } \varphi(X)=g \\ 0 & \text { if } g \notin G_{r}\end{cases}
$$

The following lemma allows us to transfer our problem to the dual of the Lie algebra. Let $\mathcal{N}^{*}$ denote the set of nilpotent elements in $\mathfrak{g}^{*}$, i.e., the set of $\tau \in \mathfrak{g}^{*}$ for which there exists $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{G})$ such that $\lim _{t \rightarrow 0}{ }^{\lambda(t)} \tau=0$.

LEMMA 3.3.2. - Fix $x \in \mathcal{B}(G)$ and $s \geqslant r$. Suppose $f \in C\left(\mathfrak{g}_{x,-s}^{*} / \mathfrak{g}_{x,(-r)^{+}}^{*}\right)$. If

$$
\Theta_{\pi}\left(\widehat{f} \circ \varphi^{-1}\right) \neq 0
$$

then $\operatorname{supp}(f) \cap\left(\mathfrak{g}_{x,(-s)^{+}}^{*}+\mathcal{N}^{*}\right) \neq \emptyset$.
The following proof is a very minor modification of the material in [25, §6.7].
Proof. - Without loss of generality, we may assume that $\mathfrak{g}_{x, s} \neq \mathfrak{g}_{x, s^{+}}$. Since $\pi$ is admissible, we can write

$$
\left.\pi\right|_{G_{x, r}}=\sum_{\sigma \in \widehat{G_{x, r}}} m(\sigma) \cdot \sigma
$$

and each irreducible representation $\sigma$ of $G_{x, r}$ occurs with finite multiplicity $m(\sigma)$.
Since $\Theta_{\pi}\left(\widehat{f} \circ \varphi^{-1}\right) \neq 0$, there exists a representation $(\sigma, W)$ of $G_{x, r}$ such that

$$
\operatorname{tr}\left(\sigma\left(\widehat{f} \circ \varphi^{-1}\right)\right) \neq 0
$$

Since $\widehat{f} \circ \varphi^{-1}$ is invariant under translation by elements of $G_{x, s^{+}}$, it follows that $\left.\sigma\right|_{G_{x, s+}}$ is trivial. Since $G_{x, s} / G_{x, s^{+}}$is abelian, there exists a basis $w_{1}, w_{2}, \ldots, w_{m}$ of $W$ and $\chi_{i} \in \mathfrak{g}_{x,-s}^{*} / \mathfrak{g}_{x,(-s)^{+}}^{*}$ for $1 \leqslant i \leqslant m$ such that

$$
\sigma(\varphi(Y)) w_{i}=\Lambda\left(\chi_{i}(Y)\right) w_{i}
$$

for all $Y \in \mathfrak{g}_{x, s}$ and $1 \leqslant i \leqslant m$. Note that from [21, §7.2] we have $\chi_{i} \in \mathfrak{g}_{x,(-s)^{+}}^{*}+\mathcal{N}^{*}$ for $1 \leqslant i \leqslant m$.

Let $\langle$,$\rangle denote the natural G$-invariant pairing of $V$ and its contragredient. Let $\widetilde{W}$ denote the dual of $W$ with dual basis $\tilde{w}_{1}, \tilde{w}_{2}, \ldots, \tilde{w}_{m}$.

Now,

$$
\begin{aligned}
0 & \neq \operatorname{tr}\left(\sigma\left(\widehat{f} \circ \varphi^{-1}\right)\right) \\
& =\text { const } \cdot
\end{aligned} \sum_{\bar{h} \in G_{x, r} / G_{x, s}} \sum_{\bar{g} \in G_{x, s} / G_{x, s}+}\left(\widehat{f} \circ \varphi^{-1}(h g)\right) \cdot \operatorname{tr}(\sigma(h g)) .
$$

Thus there exists an $\bar{h} \in G_{x, r} / G_{x, s}$ such that the inner sum is not zero. For this $h$, choose $X \in \mathfrak{g}_{x, r}$ such that $h=\varphi(X)$. Then

$$
\begin{aligned}
0 & \neq \sum_{\bar{Y} \in \mathfrak{g}_{x, s} / \mathfrak{g}_{x, s+}}\left(\widehat{f} \circ \varphi^{-1}(\varphi(X) \varphi(Y))\right) \cdot \operatorname{tr}(\sigma(\varphi(X) \varphi(Y))) \\
& =\text { const } \cdot \int_{\mathfrak{g}_{x, s}} d Y \widehat{f}(X+Y) \cdot\left(\sum_{1 \leqslant i \leqslant m}\left\langle\sigma(\varphi(X)) w_{i}, \tilde{w}_{i}\right\rangle \cdot \Lambda\left(\chi_{i}(Y)\right)\right) .
\end{aligned}
$$

So, there exists an $i$ such that

$$
\begin{aligned}
0 \neq \int_{\mathfrak{g}_{x, s}} d Y \widehat{f}(X+Y) \cdot \Lambda\left(\chi_{i}(Y)\right) & =\int_{\mathfrak{g}_{x, s}} d Y\left(\int_{\mathfrak{g}_{x,-s}^{*}} d \chi f(\chi) \cdot \Lambda(\chi(X+Y))\right) \cdot \Lambda\left(\chi_{i}(Y)\right) \\
& =\int_{\mathfrak{g}_{x, s}} d Y \int_{\mathfrak{g}_{x,-s}^{*}} d \chi f\left(\chi-\chi_{i}\right) \cdot \Lambda\left(\left(\chi-\chi_{i}\right)(X)\right) \cdot \Lambda(\chi(Y)) \\
& =\int_{\mathfrak{g}_{x,-s}^{*}} d \chi f\left(\chi-\chi_{i}\right) \cdot \Lambda\left(\left(\chi-\chi_{i}\right)(X)\right) \cdot \int_{\mathfrak{g}_{x, s}} d Y \Lambda(\chi(Y)) .
\end{aligned}
$$

The inner integral of the final displayed line above is zero unless $\chi \in \mathfrak{g}_{x,(-s)^{+}}^{*}$. Therefore, the support of $f$ must intersect $\chi_{i}+\mathfrak{g}_{x,(-s)^{+}}^{*} \subset \mathcal{N}^{*}+\mathfrak{g}_{x,(-s)^{+}}^{*}$.

### 3.4. Two hypotheses and some consequences

Both hypotheses introduced below are valid if $p$ is greater than some constant which may be determined by looking at the absolute root datum of $\mathbf{G}$.
We first assume that we can identify $\mathfrak{g}^{*}$ with $\mathfrak{g}$ in a nice way. See $[3, \S 4]$ for more information about the following hypothesis.

Hypothesis 3.4.1. - There exists a nondegenerate, bilinear, $G$-invariant, symmetric form $B$ on $\mathfrak{g}$ such that, under the associated identification of $\mathfrak{g}$ with $\mathfrak{g}^{*}$, for all $x \in \mathcal{B}(G)$ and all $r \in \mathbb{R}$ we may identify $\mathfrak{g}_{x, r}$ with $\mathfrak{g}_{x, r}^{*}$.

Remark 3.4.2. - We use the $B$ of Hypothesis 3.4 .1 to identify $\mathfrak{g}$ with $\mathfrak{g}^{*}$. When we do this, the Fourier transform of a function on $\mathfrak{g}$ is again a function on $\mathfrak{g}$.

We will also require that nilpotent orbital integrals make sense as distributions on $\mathfrak{g}$. In characteristic zero, this is proved in [22]. To the best of my knowledge, the question of convergence in positive characteristic is still open. However, an analysis of [22] shows that if $p$ is larger than some constant which can be determined from the absolute root datum of $\mathbf{G}$, then nilpotent orbital integrals converge as distributions on $\mathfrak{g}$.

Hypothesis 3.4.3.- If $\mathcal{O} \in \mathcal{O}(0)$, then for $X \in \mathcal{O}$ we may identify the tangent space to $\mathcal{O}$ at the point $X$ with $\mathfrak{g} / C_{\mathfrak{g}}(X)$. Moreover, there exists a $G$-invariant measure $d \mu_{\mathcal{O}}$ on $\mathfrak{g}$ trivially extending the (nontrivial) $G$-invariant measure on $\mathcal{O}$ such that for all $f \in C_{c}^{\infty}(\mathfrak{g})$, the integral

$$
\mu_{\mathcal{O}}(f):=\int_{\mathfrak{g}} f(Y) d \mu_{\mathcal{O}}(Y)
$$

converges.

