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Numerical analysis

Which spline spaces for design?

Quelles splines utiliser pour le design ?

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ABSTRACT

We recently determined the largest class of spaces of sufficient regularity that are suitable for design. How can we connect different such spaces, possibly with the help of connection matrices, to produce the largest class of splines usable for design? We present the answer to this question, along with some of the major difficulties encountered to establish it. We would like to stress that the results we announce are far from being a straightforward generalisation of previous work on piecewise Chebyshevian splines.

RÉSUMÉ

Nous avons récemment déterminé la plus grande classe d'espaces (de fonctions suffisamment régulières) bons pour le design. Comment connecter de tels espaces pour produire la plus grande classe de « bons » espaces de splines ? Nous donnons la réponse à cette question en pointant certaines des difficultés majeures rencontrées pour l'établir.

Version française abrégée

Sur un intervalle $[a,b]$, $a<b$, donné, nous désignerons par $\mathbb{1}$ la fonction constante égale à 1. Sur cet intervalle, choisissons

- un système $(w_0,\dots,w_{n-1})$ de *fonctions-poids* : *i.e.*, pour $0\leqslant i\leqslant n-1$, w_i est strictement positive et C^{n-1-i} sur $[a,b]$;
- un *espace de Chebyshev* (C-espace) $\mathbb{U}\subset C^0([a,b])$ de dimension 2 contenant les constantes : *i.e.*, $\mathbb{U}$ est engendré par $\mathbb{1}, U$ où la function U set strictement croissante sur $[a,b]$.

Au système $(w_0,\dots,w_{n-1})$ on associe classiquement une suite de dérivées généralisées définies par

$$L_0F:=\frac{F}{w_0},\quad L_iF:=\frac{1}{w_i}DL_{i-1}F,\quad 1\leqslant i\leqslant n-1. \tag{1}$$

Nous noterons $\mathbb{E}=QEC(w_0,\dots,w_{n-1};\mathbb{U})$ l'espace de dimension $(n+1)$ formé par les fonctions $F\in C^{n-1}([a,b])$ telles que $L_{n-1}F\in\mathbb{U}$. La notation est justifiée par le fait que $\mathbb{E}$ est un *quasi-espace de Chebyshev généralisé (en abŕegé, QEC) sur* $[a,b]$,

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au sens où tout problème d'interpolation d'Hermite en $(n+1)$ données dans $[a,b]$, autre qu'un problème de Taylor, admet une solution unique dans $\mathbb{E}$. Réciproquement, parce que nous travaillons sur un intervalle fermé borné, tout espace $\mathbb{E} \subset C^{n-1}([a,b])$, qui est un QEC de dimension $(n+1)$ peut s'écrire sous la forme $\mathbb{E} = QEC(w_0, \dots, w_{n-1}; \mathbb{U})$, et ceci d'une infinité de façons possibles dès lors que $n \geqslant 2$ (voir [12]). Pour $n \geqslant 2$ les espaces de la forme $\mathbb{E} = QEC(\mathbb{1}, w_1, \dots, w_{n-1}; \mathbb{U})$ (*i.e.*, les espaces $\mathbb{E} \subset C^{n-1}([a,b])$ qui contiennent les constantes et dont l'espace dérivé $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$ est un QEC de dimension n sur $[a,b]$) sont les espaces de fonctions régulières au sens classique de stricte dimension $(n+1)$ sur $[a,b]$ (*i.e.*, qui gardent la dimension $(n+1)$ sur tout sous-intervalle de $[a,b]$) les plus généraux que nous appellerons « *bons pour le design* ». Le critère adopté est l'existence de *floraisons*, fonctions définies sur $[a,b]^n$ à partir d'intersections de variétés osculatrices, dont les propriétés garantissent la possibilité de développer les algorithmes classiques de design et de construire des courbes sous contraintes de forme (voir [10,12]).

