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Geometry

# The lower and upper bounds of the first eigenvalues for the bi-harmonic operator on manifolds ☆

## *Bornes inférieures et supérieures des premières valeurs propres de l'opérateur bi-harmonique sur une variété*

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A B S T R A C T

In this paper, we will estimate the lower bounds and upper bounds of the first eigenvalues for bi-harmonic operators on manifolds through Reilly's and Bochner's formulae, respectively.



R É S U M É

Dans cette Note, nous donnons des minorants et majorants des premières valeurs propres de l'opérateur bi-harmonique sur une variété riemannienne, compacte, connexe, en utilisant respectivement les formules de Reilly et de Bochner.



## 1. Introduction

Let $\Omega$ be an $n$-dimensional compact connected Riemannian manifold with smooth boundary $\partial\Omega$. The following problems are called the *buckling problem*:

$$\begin{cases} \Delta^2 u = -\Lambda\,\Delta u \text{ in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

and the *clamped plate problem*:

$$\begin{cases} \Delta^2 u = \Gamma u \text{ in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

respectively, because (1.1) can describe the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary, and (1.2) can describe the characteristic vibrations of a clamped plate, where $\Delta^2$ is the bi-harmonic

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operator and $\vec{n}$ denotes the inner unit normal vector field of the boundary $\partial\Omega$. The spectra of (1.1) and (1.2) are real and purely discrete:

$$0 < \Lambda_1 \le \Lambda_2 \le \cdots \le \Lambda_k \le \cdots \uparrow +\infty, \quad 0 < \Gamma_1 \le \Gamma_2 \le \cdots \le \Gamma_k \le \cdots \uparrow +\infty.$$

For eigenvalues of problems (1.1) and (1.2) on $\Omega \subset \mathbb{R}^n$, by using the results of Li and Yau [5] and the variational characterization for eigenvalues, Levine and Protter [4] obtained the lower bound estimations

$$\frac{1}{k}\sum_{i=1}^{k}\Lambda_i \ge \frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}, \quad \frac{1}{k}\sum_{i=1}^{k}\Gamma_i \ge \frac{n}{n+4}\frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}},$$

where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. Recently, in [2], Chen, Cheng, Wang and Xia have investigated the first eigenvalues of (1.1) and (1.2) under the assumption that the Ricci curvature of $\Omega$ is bounded from below by $(n-1)$. They got that $\Lambda_1 > n$, $\Gamma_1 > n\lambda_1$, where $\lambda_1$ denotes the first eigenvalue of the Dirichlet eigenvalue problem.

One can use the Faber–Krahn inequalities of $\nu$-Euclidean type (cf. [1], p. 166) to estimate the lower bounds of the first eigenvalues for eigenvalue problems on manifolds. For example, for the Dirichlet eigenvalue problem on manifold, in [1], Chavel got $\lambda_1(\Omega) \ge \alpha^2 V(\Omega)^{-\frac{2}{\nu}}$, where $\alpha = \frac{(\nu-2)(\Omega)}{2(\nu-1)}\mathfrak{I}_\nu(\Omega)$ and $\nu > 1$ is a positive constant. This wonderful result inspires us to investigate the Faber–Krahn inequalities of $\nu$-Euclidean type for bi-harmonic operators on manifolds.

This paper is organized as follows. In Section 2, we will estimate the lower bounds of the first eigenvalues for the bi-harmonic operator on manifolds $M$, that is, using the Federer–Fleming theorem and the Reilly formula to investigate the analytic Faber–Krahn inequalities of $\nu$-Euclidean type when $\nu \ge n = \dim(M)$. In Section 3, we will use the Bochner formula to estimate the upper bounds of the first eigenvalues for the bi-harmonic operator.

## 2. The lower bounds of the first eigenvalues

In this section, our main goal is to estimate the lower bounds of the first eigenvalues for problems (1.1) and (1.2) by the analytic Faber–Krahn inequalities of $\nu$-Euclidean type when $\nu \ge n = \dim(M)$. Firstly, we will recall the Reilly formula and some preliminary knowledge of the isoperimetric constant.