We follow [17] and normalize the measure $d \mu_{\mathcal{O}}$ on $\mathcal{O}$ in the following way. Suppose $\mathcal{O} \in \mathcal{O}(0)$. Fix $X \in \mathcal{O}$. Identify the tangent space to $\mathcal{O}$ at the point $X$ with $\mathfrak{g} / C_{\mathfrak{g}}(X)$. Let $\Lambda_{X}$ denote the antisymmetric pairing on $\mathfrak{g} / C_{\mathfrak{g}}(X)$ which sends $\left(V_{1}, V_{2}\right) \in \mathfrak{g} / C_{\mathfrak{g}}(X) \times \mathfrak{g} / C_{\mathfrak{g}}(X)$ to $\Lambda\left(B\left(X,\left[V_{1}, V_{2}\right]\right)\right)$. Choose the Haar measure on $\mathfrak{g} / C_{\mathfrak{g}}(X)$ which is dual (with respect to the Fourier transform) to $\Lambda_{X}$. Since $\mathcal{O}$ is a $G$-orbit, this defines a $G$-invariant measure on $\mathcal{O}$.

We do not include the proof of the next lemma. The lemma is a straightforward generalization of a result of Waldspurger [26, §IX.4]; the proof fully uses Hypothesis 3.4.1.

Lemma 3.4.4. - Suppose that all of the hypotheses of Sections 2.2 and 3.4 are valid. Fix $r \in \mathbb{R}_{\geqslant 0}$. Suppose $\left(F^{*}, e\right) \in I_{r}^{n}$ and $X \in \mathcal{O}\left(F^{*}, e\right) \cap \mathfrak{g}_{F^{*}}$ represents $e$. If $x \in F^{*}$, then

$$
\mu_{\mathcal{O}\left(F^{*}, e\right)}\left(\left[\left(F^{*}, e\right)\right]\right)=\left[\mathfrak{g}_{x, 0}: \mathfrak{g}_{x, r^{+}}\right]^{-1 / 2} \cdot\left[C_{\mathfrak{g}_{x, 0}}(X): C_{\mathfrak{g}_{x, r^{+}}}(X)\right]^{1 / 2}
$$

Remark 3.4.5. - Note that if $\mathfrak{g}_{x, r^{+}}=\mathfrak{g}_{x, n \cdot \ell}$ for some $n \in \mathbb{N}$, then

$$
\mu_{\mathcal{O}\left(F^{*}, e\right)}\left(\left[\left(F^{*}, e\right)\right]\right)=q^{n \cdot \frac{\operatorname{dim}\left(\mathcal{O}\left(F^{*}, e\right)\right)}{2}} .
$$

We also note that the left-hand side of the equality in Lemma 3.4.4 is independent of $x$, and so the right-hand side must be as well.

The proof of the next result follows an argument of Dan Barbasch and Allen Moy [5].
Corollary 3.4.6. - Suppose that all of the hypotheses of Sections 2.2 and 3.4 are valid. The elements of

$$
\left\{\operatorname{res}_{\mathcal{D}_{r^{+}}} \mu_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{O}(0)\right\}
$$

form a basis for $\operatorname{res}_{\mathcal{D}_{r^{+}}} J(\mathcal{N})$.
Proof. - If $k$ has characteristic zero, this is known (see, for example, [13, §3]). Fix $r \in \mathbb{R}$. Fix representatives $\left(F_{\mathcal{O}}^{*}, e_{\mathcal{O}}\right) \in I_{r}^{d}$, indexed by $\mathcal{O}(0)$, for $I_{r}^{d} / \sim$. Label the elements of $\mathcal{O}(0)$ so that $\mathcal{O}_{i}<\mathcal{O}_{j}$ implies $i<j$. Define the $|\mathcal{O}(0)|$ by $|\mathcal{O}(0)|$ matrix $M(r)$ by

$$
M(r)_{i j}=\mu_{\mathcal{O}_{j}}\left(\left[\left(F_{\mathcal{O}_{i}}^{*}, e_{\mathcal{O}_{i}}\right)\right]\right)
$$

From Remark 2.5.4 we see that $M(r)$ is an upper triangular matrix. Since nilpotent orbital integrals are homogeneous (see [13, §3.1]), it follows from Lemma 3.4.4 that $M(r)$ is an invertible matrix (even when $r$ is negative). The corollary follows.

Definition 3.4.7. - Suppose that Hypothesis 3.4.1 is valid. For a distribution $T \in J(\mathfrak{g})$ we define the Fourier transform $\widehat{T} \in J(\mathfrak{g})$ of $T$ by

$$
\widehat{T}(f)=T(\widehat{f})
$$

for $f \in C_{c}^{\infty}(\mathfrak{g})$.
Under our hypotheses, for $\mathcal{O} \in \mathcal{O}(0)$ the distribution $\widehat{\mu_{\mathcal{O}}}$ is represented by a locally constant function of $\mathfrak{g}^{\text {reg }}$ (see $[16, \S 3]$ ). We abuse notation and denote both the distribution and the function which represents it by $\widehat{\mu_{\mathcal{O}}}$. The function $\widehat{\mu_{\mathcal{O}}}$ depends on how we choose $d \mu_{\mathcal{O}}$ and $\Lambda$. However, Robert Kottwitz has pointed out that the relationship we have imposed on $d \mu_{\mathcal{O}}$ and $\Lambda$ makes the function $\widehat{\mu_{\mathcal{O}}}$ independent of these choices.

### 3.5. A proof of Conjecture 1

Suppose $(\pi, V)$ is an admissible irreducible representation of $G$. Choose $r \in \mathbb{R}_{>0}$ so that $\mathfrak{g}_{r}=\mathfrak{g}_{\rho(\pi)^{+}}$. Suppose that $\varphi: \mathfrak{g}_{r} \rightarrow G_{r}$ satisfies Hypothesis 3.2.1. Suppose that Hypothesis 3.4.1 as well as all of the hypotheses of Section 2.2 are valid.

Remark 3.5.1. - Let $\tilde{\Theta}_{\pi}$ be the distribution on $\mathfrak{g}$ defined as follows. If $f \in C_{c}^{\infty}(\mathfrak{g})$, then

$$
\tilde{\Theta}_{\pi}(f):=\Theta_{\pi}\left(f_{1} \circ \varphi^{-1}\right)
$$

where $f_{1}=f \cdot\left[\mathfrak{g}_{r}\right]$. Recall that $\left[\mathfrak{g}_{r}\right]$ denotes the characteristic function of $\mathfrak{g}_{r}$. Since $\mathfrak{g}_{r}$ is a $G$-domain, we have $f_{1} \in C_{c}^{\infty}\left(\mathfrak{g}_{r}\right)$. Note that $\tilde{\Theta}_{\pi}$ is a $G$-invariant distribution on $\mathfrak{g}$ whose support is contained in $\mathfrak{g}_{r}$. Since Hypothesis 3.4.1 is valid, we can let $\widehat{\Theta}_{\pi} \in J(\mathfrak{g})$ denote the Fourier transform of $\tilde{\Theta}_{\pi}$. From Lemma 3.3.2 we have $\widehat{\Theta}_{\pi} \in \tilde{J}_{x,(-s),(-r)^{+}}$for all $x \in \mathcal{B}(G)$ and for all $s \geqslant r$. Thus $\widehat{\Theta}_{\pi} \in \tilde{J}_{(-r)^{+}}$.

THEOREM 3.5.2. - Suppose $(\pi, V)$ is an admissible irreducible representation of $G$. Choose $r \in \mathbb{R}$ so that $\mathfrak{g}_{r}=\mathfrak{g}_{\rho(\pi)^{+}}$. Suppose that $\varphi: \mathfrak{g}_{r} \rightarrow G_{r}$ satisfies Hypothesis 3.2.1. Suppose that all of the hypotheses of Sections 2.2 and 3.4 are valid. Then there exist constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ indexed by $\mathcal{O}(0)$ such that

$$
\Theta_{\pi}(\varphi(X))=\sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}}(\pi) \cdot \widehat{\mu_{\mathcal{O}}}(X)
$$

for all $X \in \mathfrak{g}_{\rho(\pi)^{+}} \cap \mathfrak{g}^{\text {reg }}$.
Proof. - Suppose $f \in C_{c}^{\infty}\left(\mathfrak{g}_{r}\right)$. We need to show that there exist complex constants $c_{\mathcal{O}}$, indexed by $\mathcal{O} \in \mathcal{O}(0)$, such that

$$
\Theta_{\pi}\left(f \circ \varphi^{-1}\right)=\sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}} \cdot \widehat{\mu_{\mathcal{O}}}(f)
$$