Fixons maintenant des nœuds intérieurs $t_0 := a < t_1 < \dots < t_q < b =: t_{q+1}$ et leur multiplicité $1 \leqslant m_k \leqslant n$, $1 \leqslant k \leqslant q$. On cherche à déterminer les espaces de splines les plus généraux qui soient « *bons pour le design* ». Le critère est à nouveau l'existence de floraisons, dont le domaine de définition naturel pour les splines est désormais une partie de $[a,b]^n$ située « autour » de la diagonale, en un sens fixé par les nœuds intérieurs et leur multiplicité. Il est bien sûr nécessaire que les espaces-sections soient eux-mêmes bons pour le design. C'est pourquoi, pour chaque $k = 0, \dots, q$, on choisit un espace $\mathbb{E}_k \subset C^{n-1}([t_k, t_{k+1}])$ contenant les constantes (on notera $\mathbb{1}_k$ la restriction de $\mathbb{1}$ à $[t_k, t_{k+1}]$) pour lequel $D\mathbb{E}_k$ est un espace QEC de dimension n sur $[t_k, t_{k+1}]$. De plus, pour chaque $k = 1, \dots, q$, on choisit une *matrice de connexion* M_k, triangulaire inférieure d'ordre $(n - m_k)$, à diagonale strictement positive. Ces données permettent de définir l'espace de splines $\mathbb{S}$ comme l'ensemble des fonctions S continues sur $[a,b]$ dont la restriction à chaque sous-intervalle $[t_k, t_{k+1}]$ est dans $\mathbb{E}_k$ et qui satisfont les conditions de raccord :

$$\left(S'(t_k^+), \dots, S^{(n-m_k)}(t_k^+)\right)^T = M_k \cdot \left(S'(t_k^-), \dots, S^{(n-m_k)}(t_k^-)\right)^T, \quad 1 \leqslant k \leqslant q. \tag{2}$$

Cet espace est de dimension $(n+1+m)$, où $m := \sum_{k=1}^q m_k$. Pour chaque $k = 0, \dots, q$, sur $[t_k, t_{k+1}]$ on peut choisir un système $(w_1^k, \dots, w_{n-1}^k)$ de fonctions-poids et un espace de Chebyshev $\mathbb{U}_k$ de dimension 2 contenant les constantes, de façon à ce que $\mathbb{E}_k = QEC(\mathbb{1}_k, w_1^k, \dots, w_{n-1}^k; \mathbb{U}_k)$. Si l'on désigne par $L_0^k = \mathrm{Id}$, $L_1^k, \dots, L_{n-1}^k$ les dérivées généralisées associées à $(\mathbb{1}_k, w_1^k, \dots, w_{n-1}^k)$, les conditions de raccord (6) s'expriment de façon équivalente sous la forme

$$\left(L_1^k(t_k^+), \dots, L_{n-m_k}^k(t_k^+)\right)^T = \widehat{M}_k \cdot \left(L_1^{k-1}(t_k^-), \dots, L_{n-m_k}^{k-1}(t_k^-)\right)^T, \quad 1 \leqslant k \leqslant q, \tag{3}$$

la matrice $\widehat{M}_k$ étant elle aussi triangulaire inférieure à diagonale strictement positive. Le résultat annoncé s'énonce comme suit.

Théorème 0.1. *L'espace de splines décrit ci-dessus est bon pour le design si et seulement si on peut trouver, pour chaque $k = 0, \dots, q$, un système $(w_1^k, \dots, w_{n-1}^k)$ de fonctions-poids sur $[t_k, t_{k+1}]$ et un espace de Chebyshev $\mathbb{U}_k$ sur $[t_k, t_{k+1}]$ (de dimension 2 et contenant les constantes), tel que $\mathbb{E}_k = QEC(\mathbb{1}_k, w_1^k, \dots, w_{n-1}^k; \mathbb{U}_k)$ de façon à ce que chacune des matrices $\widehat{M}_k$, $1 \leqslant k \leqslant q$, intervenant dans (7) soit la matrice identité d'ordre $(n - m_k)$.*

Un résultat similaire existe déjà pour le cas de splines lorsque les espaces-sections sont des espaces de Chebyshev généralisés (EC). Le passage de EC à QEC est loin d'être simple, les espaces QEC présentant de grosses difficultés, notamment parce que l'on ne peut pas s'appuyer sur les dérivées d'ordre n.