**Lemma 2.1** *(Reilly's formula). (See [7].) Let $\Omega$ be an $n$-dimensional compact connected Riemannian manifold with smooth boundary $\partial\Omega$. Given $f \in C^\infty(\bar{\Omega})$, then*

$$\int_\Omega [(\Delta f)^2 - |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f)]\mathrm{d}V = \int_{\partial\Omega} [-2(\bar{\Delta} f)\langle\nabla f, \vec{n}\rangle + (n-1)H\langle\nabla f, \vec{n}\rangle^2 + \sigma(\bar{\nabla} f, \bar{\nabla} f)]\mathrm{d}A, \tag{2.1}$$

*where $\Delta f$, $\nabla f$ and $\nabla^2 f$ are the Laplacian, gradient and Hessian of $f$, Ric the Ricci curvature of $\Omega$, $\bar{\Delta} f$ and $\bar{\nabla} f$ the Laplacian and the gradient of $f$ in $\partial\Omega$, and $\sigma$ and $H$ the second fundamental form and the mean curvature of $\partial\Omega$ with respect to the inner unit normal vector field $\vec{n}$ on $\partial\Omega$.*

We recall the following variational characterization for the first eigenvalues of the buckling problem and the clamped plate problem:

$$\Lambda_1(\Omega) = \min_{\substack{f \in H_0^2(\Omega) \\ f \not\equiv 0}} \frac{\int_\Omega (\Delta f)^2\,\mathrm{d}V}{\int_\Omega |\nabla f|^2\,\mathrm{d}V}, \tag{2.2}$$

and

$$\Gamma_1(\Omega) = \min_{\substack{f \in H_0^2(\Omega) \\ f \not\equiv 0}} \frac{\int_\Omega (\Delta f)^2\,\mathrm{d}V}{\int_\Omega f^2\,\mathrm{d}V}. \tag{2.3}$$

**Definition 2.2.** (See [1].) Let $M$ be an $n$-dimensional Riemannian manifold with $n \ge 2$. For each $\nu > 1$, the *$\nu$-isoperimetric constant* of $M$, $\mathfrak{I}_\nu(M)$, is defined to be the infimum:

$$\mathfrak{I}_\nu(M) = \inf_\Omega \frac{A(\partial\Omega)}{V(\Omega)^{1-\frac{1}{\nu}}},$$

where $\Omega$ varies over open submanifolds of $M$ possessing compact closure and $C^\infty$ boundary.

**Remark 1.** As stated in [1], the fact that $\mathfrak{I}_\nu(M) > 0$ is only possible for $n \le \nu \le \infty$. Indeed, let $\nu < n$, and consider a small geodesic ball $B(x;\epsilon)$, with center $x \in M$ and radius $\epsilon > 0$, for the isoperimetric quotient of $B(x;\epsilon)$,

$$\lim_{\epsilon\to 0} \frac{A(\partial\Omega)}{V(\Omega)^{1-\frac{1}{\nu}}} \sim \lim_{\epsilon\to 0} const.\epsilon^{\frac{n}{\nu}-1} = 0.$$

So it seems at first glance that one only has a discussion of isoperimetric constants for $\nu \geq n = \dim M$.

**Definition 2.3.** (See [1].) Let $M$ be an $n$-dimensional Riemannian manifold, $n \geq 2$. For each $\nu > 1$, the *Sobolev constant* of $M$, $\mathfrak{S}_\nu(M)$, is defined to be the infimum

$$\mathfrak{S}_\nu(M) = \inf_u \frac{\|\nabla u\|_1}{\|u\|_{\frac{\nu}{\nu-1}}},$$

where $u \in C_0^\infty(M)$.

**Lemma 2.4** *(The Federer–Fleming Theorem). The isoperimetric and Sobolev constants are equal, that is*

$$\mathfrak{J}_\nu(M) = \mathfrak{S}_\nu(M). \tag{2.4}$$

The detailed proof of the Federer–Fleming Theorem can be found in [1,3,6]. This elegant result was first proven independently in [3] by Federer and Fleming, and in [6] by Maz'ya in 1960.