From [2] we have $\widehat{f} \in \mathcal{D}_{(-r)^{+}}$. From Remark 3.5 .1 we have $\widehat{\Theta}_{\pi} \in \tilde{J}_{(-r)^{+}}$. From Corollary 3.4.6 we have $\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{D}_{(-r)}+} J(\mathcal{N})\right)=|\mathcal{O}(0)|$. Thus, from Theorem 2.1.5 (3) there exist constants $c_{\mathcal{O}} \in \mathbb{C}$, indexed by $\mathcal{O} \in \mathcal{O}(0)$, such that

$$
\widehat{\Theta}_{\pi}(\widehat{f})=\sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}} \cdot \mu_{\mathcal{O}}(\widehat{f})=\sum_{\mathcal{O} \in \mathcal{O}(0)} c_{\mathcal{O}} \cdot \widehat{\mu_{\mathcal{O}}}(f) .
$$

On the other hand, we have

$$
\widehat{\Theta}_{\pi}(\widehat{f})=\tilde{\Theta}_{\pi}(\hat{\hat{f}})=\Theta_{\pi}\left(\hat{\hat{f}} \circ \varphi^{-1}\right)
$$

Note that for all $\mathcal{O} \in \mathcal{O}(0)$ there exists a constant depending only on $\mathcal{O}$ such that

$$
\widehat{\mu_{\mathcal{O}}}(h)=\operatorname{const} \cdot \widehat{\mu_{-\mathcal{O}}}(\hat{\hat{h}})
$$

for all $h \in C_{c}^{\infty}(\mathfrak{g})$.

### 3.6. Some comments on the $c_{\mathcal{O}}(\pi)$ s

Fix an irreducible admissible representation $(\pi, V)$ of $G$. In this section we discuss how one might calculate the coefficients occurring in the Harish-Chandra-Howe local expansion. This is a modification of the approach in [5].

Fix $r>\rho(\pi)$. Assume that $\varphi: \mathfrak{g}_{r} \rightarrow G_{r}$ satisfies Hypothesis 3.2.1.
Suppose that all of the hypotheses of Sections 2.2 and 3.4 are valid.
DEFINITION 3.6.1. - For $\left(F^{*}, e\right) \in I_{-r}^{d}$, define

$$
m\left(F^{*}, e\right):=\operatorname{dim}_{\mathbb{C}}\left\{v \in V^{G_{F^{*}}^{+}} \mid \pi(\varphi(X)) v=\Lambda(-B(E, X)) v \text { for all } X \in \mathfrak{g}_{F^{*}, r}\right\}
$$

where $E \in \mathfrak{g}_{F^{*}}$ is any lift of $e$.
LEMMA 3.6.2. - Under the hypotheses discussed above we have

$$
\Theta_{\pi}\left(\left[\widehat{\left(F^{*}, e\right)}\right] \circ \varphi^{-1}\right)=m\left(F^{*}, e\right)
$$

Proof. - This follows from the proof of Lemma 3.3.2.
Remark 3.6.3. - Fix representatives $\left(F_{\mathcal{O}}^{*}, e_{\mathcal{O}}\right) \in I_{-r}^{d}$, indexed by $\mathcal{O} \in \mathcal{O}(0)$, for $I_{-r}^{d} / \sim$. Define the matrix $M(-r)$ as in the proof of Corollary 3.4.6. Define the vector $\vec{\Theta}_{\pi} \in \mathbb{Z}^{|\mathcal{O}(0)|}$ by $\left(\vec{\Theta}_{\pi}\right)_{\mathcal{O}}=m\left(F_{\mathcal{O}}^{*}, e_{\mathcal{O}}\right)$. We have

$$
c_{\mathcal{O}}(\pi)=\left(M(-r)^{-1} \cdot \vec{\Theta}_{\pi}\right)_{\mathcal{O}}
$$

Thus, in principle, we can calculate the $c_{\mathcal{O}}(\pi) \mathrm{s}$.

## 4. A proof of Conjecture 3 for positive $r$

In this section we first show that we may choose a dual basis for $\operatorname{res}_{\mathcal{D}_{r}} J\left(G_{r}\right)$ such that the elements of this basis have their support in $G_{0}$. Subject to some conditions, we then prove Conjecture 3 for positive $r$.

### 4.1. A statement of the results

First consider the case when $r=0$. Fix an alcove $C$ in $\mathcal{B}(G)$. Define the subspace $\mathcal{H}_{0}^{0}$ of $\mathcal{H}_{0}$ by

$$
\mathcal{H}_{0}^{0}:=\sum_{x \in \bar{C}} C\left(G_{x} / G_{C}\right)
$$

The sum above should be interpreted as in the introduction. In Section 4.2 .1 we prove the following theorem which is related to a conjecture of J.-L. Waldspurger [27, §3].

THEOREM 4.1.1. - If $T \in J\left(G_{0}\right)$, then

$$
\operatorname{res}_{\mathcal{H}_{0}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{\mathcal{H}_{0}^{0}} T=0
$$

Example 4.1.2. - For $\mathbf{S L}_{2}(k)$, Theorem 4.1.1 implies that

$$
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{H}_{0}} J(\mathcal{U})\right) \leqslant 3
$$

Note that $\mathbf{S L}_{2}(k)$ has more than three unipotent orbits. This is markedly different from what our experience on the Lie algebra suggests ought to be true; it reflects the fact that, unlike nilpotent orbits, unipotent orbits are not homogeneous (see [13, §3.1]).

Now suppose $r \in \mathbb{R}_{>0}$. The sums below should be interpreted as in the introduction.
DEFINITION 4.1.3. - For $s \geqslant 0$ define $\mathcal{H}_{s^{+}}$by

$$
\mathcal{H}_{s^{+}}:=\sum_{x \in \mathcal{B}(G)} C_{c}\left(G / G_{x, s^{+}}\right)
$$

Define the subspace $\mathcal{H}_{s^{+}}^{0}$ of $\mathcal{H}_{s^{+}}$by

$$
\mathcal{H}_{s^{+}}^{0}:=\sum_{x \in \mathcal{B}(G)} C\left(G_{x, 0} / G_{x, s^{+}}\right)
$$

Define the subspace $\mathcal{H}_{s^{+}}^{s}$ of $\mathcal{H}_{s^{+}}$by

$$
\mathcal{H}_{s^{+}}^{s}:=\sum_{x \in \mathcal{B}(G)} C\left(G_{x, s} / G_{x, s^{+}}\right)
$$

From [2] and [12, Lemma 5.5.3] there exists $0 \leqslant s<r$ such that $\mathcal{H}_{r}=\mathcal{H}_{s^{+}}$and $G_{r}=G_{s^{+}}$. Thus, for purposes of investigating Conjecture 3 for positive $r$, we may assume that $r$ is nonnegative and prove the following theorem.

THEOREM 4.1.4. - Suppose all of the hypotheses of Sections 2.2 and 4.3 are valid. Suppose $r \in \mathbb{R}_{\geqslant 0}$.

1. Suppose $T \in J\left(G_{r^{+}}\right)$. We have

$$
\operatorname{res}_{\mathcal{H}_{r^{+}}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{\mathcal{H}_{r^{+}}^{0}} T=0 .
$$

2. Suppose $T \in J\left(G_{r^{+}}\right)$. We have

$$
\operatorname{res}_{\mathcal{H}_{r^{+}}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{\mathcal{H}_{r+}^{r}} T=0
$$

3. If $\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{H}_{r}+} J(U)\right)=|\mathcal{O}(0)|$, then

$$
\operatorname{res}_{\mathcal{H}_{r^{+}}} J\left(G_{r^{+}}\right)=\operatorname{res}_{\mathcal{H}_{r^{+}}} J(\mathcal{U})
$$

Remark 4.1.5. - The proof of statement (1) does not require any restrictions on $\mathbf{G}$ and $k$.
Remark 4.1.6. - If the unipotent orbital integrals converge for $f \in \mathcal{H}_{s^{+}}$, then the condition that

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}}\left(\operatorname{res}_{\mathcal{H}_{s^{+}}} J(\mathcal{U})\right)=|\mathcal{O}(0)| \tag{5}
\end{equation*}
$$

is known to hold under our hypotheses. This follows from Lemma 4.4.1 (3) by arguing as in Corollary 3.4.6.

Remark 4.1.7. - If condition (5) and the hypotheses of Sections 2.2 and 4.3 are valid, then Theorem 4.1.4 (3) implies Conjecture 3 (for positive $r$ ).