1. Introduction

Determining the largest class of spline spaces that can be used for design is obviously a crucial question in CAGD, but it is also important in other domains, *e.g.*, it is strongly connected with all possibilities of interpolating with splines. It can also be viewed as determining the largest class of spline spaces that can be used in a very lively recent research area, namely Isogeometric Analysis, where design tools have been successfully introduced in many recent papers.

To justify the framework presented in Section 3, we draw the reader's attention to two points. First, a "good" spline space is expected to permit the development of all classical design algorithms, *e.g.*, knot insertion. This makes it necessary for the section-spaces themselves to be "good for design". This explains why Section 2 is devoted to a brief presentation of the major results concerning Quasi-Extended Chebyshev spaces, see [10] and [12]. Second, the presence of connection matrices in our spline spaces is justified by the fact that requiring that left and right derivatives coincide at a given knot up to some order has no special meaning for parametric spline curves. Moreover, the entries of such matrices behave as efficient shape parameters, and they can be very useful for many purposes, *e.g.*, for suppressing oscillations in interpolating splines.

2. Quasi-Extended Chebyshev spaces

Given a non-trivial real interval I and an integer $n \geqslant 1$, let $\mathbb{E} \subset C^{n-1}(I)$ be an $(n+1)$-dimensional space. It is said to be of *strict dimension* $(n+1)$ if it remains of dimension $(n+1)$ by restriction to any non-trivial subinterval of I. We say that

$\mathbb{E}$ is a *Quasi-Extended Chebyshev space* (for short, QEC-space) *on* I when any Hermite interpolation problem in $(n+1)$ data *that is not a Taylor interpolation problem* has a unique solution in $\mathbb{E}$ [10]. By comparison, $\mathbb{E}$ is an $(n+1)$-dimensional *Extended Chebyshev space* (EC-space) on I when any Hermite interpolation problem in $(n+1)$ data has a unique solution in $\mathbb{E}$. This includes Taylor interpolation problems and therefore being an $(n+1)$-dimensional EC-space on I requires $\mathbb{E}$ to be contained in $C^n(I)$. The interest of QEC-spaces lies in the following result:

Theorem 2.1. *Given $n \geqslant 2$, let $\mathbb{E} \subset C^{n-1}(I)$ be of strict dimension $(n+1)$ and let it contain constants. The following two properties are then equivalent:*

(i) *$\mathbb{E}$ is "good for design", in the sense that $\mathbb{E}$ possesses blossoms;*
(ii) *the space $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$ is an n-dimensional QEC-space on I.*

The reader interested in the exact definition of blossoms can refer to [10]. Let us just recall that these powerful tools are defined in a geometrical way via intersection of osculating flats. Our terminology "*good for design*" is highly justified by the fact that, when (i) is satisfied, each function $F \in \mathbb{E}$ "*blossoms*" into a function $f : I^n \to \mathbb{R}$, called *the blossom of* F which satisfies the following properties:

$(\mathrm{B})_1$ symmetry: f is symmetric on I^n;
$(\mathrm{B})_2$ diagonal property: for all $x \in I$, $f(x^{[n]}) = F(x)$, $x^{[n]}$ standing for x repeated n times;
$(\mathrm{B})_3$ pseudoaffinity property: given any $y_1, \dots, y_{n-1}, a, b \in I$, with $a < b$, there exists a strictly increasing function $\beta(y_1, \dots, y_{n-1}; a, b; .) : I \to \mathbb{R}$ (independent of F) such that:

$$f(y_1, \dots, y_{n-1}, x) = \big[1 - \beta(y_1, \dots, y_{n-1}; a, b; x)\big] f(y_1, \dots, y_{n-1}, a) \\ + \beta(y_1, \dots, y_{n-1}; a, b; x) f(y_1, \dots, y_{n-1}, b), \quad x \in I. \tag{4}$$

The latter three properties are crucial because they permit the development of all the classical geometric design algorithms in $\mathbb{E}$ (*e.g.*, de Casteljau algorithms), and they guarantee shape preserving properties [8].

Similar results (Theorem 2.1 and $(\mathrm{B})_i$, $i = 1, 2, 3$) were first obtained with EC instead of QEC [9]. Extending them to QEC-spaces was not at all straightforward: it required the elaboration of a whole sophisticated theory of QEC-spaces, for which we refer the reader to [10] by comparison with [9].