**Theorem 2.5.** *Let $\Omega$ be an $n$-dimensional compact connected Riemannian manifold with smooth boundary $\partial\Omega$. Assume that the Ricci curvature of $\Omega$ is bounded from below by a constant $C$, and the isoperimetric constant $\mathfrak{J}_\nu(\Omega)$ of $\Omega$ is positive for some $\nu > 1$. Let $\Lambda_1(\Omega)$ and $\Gamma_1(\Omega)$ be the first eigenvalues of the problems (1.1) and (1.2). Then*

$$\Lambda_1(\Omega) \geq \frac{1}{4}\left(\frac{\mathfrak{J}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2 + C, \tag{2.5}$$

*and*

$$\Gamma_1(\Omega) \geq \left[\frac{1}{4}\left(\frac{\mathfrak{J}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2 + C\right]\lambda_1(\Omega), \tag{2.6}$$

*where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet eigenvalue problem.*

**Proof.** For any $u \in H_0^2(\Omega)$, without loss of generality, we choose the normal coordinate $(U; x^1, \cdots, x^n)$ for any point $P \in \Omega$. By a direct calculation, we can get

$$|\nabla|\nabla u||^2(P) = \sum_{q=1}^n \left[\frac{(\sum_{i=1}^n u_i u_{iq})^2}{\sum_{i=1}^n u_i^2}\right], \tag{2.7}$$

where $u_j = \frac{\partial u}{\partial x^j}$, $u_{jp} = \frac{\partial^2 u}{\partial x^j \partial x^p}$. From the Cauchy–Schwarz inequality, it follows that

$$\sum_{q=1}^n\Big(\sum_{i=1}^n u_i u_{iq}\Big)^2 \leq \Big[\sum_{q=1}^n\Big(\sum_{i=1}^n u_i^2\Big)\Big(\sum_{i=1}^n u_{iq}^2\Big)\Big] = \Big(\sum_{i=1}^n u_i^2\Big)\Big(\sum_{i,q=1}^n u_{iq}^2\Big). \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$|\nabla|\nabla u||^2(P) \leq \sum_{i,q=1}^n u_{iq}^2 = |\nabla^2 u|^2(P),$$

this implies

$$|\nabla|\nabla u||^2 \leq \sum_{i,q=1}^n u_{iq}^2 = |\nabla^2 u|^2 \tag{2.9}$$

on $\Omega$ for any $u \in H_0^2(\Omega)$.

For any $u \in C_0^\infty(\Omega)$, let the test function $f(u) = |\nabla u|^2$, then, we first have by the Hölder inequality that

$$\int_\Omega |f| \mathrm{d}V \leq \left\{\int_\Omega |f|^{\frac{\nu}{\nu-1}} \mathrm{d}V\right\}^{\frac{\nu-1}{\nu}} \left\{\int_\Omega 1\, \mathrm{d}V\right\}^{\frac{1}{\nu}} = \left\{\int_\Omega |f|^{\frac{\nu}{\nu-1}} \mathrm{d}V\right\}^{\frac{\nu-1}{\nu}} V(\Omega)^{\frac{1}{\nu}}. \tag{2.10}$$

According to the Federer–Fleming Theorem (2.4), we derive

$$\mathfrak{I}_\nu(\Omega)\left\{\int_\Omega |f|^{\frac{\nu}{\nu-1}}\,dV\right\}^{\frac{\nu-1}{\nu}} \le \int_\Omega |\nabla f|\,dV,$$

which together with (2.10) gives us

$$\int_\Omega |f|\,dV \le \frac{V(\Omega)^{\frac{1}{\nu}}}{\mathfrak{I}_\nu(\Omega)}\int_\Omega |\nabla f|\,dV. \tag{2.11}$$

From the Cauchy–Schwarz inequality, it is obvious that

$$\int_\Omega |\nabla f|dV = 2\int_\Omega |\nabla u||\nabla|\nabla u||dV \le 2\left\{\int_\Omega |\nabla u|^2\,dV\right\}^{\frac{1}{2}}\left\{\int_\Omega |\nabla|\nabla u||^2\,dV\right\}^{\frac{1}{2}}. \tag{2.12}$$

Combining (2.11) and (2.12) yields

$$\int_\Omega |\nabla u|^2\,dV \le 2\frac{V(\Omega)^{\frac{1}{\nu}}}{\mathfrak{I}_\nu(\Omega)}\left\{\int_\Omega |\nabla u|^2\,dV\right\}^{\frac{1}{2}}\left\{\int_\Omega |\nabla|\nabla u||^2\,dV\right\}^{\frac{1}{2}},$$

since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, the above relation holds also for any $u \in H_0^2(\Omega)$, which implies

$$\frac{\int_\Omega |\nabla|\nabla u||^2\,dV}{\int_\Omega |\nabla u|^2\,dV} \ge \frac{1}{4}\left(\frac{\mathfrak{I}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2,$$

thus, by (2.9), we get

$$\frac{\int_\Omega |\nabla^2 u|^2\,dV}{\int_\Omega |\nabla u|^2\,dV} \ge \frac{1}{4}\left(\frac{\mathfrak{I}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2. \tag{2.13}$$