### 4.2. A dual basis among the compactly supported functions

The proofs of this subsection are derived from a beautiful paper of Dan Barbasch and Allen Moy [4]. The presentation below has also been influenced by a preprint of Allen Moy and Gopal Prasad [20].

For $g \in G$ and $x \in \mathcal{B}(G)$, we let $\mathrm{d}_{g}(x)$ denote the distance that $g$ moves $x$. The function $\mathrm{d}_{g}: \mathcal{B}(G) \rightarrow \mathbb{R} \geqslant 0$ is called the displacement function (for $g$ ). In this subsection, we will repeatedly use the following result (see, for example, [12, Corollary 3.3.3]).

Lemma 4.2.1. - Suppose $g \in G$. If there exists an $x \in \mathcal{B}(G)$ such that $g x=x$, then for all $y \in \mathcal{B}(G)$ we have

$$
\left.\mathrm{d}_{g}\right|_{(y, g y)}<\mathrm{d}_{g}(y)=\mathrm{d}_{g}(g y) .
$$

### 4.2.1. A proof of Theorem 4.1.1

We begin with a "descent and recovery" result.
Fix an alcove $C$ in $\mathcal{B}(G)$. Let $m_{g}$ denote the minimum of $\mathrm{d}_{g}$ restricted to $\bar{C}$.
Remark 4.2.2. - Note that if $h \in g \cdot G_{C}$, then $\mathrm{d}_{h}=\mathrm{d}_{g}$ on $\bar{C}$ so that $m_{h}=m_{g}$.
Lemma 4.2.3. - Suppose $u \in \mathcal{U}$ and $T \in J\left(G_{0}\right)$. If $m_{u}>0$, then there exist a finite set $\mathcal{V} \subset \mathcal{U}$ and a constant $c \in \mathbb{Q}$ such that

1. $m_{u}>m_{v}$ for all $v \in \mathcal{V}$, and
2. $T\left(\left[u \cdot G_{C}\right]\right)=c \cdot \sum_{v \in \mathcal{V}} T\left(\left[v \cdot G_{C}\right]\right)$.

Proof. - Since $\bar{C}$ is compact, there exists $y \in \bar{C}$ such that $\mathrm{d}_{u}(y)=m_{u}$.
We first note that $y \in \bar{C} \backslash C$. Indeed, suppose $y \in C$. Since $m_{u}>0$, we have $y \neq u y$. But then for all $z \in(y, u y) \cap C$ we have $d_{u}(z)<d_{u}(y)=m_{u}$ from Lemma 4.2.1, a contradiction.

Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing $C$. Let $\mathbf{S}$ be the maximal $k$-split torus of $\mathbf{G}$ corresponding to $\mathcal{A}$. Let $S=\mathbf{S}(k)$. From the Bruhat decomposition of $G$ there exist $n \in N_{\mathbf{G}}(\mathbf{S})(k)$ and $b_{i} \in G_{C}$ so that $u=b_{1}^{-1} \cdot n \cdot b_{2}$. Let $u^{\prime}={ }^{b_{1}} u \in \mathcal{U}$. We have the following facts.

Remark 4.2.4. -

1. We have $T\left(\left[u \cdot G_{C}\right]\right)=T\left(\left[u^{\prime} \cdot G_{C}\right]\right)$.
2. We have $\mathrm{d}_{u^{\prime}}(x)=\mathrm{d}_{u}(x)$ for all $x \in \bar{C}$. In particular, $m_{u}=m_{u^{\prime}}$.
3. Finally, $\left[u^{\prime} y, y\right]=[n y, y]$ and $u^{\prime} y=n y \in \mathcal{A}$.

Remark 4.2.4 (3) and another application of Lemma 4.2 .1 show that $[y, n y] \cap \bar{C}=\{y\}$. Let $F$ be the first facet in $\mathcal{A}$ that $(y, n y]$ passes through as we travel from $y$ to $n y$. Note that $y \in \bar{F} \backslash F$.

For points $x_{1}, x_{2} \in \mathcal{A}$, let $\left(x_{1}-x_{2}\right) \in \mathbf{X}_{*}^{k}(\mathbf{S}) \otimes \mathbb{R}$ denote the corresponding vector. Choose points $x_{C} \in C$ and $x_{F} \in F$ so that $\alpha\left(x_{C}-x_{F}\right) \neq 0$ for all $\alpha \in \Phi(S)$. Let $D$ be an alcove in $\mathcal{A}$ such that

1. $F \subset \bar{D}$ and
2. $D \cap\left(x_{C}, x_{F}\right) \neq \emptyset$.

Note that the first condition on $D$ implies that $C \neq D$.
Define

$$
Q:=\left\{\psi \in \Psi(\mathcal{A})|\psi|_{D}>0 \text { and }\left.\psi\right|_{C}<0\right\} .
$$

We have $\left.\psi\right|_{F}>0$ for all $\psi \in Q$. Indeed, since $\left.\psi\right|_{D}>0$ and $F \subset \bar{D}$, if $\psi \in Q$, then $\left.\psi\right|_{F} \geqslant 0$. If for some $\psi^{\prime} \in Q$ we have $\left.\psi^{\prime}\right|_{F}=0$, then $\psi^{\prime}\left(x_{F}\right)=0$ and $\psi^{\prime}\left(x_{C}\right)<0$. Since $D \cap\left(x_{C}, x_{F}\right) \neq \emptyset$, we conclude that $\left.\psi^{\prime}\right|_{D} \leqslant 0$, a contradiction.

Since $y \in \bar{D} \cap \bar{C}$, we have $\psi(y)=0$ for all $\psi \in Q$. Consequently, this and the previous paragraph imply that for $\psi \in Q$ we have $\left(n^{-1} \psi\right)(y)=\psi(n y)>0$. Therefore, for $\psi \in Q$ we have

$$
\begin{equation*}
n^{-1} \cdot U_{\psi} \cdot n=U_{n^{-1} \psi} \subset G_{y}^{+} \subset G_{C} . \tag{6}
\end{equation*}
$$

We also have $U_{\psi}^{+} \subset G_{y}^{+} \subset G_{C} \cap G_{D}$ for all $\psi \in Q$.
Let $\Phi^{+} \subset \Phi(S)$ denote the set of positive roots with respect to $\left(x_{F}-x_{C}\right)$. Let

$$
\Phi^{-}=\Phi(S) \backslash \Phi^{+} .
$$

If $\psi \in Q$, then $\dot{\psi} \in \Phi^{+}$. Let $U^{ \pm}$denote the group generated by the root groups $U_{a}$ for $a \in \Phi^{ \pm}$. Let $P^{ \pm}=N_{G}\left(U^{ \pm}\right)$. The groups $P^{+}$and $P^{-}$are opposite minimal parabolic subgroups of $G$, and if $M=P^{+} \cap P^{-}$, then $P^{ \pm}$has a Levi decomposition $P^{ \pm}=M N^{ \pm}$. Since $S \subset M$, both $G_{C}$ and $G_{D}$ have Iwahori decompositions with respect to $P^{+}$. So we have $G_{D}=\left(G_{D} \cap P^{-}\right) \cdot\left(G_{D} \cap\right.$ $\left.U^{+}\right)$. We also have $G_{D} \cap P^{-} \subset G_{C} \cap P^{-}$and

$$
G_{D} \cap U^{+}=\left(G_{C} \cap U^{+}\right) \cdot\left(\prod_{\psi \in Q} U_{\psi}\right)
$$

with the product over $\psi \in Q$ in any order. Therefore

$$
G_{C} \cdot G_{D}=G_{C} \cdot\left(\prod_{\psi \in Q} U_{\psi}\right) .
$$

From the above paragraph, Remark 4.2.4 (1), and Eq. (6) we have

$$
\begin{aligned}
T\left(\left[u \cdot G_{C}\right]\right) & =T\left(\left[u^{\prime} \cdot G_{C}\right]\right) \\
& =\text { const } \cdot \sum_{\bar{g} \in\left(\prod_{\psi \in Q} U_{\psi}\right) /\left(\prod_{\psi \in Q} U_{\psi}^{+}\right)} T\left(\left[g^{-1} \cdot n \cdot G_{C} \cdot g\right]\right) \\
= & \text { const } \cdot \sum_{\bar{g} \in\left(\prod_{\psi \in Q} U_{\psi}\right) /\left(\prod_{\psi \in Q} U_{\psi}^{+}\right)} T\left(\left[n \cdot G_{C} \cdot g\right]\right) \\
= & \text { const } \cdot T\left(\left[n \cdot G_{C} \cdot G_{D}\right]\right) \\
= & \text { const } \cdot \sum_{\bar{\alpha} \in n G_{C} G_{D} / G_{D}} T\left(\left[\alpha \cdot G_{D}\right]\right) .
\end{aligned}
$$