Example. Let us recall a simple procedure to build QEC-spaces on I [10]. Take

– any *system of weight functions on* I, *i.e.*, any sequence $(w_0, \dots, w_{n-1})$ such that, for $0 \leqslant i \leqslant n-1$, w_i is positive and C^{n-1-i} on I;
– any two-dimensional space $\mathbb{U} \subset C^0(I)$ supposed to be a *Chebyshev space on* I (*i.e.*, any non-zero element in $\mathbb{U}$ vanishes at most once in I, not counting possible multiplicities) containing constants, that is, any space spanned by $\mathbb{1}, U$ where U is strictly monotone on I.

As is classical, one can define differential operators $L_0, \dots, L_{n-1}$ on $C^{n-1}(I)$ as follows:

$$L_0 F := \frac{F}{w_0}, \quad L_i F := \frac{1}{w_i} D L_{i-1} F, \quad 1 \leqslant i \leqslant n-1. \tag{5}$$

Then the set of all functions $F \in C^{n-1}(I)$ such that $L_{n-1} F \in \mathbb{U}$ is an $(n+1)$-dimensional QEC-space on I, denoted $QEC(w_0, \dots, w_{n-1}; \mathbb{U})$. For instance, given any positive numbers $p, q > n-1$, the linear space $\mathbb{E}_n^{p,q}$ spanned by the functions $1, x, \dots, x^{n-2}, (1-x)^p, x^q$ is a QEC-space on $[0, 1]$, see [2,3,6,5].

We recently established the following converse property of which the proof strongly relies on Theorem 2.1 and on the pseudoaffinity property of blossoms [12].

Theorem 2.2. *Let $\mathbb{E}$ be an $(n+1)$-dimensional QEC-space on a closed bounded interval I. Then there exists infinitely many ways to write $\mathbb{E}$ as $\mathbb{E} = QEC(w_0, \dots, w_{n-1}; \mathbb{U})$.*

3. The result

Given $I = [a, b]$, $a < b$, the ingredients to build our *piecewise quasi-Chebyshevian spline space* $\mathbb{S}$ are:

– a sequence of interior knots: $a < t_1 < \cdots < t_q < b$ and an associated sequence of multiplicities m_k, with $1 \leqslant m_k \leqslant n$ for $1 \leqslant k \leqslant q$;
– a sequence of section spaces $\mathbb{E}_k$, $0 \leqslant k \leqslant q$: for each k, $\mathbb{E}_k$ contains the constant function $\mathbb{1}_k$ and $D\mathbb{E}_k$ is an n-dimensional QEC-space on $[t_k, t_{k+1}]$, where $t_0 := a$, $t_{q+1} := b$;

– a sequence of connection matrices M_k, $1 \leqslant k \leqslant q$: for each k, M_k is a lower triangular square matrix of order $(n - m_k)$ with positive diagonal entries.

The *spline space* $\mathbb{S}$ is then defined as the space of all continuous functions $S : I \to \mathbb{R}$ such that

1) for $k = 0, \dots, q$, the restriction of S to $[t_k, t_{k+1}]$ belongs to $\mathbb{E}_k$;
2) for $k = 1, \dots, q$, the following connection condition is fulfilled:

$$\left(S'(t_k^+), \dots, S^{(n-m_k)}(t_k^+)\right)^T = M_k \cdot \left(S'(t_k^-), \dots, S^{(n-m_k)}(t_k^-)\right)^T. \tag{6}$$

Observe that, if we want the connection at an interior knot to be expressed in terms of left and right derivatives at this knot, the assumption $m_k \geqslant 1$ is made necessary by the fact that we are dealing with QEC-spaces. For EC-spaces, we could allow multiplicities equal to 0, see [14]. How to define the connections with possibly zero multiplicities is one of the major difficulties encountered when the sections are QEC-spaces. On purpose, we avoided it in order to make our statement in Theorem 3.1 simpler, and modelled on the result established in [14]. Nevertheless, the proof of Theorem 3.1 strongly uses zero multiplicities.