From the Reilly formula (2.1) and $u|_{\partial\Omega} = \frac{\partial u}{\partial \vec{n}}|_{\partial\Omega} = 0$, we have

$$\int_\Omega \left[(\Delta u)^2 - |\nabla^2 u|^2 - \mathrm{Ric}(\nabla u, \nabla u)\right]dV = 0,$$

this equality and the Ricci curvature of $\Omega$ is bounded from below by a constant $C$ imply

$$\int_\Omega |\nabla^2 u|^2\,dV = \int_\Omega (\Delta u)^2\,dV - \int_\Omega \mathrm{Ric}(\nabla u, \nabla u)dV \le \int_\Omega (\Delta u)^2\,dV - C\int_\Omega |\nabla u|^2\,dV,$$

it follows from this inequality and (2.13) that

$$\frac{\int_\Omega (\Delta u)^2\,dV}{\int_\Omega |\nabla u|^2\,dV} \ge \frac{1}{4}\left(\frac{\mathfrak{I}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2 + C, \tag{2.14}$$

and

$$\frac{\int_\Omega (\Delta u)^2\,dV}{\int_\Omega u^2\,dV} \ge \left[\frac{1}{4}\left(\frac{\mathfrak{I}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2 + C\right]\frac{\int_\Omega |\nabla u|^2\,dV}{\int_\Omega u^2\,dV}. \tag{2.15}$$

From the Poincaré inequality and the fact that $u$ is non-zero and $u|_{\partial\Omega} = 0$, we can obtain

$$\frac{\int_\Omega |\nabla u|^2\,dV}{\int_\Omega u^2\,dV} \ge \lambda_1(\Omega), \tag{2.16}$$

where the equality holds if and only if $u$ is the first eigenfunction of the Dirichlet eigenvalue problem. Then, it follows from (2.2), (2.3), (2.14), (2.15) and (2.16) that

$$\Lambda_1(\Omega) = \min_{\substack{u\in H_0^2(\Omega)\\ u\not\equiv 0}} \frac{\int_\Omega (\Delta u)^2\,dV}{\int_\Omega |\nabla u|^2\,dV} \ge \frac{1}{4}\left(\frac{\mathfrak{I}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}}\right)^2 + C,$$

and

$$\Gamma_1(\Omega) = \min_{\substack{f \in H_0^2(\Omega) \\ f \not\equiv 0}} \frac{\int_\Omega (\Delta u)^2 \, dV}{\int_\Omega u^2 \, dV} \geq \left[ \frac{1}{4} \left( \frac{\mathfrak{I}_\nu(\Omega)}{V(\Omega)^{\frac{1}{\nu}}} \right)^2 + C \right] \lambda_1(\Omega).$$

We complete the proof. □

Similarly with the analytic Faber–Krahn inequality of $\nu$-Euclidean type for the Dirichlet eigenvalue problem on manifold (cf. [1, p. 166]), we refer to the inequalities (2.5) and (2.6) as the analytic Faber–Krahn inequalities of $\nu$-Euclidean type.

As natural applications of the above results, concrete lower bounds of the first eigenvalues of the above problems on unit ball and unit sphere will be obtained.

**Example 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a unit ball, that is, $\Omega = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$. It is well known that $\mathfrak{I}_n(\Omega) = n\omega_n^{\frac{1}{n}}$ for any connected domain $\Omega \subset R^n$. Then from (2.5) and (2.6), we can get:

$$\Lambda_1(\Omega) \geq \frac{n^2}{4}, \; \Gamma_1(\Omega) \geq \frac{n^4}{16}.$$

**Example 2.2.** Let $S^n$ be a unit sphere with sectional curvature 1, and $\Omega \subseteq S^n$ (small enough) be a relatively compact domain with smooth boundary $\partial\Omega$. Then the Ricci curvature of $S^n$ is $n-1$. From [8, Theorem 1.4], we know that for any connected domain $\Omega \subset S^n$, $n = 2, 3, 4, 5$,