Since $T \in J\left(G_{0}\right)$ and $G_{0} \subset G_{D} \cdot \mathcal{U}$, for $\alpha \in n \cdot G_{C} \cdot G_{D}$ we have $T\left(\left[\alpha \cdot G_{D}\right]\right)=0$ unless $\alpha \cdot G_{D} \cap \mathcal{U} \neq \emptyset$. We can therefore choose a finite subset $\mathcal{V} \subset \mathcal{U} \cap\left(n \cdot G_{C} \cdot G_{D}\right)$ such that

$$
T\left(\left[u \cdot G_{C}\right]\right)=\text { const } \cdot \sum_{v \in \mathcal{V}} T\left(\left[v \cdot G_{D}\right]\right) .
$$

Note that for all $\alpha \in n \cdot G_{C} \cdot G_{D}$, we have $\alpha y=n y$. From Lemma 4.2.1 there exists a $z \in \bar{D} \cap(y, n y)$ such that $\mathrm{d}_{v}(z)<\mathrm{d}_{v}(y)$ for all $v \in \mathcal{V}$. Therefore, from Remark 4.2.4 (2) it follows that

$$
\min _{x \in \bar{D}} \mathrm{~d}_{v}(x) \leqslant \mathrm{d}_{v}(z)<\mathrm{d}_{v}(y)=\mathrm{d}_{n}(y)=\mathrm{d}_{u^{\prime}}(y)=m_{u}
$$

for all $v \in \mathcal{V}$.

Since $G$ acts transitively on the alcoves of $\mathcal{B}(G)$, there exists an element of $G$ taking $D$ to $C$. Since $\mathrm{d}_{g h}(g x)=\mathrm{d}_{h}(x)$ for all $h, g \in G$ and $x \in \mathcal{B}(G)$, the lemma follows from the $G$-invariance of $T$.

We can now prove Theorem 4.1.1.
Proof of Theorem 4.1.1. - Fix $T \in J\left(G_{0}\right)$.
$" \Rightarrow ":$ Since $\mathcal{H}_{0}^{0} \subset \mathcal{H}_{0}$, if $\operatorname{res}_{\mathcal{H}_{0}} T=0$, then $\operatorname{res}_{\mathcal{H}_{0}^{0}} T=0$.
" $\Leftarrow "$ " Suppose $f \in \mathcal{H}_{0}$. We want to show that $T(f)$ is completely determined by $\operatorname{res}_{\mathcal{H}_{0}^{0}} T$. Since $T$ is linear, $G$-invariant, and supported in $G_{0} \subset G_{C} \cdot \mathcal{U}$, without loss of generality we can assume that $f=\left[u \cdot G_{C}\right]$ where $u \in \mathcal{U}$.

Suppose $m_{u}=0$. Since $\bar{C}$ is compact, there exists $y \in \bar{C}$ such that $\mathrm{d}_{u}(y)=m_{u}$. Therefore, $u y=y$ and so $u \in G_{y}$ from [12, Lemma 4.2.1]. Thus $\left[u \cdot G_{C}\right] \in C\left(G_{y} / G_{C}\right) \subset \mathcal{H}_{0}^{0}$.

If $m_{u}>0$, then we wish to iteratively apply Lemma 4.2.3 until we can write

$$
T\left(\left[u \cdot G_{C}\right]\right)=\sum_{v \in \mathcal{V}} c(v) \cdot T\left(\left[v \cdot G_{C}\right]\right)
$$

with $\mathcal{V}$ a finite subset of $\mathcal{U}$ and $m_{v}=0$ for all $v \in \mathcal{V}$. Suppose that we cannot do this. From Lemma 4.2.3 there exists a sequence $\left\{u_{i}\right\}$ of unipotent elements such that $m_{u_{i}}>m_{u_{(i+1)}}>0$ for all $i \in \mathbb{N}$. From the Bruhat decomposition there exist $n_{i} \in N_{\mathbf{G}}(\mathbf{S})(k)$ and $b_{i}, b_{i}^{\prime} \in G_{C}$ so that $u_{i}=b_{i} \cdot n_{i} \cdot b_{i}^{\prime}$ for all $i \in \mathbb{N}$. We have $\mathrm{d}_{u_{i}}(x)=\mathrm{d}_{n_{i}}(x)$ for all $x \in \bar{C}$, thus $m_{n_{i}}>m_{n_{(i+1)}}>0$ for all $i \in \mathbb{N}$. This contradicts the fact that $\left\{m_{n}: n \in N_{\mathbf{G}}(\mathbf{S})(k)\right\}$ is a discrete subset of $\mathbb{R}$.

### 4.2.2. A proof of Theorem 4.1.4 (1)

Fix $r \geqslant 0$.
Suppose $F^{*} \in \mathcal{F}(r)$ and $h \in G$. Since $\overline{F^{*}}$ is compact, the restriction of the displacement function $d_{h}$ to $\overline{F^{*}}$ attains its (nonnegative) minimum on $\overline{F^{*}}$.

Remark 4.2.5. - If $g \in h \cdot G_{F^{*}}^{+}$, then $\mathrm{d}_{g}=\mathrm{d}_{h}$ on $\overline{F^{*}}$. (We could write $\mathrm{d}_{h \cdot G_{F^{*}}^{+}}$to more accurately reflect this fact, but this is cumbersome.) Thus, $\min _{x \in \overline{F^{*}}} \mathrm{~d}_{g}(x)$ is independent of our choice of $g \in h \cdot G_{F^{*}}^{+}$.

Lemma 4.2.6. - Suppose $F^{*} \in \mathcal{F}(r), u \in \mathcal{U}$, and $T \in J\left(G_{r^{+}}\right)$. Let $m=\min _{x \in \overline{F^{*}}} \mathrm{~d}_{u}(x)$. If $m>0$ and there exists $y \in F^{*}$ such that $\mathrm{d}_{u}(y)=m$, then there exist $v \in \mathcal{U}$, a constant $c(v) \in \mathbb{Q}$, and a generalized $r$-facet $F_{v}^{*}$ such that

1. $\min _{x \in \overline{F^{*}}} \mathrm{~d}_{u}(x)>\min _{x \in \overline{F_{v}^{*}}} \mathrm{~d}_{v}(x)$, and
2. $T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=c(v) \cdot T\left(\left[v \cdot G_{F_{v}^{*}}^{+}\right]\right)$.

Proof. - Choose $y \in F^{*}$ such that $\mathrm{d}_{u}(y) \leqslant \mathrm{d}_{u}(x)$ for all $x \in \overline{F^{*}}$. Let $\mathbf{S}$ be a maximal $k$-split torus of $\mathbf{G}$ such that $y$ is an element of the apartment $\mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(G)$. Let $S=\mathbf{S}(k)$ and $\mathcal{A}=\mathcal{A}(\mathbf{S}, k)$. Let $C$ be a 0 -alcove (i.e., an affine chamber) in $\mathcal{A}$ such that $y \in \bar{C}$.

From the Bruhat decomposition there exist $n \in N_{\mathbf{G}}(\mathbf{S})(k)$ and $b_{i} \in G_{C}$ such that $u=b_{1} \cdot n \cdot b_{2}$. Define $u^{\prime} \in \mathcal{U}$ by $u^{\prime}={ }_{1}^{b_{1}^{-1}} u=n \cdot b_{2} \cdot b_{1}$. We have the following facts.

Remark 4.2.7.-

1. Since $b_{1} \in N_{G}\left(G_{y, r}\right) \cap N_{G}\left(G_{y, r^{+}}\right)$, from Remark 1.5 .6 we have $b_{1} \in \operatorname{stab}_{G}\left(F^{*}\right)$.
2. We have $T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=T\left(\left[u^{\prime} \cdot G_{F^{*}}^{+}\right]\right)$.
3. Since $y \in \bar{C}$, we have $u^{\prime} y=n b_{2} b_{1} y=n y \in \mathcal{A}$.
4. Finally, $\mathrm{d}_{u^{\prime}}(y)=\mathrm{d}_{u}(y)=m \leqslant \mathrm{~d}_{u}(x)$ for all $x \in \overline{F^{*}}$.