According to Theorems 2.1 and 2.2, for each $k = 0, \dots, q$, one can choose a system $(w_1^k, \dots, w_{n-1}^k)$ of weight functions on $[t_k, t_{k+1}]$ and a two-dimensional Chebyshev space $\mathbb{U}_k$ on $[t_k, t_{k+1}]$ containing the constant function $\mathbb{1}_k$ such that $\mathbb{E}_k = QEC(\mathbb{1}_k, w_1^k, \dots, w_{n-1}^k; \mathbb{U}_k)$. For each such choice, when denoting by $L_0^k = \mathrm{Id}$, $L_1^k, \dots, L_{n-1}^k$ the differential operators on $C^{n-1}([t_k, t_{k+1}])$ associated with $(\mathbb{1}_k, w_1^k, \dots, w_{n-1}^k)$, the connection conditions (6) can be replaced by

$$\left(L_1^k(t_k^+), \dots, L_{n-m_k}^k(t_k^+)\right)^T = \widehat{M}_k \cdot \left(L_1^{k-1}(t_k^-), \dots, L_{n-m_k}^{k-1}(t_k^-)\right)^T, \quad 1 \leqslant k \leqslant q, \tag{7}$$

where $\widehat{M}_k$ is in turn a lower triangular matrix of order $(n - m_k)$ with positive diagonal entries. The result we are announcing is then as follows.

Theorem 3.1. *The following two statements are equivalent:*

(i) *the piecewise quasi-Chebyshevian spline space $\mathbb{S}$ is good for design (i.e., it possesses blossoms);*
(ii) *for each $k = 0, \dots, q$, there exists a system $(w_1^k, \dots, w_{n-1}^k)$ of weight functions on $[t_k, t_{k+1}]$ and a two-dimensional C-space $\mathbb{U}_k$ on $[t_k, t_{k+1}]$ containing the constant function $\mathbb{1}_k$ such that $\mathbb{E}_k = QEC(\mathbb{1}_k, w_1^k, \dots, w_{n-1}^k; \mathbb{U}_k)$ and such that each matrix $\widehat{M}_k$ in (7) is the identity matrix of order $(n - m_k)$.*

Again, we omit the precise definition of blossoms for splines, simply mentioning that, when (i) holds, each spline $S \in \mathbb{S}$ "*blossoms*" into a function s (the blossom of S), of which the natural domain of definition is a restricted set $\mathbb{A}_n(\mathbb{K})$ of n-tuples said to be *admissible* w.r.t. the knot-vector $\mathbb{K} := (\xi_{-n}, \dots, \xi_{m+n+1}) := ({t_0}^{[n+1]}, {t_1}^{[m_1]}, \dots, {t_q}^{[m_q]}, {t_{q+1}}^{[n+1]})$, where $x^{[k]}$ stands for x repeated k times, and with $m := \sum_{k=1}^q m_k$.

4. Sketch of the proof – illustration

Theorem 3.1 extends to piecewise quasi-Chebyshevian splines a result obtained for piecewise Chebyshevian splines in [14]. We already mentioned that replacing one single EC-space by a QEC-space was not an easy task. We are now interested in mixing different QEC-spaces to obtain the largest class of splines usable for design, under the possible presence of connection matrices. Clearly, this cannot be a trivial adaptation of existing results. Though the general guideline follows the ideas developed in [14], using QEC-section spaces raises two main problems recalled below.

• When (i) holds, by nature blossoms are symmetric on $\mathbb{A}_n(\mathbb{K})$ and they satisfy the diagonal property. A major difficulty consists in proving that they also satisfy the crucial pseudoaffinity property $(\mathrm{B})_3$, of course limited to $\mathbb{A}_n(\mathbb{K})$. The tricky proof of $(\mathrm{B})_3$ in QEC-spaces involved difficult generalised convexity arguments, see [10]. The spline framework will require that we go deeper into the arguments in question, comparing generalised convexity on the left and on the right of each interior knot. We would like to stress that pseudoaffinity is THE property justifying that a spline space is considered good for design when it possesses blossoms [11]. Indeed, it permits all the geometric design algorithms and leads to the important intermediate result stated below.