We observe that $[y, n y] \cap \overline{F^{*}}=\{y\}$. Indeed, suppose there exists a $z \in(y, n y)$ such that $z \in \overline{F^{*}}$. From Remark 4.2.7 (3) and Lemma 4.2.1 we have $\mathrm{d}_{u^{\prime}}(z)<\mathrm{d}_{u^{\prime}}(y)$. Consequently

$$
\mathrm{d}_{u}\left(b_{1} z\right)=\operatorname{dist}\left(b_{1}^{-1} u z, z\right)=\operatorname{dist}\left(u^{\prime} z, z\right)=\mathrm{d}_{u^{\prime}}(z)<\mathrm{d}_{u^{\prime}}(y)
$$

From Remark 4.2.7 (4) we have $m=\mathrm{d}_{u^{\prime}}(y)$ and from Remark 4.2.7 (1) we have $b_{1} z \in \overline{F^{*}}$. This contradicts the minimality of $m$.

Let $F_{1}^{*} \in \mathcal{F}(r)$ be the first generalized $r$-facet that $\left(y, u^{\prime} y\right]=(y, n y]$ passes through as we travel from $y$ to $n y$. Note that $F_{1}^{*} \cap \mathcal{A} \neq \emptyset$. From Remark 1.5 .6 we have $F^{*} \subset \overline{F_{1}^{*}}$ and so $F^{*} \cap \mathcal{A} \subset \overline{F_{1}^{*}} \cap \mathcal{A}$. From Remark 1.5.6 we have $G_{F^{*}}^{+} \subset G_{F_{1}^{*}}^{+}$. Let

$$
Q:=\left\{\psi \in \Psi(\mathcal{A})|\psi|_{F_{1}^{*} \cap \mathcal{A}}>r \text { and }\left.\psi\right|_{F^{*} \cap \mathcal{A}}=r\right\}
$$

Then

$$
G_{F_{1}^{*}}^{+}=G_{F^{*}}^{+} \cdot \prod_{\psi \in Q} U_{\psi}
$$

where the product over $Q$ may be taken in any order. Fix $\psi \in Q$. Since $\left(n^{-1} \psi\right)(y)=\psi(n y)>r$, we have ${ }^{n^{-1}} U_{\psi}=U_{n^{-1} \psi} \subset G_{F^{*}}^{+}$. We also have $U_{\psi}^{+} \subset G_{F^{*}}^{+}$. Therefore, from Remark 4.2.7 (2) we have

$$
\begin{aligned}
T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)= & T\left(\left[u^{\prime} \cdot G_{F^{*}}^{+}\right]\right) \\
= & \sum_{\bar{g} \in\left(\prod_{\psi \in Q}\right.} \sum_{\left.U_{\psi}\right) /\left(\prod_{\psi \in Q} U_{\psi}^{+}\right)} T\left(\left[g^{-1} \cdot n \cdot b_{2} \cdot b_{1} \cdot G_{F^{*}}^{+} \cdot g\right]\right) \\
= & \sum_{\bar{g} \in\left(\prod_{\psi \in Q} U_{\psi}\right) /\left(\prod_{\psi \in Q} U_{\psi}^{+}\right)} T\left(\left[u^{\prime} \cdot G_{F^{*}}^{+} \cdot g\right]\right) \\
= & \text { const } \cdot \sum \quad T\left(\left[u^{\prime} \cdot G_{F_{1}^{*}}^{+}\right]\right) .
\end{aligned}
$$

Note that for all $z \in F_{1}^{*} \cap\left(y, u^{\prime} y\right) \neq \emptyset$ we have from Lemma 4.2.1 that $\mathrm{d}_{u^{\prime}}(z)<\mathrm{d}_{u^{\prime}}(y)$. Therefore, from Remark 4.2.7 (4) we have

$$
\min _{x \in \overline{F_{1}^{*}}} \mathrm{~d}_{u^{\prime}}(x)<\mathrm{d}_{u^{\prime}}(y)=m=\min _{x \in \overline{F^{*}}} \mathrm{~d}_{u}(x)
$$

We now consider the case when the restriction of $\mathrm{d}_{u}$ to $\overline{F^{*}}$ does not obtain its minimum on $F^{*}$.
Lemma 4.2.8. - Suppose $F^{*} \in \mathcal{F}(r), u \in \mathcal{U}$, and $T \in J\left(G_{r^{+}}\right)$. Let $m=\min _{x \in \overline{F^{*}}} \mathrm{~d}_{u}(x)$. If $\mathrm{d}_{u}(x)>m$ for all $x \in F^{*}$, then there exist a finite set $\mathcal{V} \subset \mathcal{U}$ and a generalized $r$-facet $F_{1}^{*} \subset \overline{F^{*}} \backslash F^{*}$ such that

1. there exists $y \in F_{1}^{*}$ such that $\mathrm{d}_{v}(y)=\min _{x \in \overline{F_{1}^{*}}} \mathrm{~d}_{v}(x)=m$ for all $v \in \mathcal{V}$, and
2. $T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=\sum_{v \in \mathcal{V}} T\left(\left[v \cdot G_{F_{1}^{*}}^{+}\right]\right)$.

Proof. - Choose $y \in \overline{F^{*}} \backslash F^{*}$ such that $m=\mathrm{d}_{u}(y)$. There exists $F_{1}^{*} \in \mathcal{F}(r)$ such that $y \in F_{1}^{*}$. From Remark 1.5.6 we have $F_{1}^{*} \subset \overline{F^{*}}$. From Remark 1.5 .6 we have

$$
T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=\sum_{\bar{\alpha} \in G_{F^{*}}^{+} / G_{F_{1}^{*}}^{+}} T\left(\left[u \cdot \alpha \cdot G_{F_{1}^{*}}^{+}\right]\right)
$$

Note that for all $\alpha \in G_{F^{*}}^{+}$we have $\mathrm{d}_{u}(x)=\mathrm{d}_{u \alpha}(x)$ for all $x \in \overline{F^{*}}$. Note that the support of $T$ is contained in $G_{r^{+}} \subset \mathcal{U} \cdot G_{F_{1}^{*}}^{+}$. Therefore, if $u \cdot \alpha \cdot G_{F_{1}^{*}}^{+} \cap \mathcal{U}=\emptyset$, then $T\left(\left[u \cdot \alpha \cdot G_{F_{1}^{*}}^{+}\right]\right)=0$. Thus, there exists a finite set $\mathcal{V} \subset \mathcal{U}$ such that

$$
T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=\sum_{v \in \mathcal{V}} T\left(\left[v \cdot G_{F_{1}^{*}}^{+}\right]\right)
$$

and $\mathrm{d}_{v}(y)=\min _{x \in \overline{F_{1}^{*}}} \mathrm{~d}_{v}(x)=m$ for all $v \in \mathcal{V}$.
Corollary 4.2.9. - Suppose $F^{*} \in \mathcal{F}(r), u \in \mathcal{U}$, and $T \in J\left(G_{r^{+}}\right)$. Let $m=\min _{x \in \overline{F^{*}}} \mathrm{~d}_{u}(x)$. If $m>0$, then there exist a finite set $\mathcal{V} \subset \mathcal{U}$, constants $c(v) \in \mathbb{Q}$ indexed by $v \in \mathcal{V}$, and generalized $r$-facets $F_{v}^{*}$ indexed by $v \in \mathcal{V}$ such that

1. $m>\min _{x \in \overline{F_{v}^{*}}} \mathrm{~d}_{v}(x)$ for all $v \in \mathcal{V}$, and
2. $T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=\sum_{v \in \mathcal{V}} c(v) \cdot T\left(\left[v \cdot G_{F_{v}^{*}}^{+}\right]\right)$.

Proof. - If the restriction of $\mathrm{d}_{u}$ to $\overline{F^{*}}$ attains its minimum on $F^{*}$, then this follows from Lemma 4.2.6.

If the restriction of $\mathrm{d}_{u}$ to $\overline{F^{*}}$ does not attain its minimum on $F^{*}$, then from Lemma 4.2.8 there exist a finite subset $\mathcal{V} \subset \mathcal{U}, F_{1}^{*} \in \mathcal{F}(r)$, and $y \in F_{1}^{*}$ such that $F_{1}^{*} \subset \overline{F^{*}}$, $\mathrm{d}_{v}(y)=\min _{x \in \overline{F_{1}^{*}}} \mathrm{~d}_{v}(x)=m$ for all $v \in \mathcal{V}$, and $T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=\sum_{v \in \mathcal{V}} T\left(\left[v \cdot G_{F_{1}^{*}}^{+}\right]\right)$. The result follows from Lemma 4.2.6.

We can now begin our proof of Theorem 4.1.4.
Proof of Theorem 4.1.4 (1). - Fix $T \in J\left(G_{r}^{+}\right)$.
$" \Rightarrow ":$ Since $\mathcal{H}_{r^{+}}^{0} \subset \mathcal{H}_{r^{+}}$, if $\operatorname{res}_{\mathcal{H}_{r^{+}}} T=0$, then $\operatorname{res}_{\mathcal{H}_{r+}^{0}} T=0$.
" $\Leftarrow "$ : Suppose $f \in \mathcal{H}_{r^{+}}$. We want to show that $T(f)$ is completely determined by $\operatorname{res}_{\mathcal{H}_{r+}^{0}} T$.
For a fixed generalized $r$-facet $F^{*} \in \mathcal{F}(r)$ we have $G_{r^{+}} \subset \mathcal{U} \cdot G_{F^{*}}^{+}$. Since $T$ is linear and $T$ has support in $G_{r^{+}}$, without loss of generality we can assume that $f=\left[u \cdot G_{F^{*}}^{+}\right]$where $u \in \mathcal{U}$ and $F^{*} \in \mathcal{F}(r)$.

Let $m=\min _{x \in \overline{F^{*}}} \mathrm{~d}_{u}(x)$.
Suppose $m=0$. From Lemma 4.2 .8 we can assume that there is a $y \in F^{*}$ such that $\mathrm{d}_{u}(y)=0$. From [12, Lemma 4.2.1], we have $u \in G_{y}$. Thus $\left[u \cdot G_{F^{*}}^{+}\right] \in C\left(G_{y} / G_{y, r^{+}}\right) \subset \mathcal{H}_{r^{+}}^{0}$.

If $m>0$, then we wish to iteratively apply Corollary 4.2 .9 until we can write

$$
T\left(\left[u \cdot G_{F^{*}}^{+}\right]\right)=\sum_{v \in \mathcal{V}} c(v) \cdot T\left(\left[v \cdot G_{F_{v}^{*}}^{+}\right]\right)
$$

with $\mathcal{V}$ a finite subset of $\mathcal{U}, F_{v}^{*} \in \mathcal{F}(r)$ for all $v \in \mathcal{V}$, and $\min _{x \in \overline{F_{v}^{*}}} \mathrm{~d}_{v}(x)=0$ for all $v \in \mathcal{V}$; we would then be finished. Suppose that we cannot do this. Then from Corollary 4.2.9 there exists a sequence of triples $\left(u_{i}, F_{i}^{*}, y_{i}\right) \in \mathcal{U} \times \mathcal{F}(r) \times \mathcal{B}(G)$ such that $y_{i} \in F_{i}^{*}$ and

$$
\mathrm{d}_{u_{i}}\left(y_{i}\right)=\min _{x \in \bar{F}_{i}^{*}} \mathrm{~d}_{u_{i}}(x)>\min _{x \in \bar{F}_{(i+1)}^{*}} \mathrm{~d}_{u_{(i+1)}}(x)=\mathrm{d}_{u_{(i+1)}}\left(y_{(i+1)}\right)>0
$$

for all $i \in \mathbb{N}$.
Fix a maximal $k$-split torus $\mathbf{S}$ in $\mathbf{G}$. Let $S=\mathbf{S}(k)$ and let $\mathcal{A}$ be the apartment in $\mathcal{B}(G)$ corresponding to $S$. Fix a 0 -alcove (i.e., an affine chamber) $C$ in $\mathcal{A}$. Since $\mathrm{d}_{g_{h}}(g x)=\mathrm{d}_{h}(x)$ for all $h, g \in G$ and $x \in \mathcal{B}(G)$, we can assume that $y_{i} \in \bar{C}$ for all $i \in \mathbb{N}$. We also have the finite
disjoint union

$$
\bar{C}=\prod_{\substack{F^{* *} \in \mathcal{F}(r) \\ F^{*} \cap \bar{C} \neq \emptyset}}\left(F^{*} \cap \bar{C}\right) .
$$

Therefore, there exists a subsequence $\left(u_{i}, F^{*}, y_{i}\right)$ of our original sequence of triples with $F^{*} \in \mathcal{F}(r)$ such that $F^{*} \cap \bar{C} \neq \emptyset$. Note that $y_{i} \in F^{*} \cap \bar{C}$. From the Bruhat decomposition there exist $n_{i} \in N_{\mathbf{G}}(\mathbf{S})(k)$ and $b_{i}, b_{i}^{\prime} \in G_{C}$ so that $u_{i}=b_{i} \cdot n_{i} \cdot b_{i}^{\prime}$ for all $i \in \mathbb{N}$. We have

$$
\mathrm{d}_{u_{i}}(x)=\mathrm{d}_{n_{i}}(x)
$$

for all $x \in \overline{F^{*}} \cap \bar{C}$. So

$$
\min _{x \in \overline{F^{*} \cap \bar{C}}} \mathrm{~d}_{n_{i}}(x)=\mathrm{d}_{u_{i}}\left(y_{i}\right)>\mathrm{d}_{u_{(i+1)}}\left(y_{(i+1)}\right)=\min _{x \in \overline{F^{*} \cap \bar{C}}} \mathrm{~d}_{n_{(i+1)}}(x)>0
$$

for all $i \in \mathbb{N}$. Since

$$
\left\{\min _{x \in \overline{F^{*} \cap \bar{C}}} \mathrm{~d}_{n}(x): n \in N_{\mathbf{G}}(\mathbf{S})(k)\right\}
$$

is a discrete subset of $\mathbb{R}$, this is a contradiction.

### 4.3. Some hypotheses on the map $\exp _{t}$

The hypotheses of this subsection place some restrictions on $k$ and $\mathbf{G}$; the hypotheses are both valid if $p$ is larger than some constant which can be determined by examining the absolute root datum of $\mathbf{G}$.

Recall that for $x \in \mathcal{B}(G)$ and $t \in \mathbb{R}$ we call a coset of $\mathfrak{g}_{x, t} / \mathfrak{g}_{x, t^{+}}$(respectively, a coset of $G_{x, t} / G_{x, t^{+}}$) degenerate if the coset has nontrivial intersection with $\mathcal{N}$ (respectively, with $\mathcal{U}$ ).

DEFINITION 4.3.1. - For $F \in \mathcal{F}(0)$ we let $\mathcal{N}_{F}$ (respectively, $\mathcal{U}_{F}$ ) denote the set of degenerate elements in $\mathfrak{g}_{F} / \mathfrak{g}_{F}^{+}$(respectively, in $G_{F} / G_{F}^{+}$).

For more information about the following hypothesis, see, for example, $[6, \S 5.5]$.
HYpothesis 4.3.2. - Suppose that Hypothesis 2.2 .5 is valid.

1. For all $x \in \mathcal{B}(G)$, for all $t \in \mathbb{R}_{\geqslant 0}$, and for all $X \in \mathcal{N}$ we have

$$
X \in \mathfrak{g}_{x, t} \quad \text { if and only if } \quad \exp _{t}(X) \in G_{x, t}
$$

2. The map $\exp _{t}: \mathcal{N} \rightarrow \mathcal{U}$ is bijective.
3. For all $F \in \mathcal{F}(0)$ the map $\exp _{t}$ induces a $G_{F}(\mathfrak{f})$-equivariant bijection $\overline{\exp }_{t}: \mathcal{N}_{F} \rightarrow \mathcal{U}_{F}$.

See [ $1, \S 1]$ for more information about the following hypothesis.
HYPOTHESIS 4.3.3. - For all $x \in \mathcal{B}(G)$ and for all $r>0$ there exists a map $\phi_{x}: \mathfrak{g}_{x, r} \rightarrow G_{x, r}$ such that for all $t \geqslant 0$, for all $u \in \mathcal{U} \cap\left(G_{x, t} \backslash G_{x, t^{+}}\right)$, and for all $X \in \mathfrak{g}_{x, r}$ we have

$$
\phi_{x}\left({ }^{u} X-X\right)=u \cdot \phi_{x}(X) \cdot u^{-1} \cdot\left(\phi_{x}(X)\right)^{-1} \text { modulo } G_{x,(t+r)^{+}}
$$

### 4.4. Parameterization of unipotent orbits

If $r \in \mathbb{R} \geqslant 0$, then from Hypothesis 4.3.2 and Remark 2.5.11 we have a bijective correspondence between $I_{r}^{d} / \sim$ and $\mathcal{U} / G$ given by the map sending $\left(F^{*}, e\right) \in I_{r}^{d}$ to $\exp _{t}\left(\mathcal{O}\left(F^{*}, e\right)\right)$. We need more information about this parameterization; in particular, we will require the analogue of Remark 2.5.4.