Theorem 4.1. *If (i) of Theorem 3.1 holds, then $\mathbb{S}$ possesses a quasi-B-spline basis that is its optimal normalised totally positive basis. Conversely, if $\mathbb{S}$ and any spline space derived from $\mathbb{S}$ by knot insertion possess B-spline bases, then (i) holds.*

The presence of *quasi-B-spline basis* is crucial for local control of the curves. Let us recall that it means a normalised sequence $N_\ell \in \mathbb{S}$, $-n \leqslant \ell \leqslant m$, (*i.e.*, $\sum_{\ell=-n}^m N_\ell = \mathbb{1}$), each N_ℓ being positive on the interior of its support $[\xi_\ell, \xi_{\ell+n+1}]$, and satisfying some additional condition of zeroes at the endpoints of its support. The term "quasi" refers to the fact that the count of zeroes takes into account that we are dealing with QEC-spaces, not with EC-spaces. The total positivity of such bases ensures shape-preserving control (see [4]), and optimality should simply be understood as "the best possible for design" [1]. For the links with blossoms, see [8].

• For simplicity, we have assumed all interior multiplicities to satisfy $m_k \geqslant 1$. Still, the proof makes it essential to allow $m_k = 0$ to check if the generalised convexity properties on the left and on the right of an interior knot are compatible. Below we give an example illustrating what kind of nth connection conditions should be used when dealing with QEC-spaces.

Example. Let us give an elementary example. Given any real numbers $p_k, q_k > n-1$, $0 \leqslant k \leqslant q$, here we consider a spline space $\mathbb{S}$ with kth section space $\mathbb{E}_n^{p_k,q_k}$ after the convenient affine change of variable.

1- To remain inside the framework described at the beginning of Section 3 ($m_k \geqslant 1$ for all k), consider the case where all interior knots are simple. Given any positive numbers a_k, $1 \leqslant k \leqslant q$, assume the splines to be C^{n-2} on I and to meet the additional requirement:

$$S^{(n-1)}(t_k^+) = a_k S^{(n-1)}(t_k^-), \quad 1 \leqslant k \leqslant q. \tag{8}$$

Using Theorems 3.1 and 2.2, one can easily prove that the spline space $\mathbb{S}$ is then good for design. In particular it therefore possesses a quasi-B-spline basis which is its optimal normalised totally positive basis. Note that the case where all a_k's are equal to 1 and where all p_k, q_k's are integers corresponds to the so-called *variable degree polynomial splines* [2,6], which motivated the study of general QEC spaces.

2- To illustrate multiplicities equal to 0, it suffices to consider the case where $m_k = 0$ for all $k = 1, \ldots, q$. Assume again that the splines are C^{n-2} on I. Each interior knot t_k is then allocated two additional positive numbers b_k, c_k. Along with (8), the splines in $\mathbb{S}$ are now supposed to satisfy:

$$S^{(n-1)}(t_{k+1}^-) = -b_k S^{(n-1)}(t_{k-1}^+) + c_k S^{(n-1)}(t_k^-) \text{ for } k = 1, \ldots, q. \tag{9}$$

The positive numbers b_k, c_k should be chosen so that, when S ranges over $\mathbb{S}$, the continuous function on $[a, b]$ defined by $S^{(n-1)}(x)/\prod_{i=1}^k a_i$ on $[t_k^+, t_{k+1}^-]$ belongs to a C-space on $[a, b]$, according to a procedure explained in [13]. This yields a spline space good for design, in the sense that it possesses blossoms satisfying the pseudoaffinity property on the set $\mathbb{A}_n(\mathbb{K})$, that is, here, $[a, b]^n$.

5. Conclusion

Allowing the presence of connection matrices in piecewise quasi-Chebyshevian splines can be very useful, as testified by the example of geometrically continuous variable degree polynomial splines with four-dimensional sections, which was investigated in [7]. Theorem 3.1 presents a twofold interest:

- first, it provides us with a constructive recipe to build all spline spaces which can be used for design;
- for given QEC-section spaces, it enables us to answer the following question: a sequence of connection matrices M_k, $1 \leqslant k \leqslant q$, being given, are the corresponding splines suitable for design or not? This is made possible because we are able to describe all possibilities to write the QEC section-spaces via systems of weight functions [12].

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