Suppose that all of the hypotheses of Sections 2.2 and 4.3 are valid. Let $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ be a Lie algebra homomorphism defined over $k$. For $r \in \mathbb{R}, i \in \mathbb{Z}$, and $x \in \mathcal{B}\left(C_{\mathbf{G}}\left(\lambda_{\phi}\right), k\right)$, we define

$$
\mathfrak{g}_{x, r}^{\phi}(i):=\left\{\left.X \in \mathfrak{g}_{x, r}\right|^{\lambda_{\phi}(t)} X=t^{i} X \text { for all } t \in k^{\times}\right\}
$$

and

$$
\mathfrak{g}_{x, r}^{\phi}(\geqslant i):=\sum_{j \geqslant i} \mathfrak{g}_{x, r}^{\phi}(j) .
$$

Let $V_{x, r}^{\phi}(i)$ (respectively, $V_{x, r}^{\phi}(\geqslant i)$ ) denote the image of $\mathfrak{g}_{x, r}^{\phi}(i)$ (respectively, the image of $\mathfrak{g}_{x, r}^{\phi}(\geqslant i)$ ) in $V_{x, r}$.

Lemma 4.4.1. - Fix $r \geqslant 0$. Suppose that all of the hypotheses of Sections 2.2 and 4.3 are valid. Suppose $\left(F^{*}, e\right) \in I_{r}^{d}$. Choose $X \in \mathcal{O}\left(F^{*}, e\right) \cap \mathfrak{g}_{F^{*}}$ whose image in $V_{F^{*}}$ is $e$. There exists a Lie algebra homomorphism $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ defined over $k$ such that $F^{*} \subset \mathcal{B}(\phi, r)$ and $X=\phi\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$. Moreover, if $u=\exp _{t}(X)$, then we have

1. If $r>0$ (respectively, If $r=0$ ), then the image of $u$ in $G_{F^{*}} / G_{F^{*}}^{+}$is e (respectively, is $\left.\overline{\exp }_{t}(e)\right)$.
2. For all $i \geqslant 1$, for all $s \in \mathbb{R}$, and for all $x \in F^{*}$ the map

$$
(\operatorname{Ad}(u)-1): \mathfrak{g}_{x, s} \rightarrow \mathfrak{g}_{x, s+r}
$$

induces a surjective map from $V_{x, s}^{\phi}(i-2)$ to $V_{x, s+r}^{\phi}(i)$ if $r>0$ (respectively, from $V_{x, s}^{\phi}(\geqslant(i-2))$ to $V_{x, s}^{\phi}(\geqslant i)$ if $\left.r=0\right)$.
3. If $\mathcal{O}$ is a unipotent orbit such that $\mathcal{O} \cap u G_{F^{*}}^{+} \neq \emptyset$, then ${ }^{G} u \subset \overline{\mathcal{O}}$.
4. For all $x \in F^{*}$, we have ${ }^{G} u \cap u G_{F^{*}}^{+}={ }^{G_{x}^{+}} u$.

Remark 4.4.2. - Note that $u \in \exp _{t}\left(\mathcal{O}\left(F^{*}, e\right)\right) \cap G_{F^{*}}$.
Proof. - From [11, Lemma 5.3.3] there exists a Lie algebra homomorphism $\phi: \mathfrak{s l}_{2} \rightarrow \mathfrak{g}$ defined over $k$ such that $X=\phi\left(\begin{array}{cc}0 & 1 \\ 0 & 0\end{array}\right)$ and $F^{*} \subset \mathcal{B}(\phi, r)$.

First suppose that $r=0$. From Hypotheses 4.3 .2 we have that (1) holds. Claim (2) is a consequence of the fact (from Hypothesis 2.2.2 and $\mathfrak{s l}_{2}(\mathfrak{f})$-representation theory) that for all $j \geqslant 1$ the map $\operatorname{ad}(X): \mathfrak{g}_{x, s} \rightarrow \mathfrak{g}_{x, s}$ induces a surjective map from $V_{x, s}^{\phi}(j-2)$ to $V_{x, s}^{\phi}(j)$. We now consider (3). Suppose $u^{\prime} \in \mathcal{O} \cap u G_{F^{*}}^{+}$. From Hypothesis 4.3.2 there exists an $X^{\prime} \in \mathfrak{g}$ such that $\exp _{t}\left(X^{\prime}\right)=u^{\prime}$ and $X^{\prime} \in X+\mathfrak{g}_{F^{*}}^{+}$. From Remark 2.5.4 we have ${ }^{G} X$ is contained in the closure of ${ }^{G} X^{\prime}$. It follows from Hypothesis 2.2.5 that ${ }^{G} u \subset \overline{\mathcal{O}}$. Finally, if ${ }^{g} u \in u G_{F^{*}}^{+}$, then the reasoning above shows that ${ }^{g} X \in X+\mathfrak{g}_{F^{*}}^{+}$. From Remark 2.5.4 there exists an $h \in G_{x}^{+}$such that ${ }^{h} X={ }^{g} X$ and so (4) follows.

Now suppose that $r>0$. Since $r>0$, it follows from Hypothesis 4.3.2 and Hypothesis 2.2.5 that $\operatorname{ad}(X)$ and $(\operatorname{Ad}(u)-1)$ induce the same map from $V_{x, s}$ to $V_{x, s+r}$ for all $s \in \mathbb{R}$. It follows that the image of $u$ in $G_{F^{*}} / G_{F^{*}}^{+}$is $e$, and so (1) and (2) are true. We now show that (3) is valid. Choose $u^{\prime} \in \mathcal{O} \cap u G_{F^{*}}^{+}$. From Hypothesis 4.3.2 there exists $X^{\prime} \in \mathfrak{g}_{F^{*}}$ such that $\exp _{t}\left(X^{\prime}\right)=u^{\prime}$. But $X^{\prime}$ and $u^{\prime}$ have the same image in $V_{F^{*}}$. Thus $X^{\prime} \in X+\mathfrak{g}_{F^{*}}^{+}$. The result now follows from Remark 2.5.4 and Hypothesis 2.2.5. Statement (4) is proved similarly.

### 4.5. A (sketch of the) proof of Theorem 4.1.4 (2) and (3)

The remainder of the proof of Theorem 4.1.4 is nearly identical to the proof of Theorem 2.1.5 given in Section 2. For this reason, we only sketch the proof.

Suppose $r \in \mathbb{R}_{\geqslant 0}$. Recall from Theorem 4.1.4 (1) (which has been proved) that for $T \in J\left(G_{r^{+}}\right)$, we have

$$
\operatorname{res}_{\mathcal{H}_{r^{+}}} T=0 \quad \text { if and only if } \quad \operatorname{res}_{\mathcal{H}_{r^{+}}^{0}} T=0
$$

Thus, the proof of Theorem 4.1.4 (2) and (3) has been reduced to a question about functions supported on $G_{0}$. We now show how to complete the proof of this theorem. We begin with an analogue of Lemma 2.3.1, the descent and recovery lemma.

Lemma 4.5.1. - Suppose that all of the hypotheses of Sections 2.2 and 4.3 are valid. Suppose $x \in \mathcal{B}(G)$ and $0 \leqslant t<r$. Let $\mathbf{S}$ be a maximal $k$-split torus of $\mathbf{G}$ such that $x \in \mathcal{A}(\mathbf{S}, k)$. If $v \in\left(\mathcal{U} G_{x, r^{+}}\right) \cap\left(G_{x, t} \backslash G_{x, t^{+}}\right)$, then there exist $u \in \mathcal{G}_{x}\left(v G_{x, r^{+}}\right)$and $\lambda \in \mathbf{X}_{*}^{k}(\mathbf{S})$ such that for all sufficiently small $\varepsilon>0$ we have

1. $u G_{x, r^{+}} \subset G_{x+\varepsilon \cdot \lambda, t^{+}}$, and
2. $u G_{x+\varepsilon \cdot \lambda, r^{+}} \subset^{G_{x,(r-t)}}\left(u G_{x, r^{+}}\right)$.

We offer some comments on the proof of Lemma 4.5.1. The essential tool in the proof of Lemma 2.3.1 is the use of the theory of $\mathfrak{s l}_{2}(\mathfrak{f})$-representations. Lemma 4.4.1 (2) shows us how to replace this part of the proof. Also, the role of Hypothesis 2.2 .8 will be played by Hypothesis 4.3.3.

The proof of Theorem 4.1.4 (2) is now a straightforward translation of the proof in Section 2.4. Moreover, thanks to parts (3) and (4) of Lemma 4.4.1, the proof of Theorem 4.1.4 (3) can be extracted from Section 2.6.

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