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-\frac{1}{n}}} \geq n\,\omega_n^{\frac{1}{n}} \left(1 - \tau V(\Omega)^{\frac{2}{n}}\right)^{\frac{1}{n}},$$

where $\tau = \frac{n(n-1)}{2(n+2)\omega_n^{\frac{2}{n}}}$. According to Definition 2.2, we derive: $\mathfrak{I}_n(\Omega) \geq n\,\omega_n^{\frac{1}{n}}(1 - \tau V(\Omega)^{\frac{2}{n}})^{\frac{1}{n}}$. Then from (2.5) and (2.6), we have:

$$\Lambda_1(\Omega) \geq \frac{n^2}{4} \left[ \frac{\omega_n}{V(\Omega)} \left(1 - \tau V(\Omega)^{\frac{2}{n}}\right) \right]^{\frac{2}{n}} + n - 1, \qquad \Gamma_1(\Omega) \geq \left\{ \frac{n^2}{4} \left[ \frac{\omega_n}{V(\Omega)} \left(1 - \tau V(\Omega)^{\frac{2}{n}}\right) \right]^{\frac{2}{n}} + n - 1 \right\} \lambda_1.$$

## 3. The upper bounds of the first eigenvalues

In this section, we will estimate the upper bounds of the first eigenvalues for problems (1.1) and (1.2). Firstly, we recall the Bochner–Weitzenböck formula for later use.

**Lemma 3.1** *(Bochner–Weitzenböck formula). Let $\Omega$ be an n-dimensional complete Riemannian manifold. Given any $u \in C^\infty(\Omega)$, then*

$$\frac{1}{2}(\Delta|\nabla u|^2) = \langle \nabla u, \nabla(\Delta u)\rangle + |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u),$$

*where $\Delta u$, $\nabla u$ and $\nabla^2 u$ are the Laplacian, gradient and Hessian of u, Ric the Ricci curvature of $\Omega$.*

**Theorem 3.2.** *Let $\Omega$ be an n-dimensional compact connected Riemannian manifold with smooth boundary $\partial\Omega$. Assume that the Ricci curvature of $\Omega$ is bounded from below by a constant C. Let $\Lambda_1(\Omega)$ and $\Gamma_1(\Omega)$ be the first eigenvalues of the problems (1.1) and (1.2). If the first eigenfunction corresponding to the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet eigenvalue problem on domain $\Omega$ satisfies $u_1 \in H_0^2(\Omega)$. Then*

$$\Lambda_1(\Omega) < n(\lambda_1(\Omega) - C), \; \Gamma_1(\Omega) < n\lambda_1(\Omega)(\lambda_1(\Omega) - C).$$

**Proof.** It follows from $u|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$ that $(\nabla u)|_{\partial\Omega} = 0$, then, by the divergence theorem, we have:

$$\int_\Omega \Delta|\nabla u|^2 \, dV = 2\int_{\partial\Omega} |\nabla u|\langle \nabla|\nabla u|, \vec{n}\rangle \, dA = 0. \tag{3.1}$$

From the Bochner–Weitzenböck formula and (3.1), we have:

$$\lambda_1 \int_\Omega |\nabla u_1|^2 \, dV = \int_\Omega [|\nabla^2 u_1|^2 + \mathrm{Ric}(\nabla u_1, \nabla u_1)] \, dV. \tag{3.2}$$

By the Schwarz inequality, we get

$$|\nabla^2 u_1|^2 \geq \frac{1}{n}(\Delta u_1)^2, \tag{3.3}$$

with equality holding if and only if

$$\nabla^2 u_1 = \frac{\Delta u_1}{n}\langle , \rangle .$$

From (3.2), (3.3) and the Ricci curvature is bounded from below by a constant $C$, we have

$$(\lambda_1 - C)\int\limits_\Omega |\nabla u_1|^2 \, dV \geq \frac{1}{n}\int\limits_\Omega (\Delta u_1)^2 \, dV,$$

which together with (2.3) and (2.4) implies

$$\Lambda_1(\Omega) \leq \frac{\int_\Omega (\Delta u_1)^2 \, dV}{\int_\Omega |\nabla u_1|^2 \, dV} \leq n(\lambda_1 - C), \ \Gamma_1(\Omega) \leq \frac{\int_\Omega (\Delta u_1)^2 \, dV}{\int_\Omega (u_1)^2 \, dV} \leq n\lambda_1(\lambda_1 - C).$$

Let us show that neither $\Gamma_1(\Omega) = n(\lambda_1 - C)$ nor $\Gamma_1(\Omega) = n\lambda_1(\lambda_1 - C)$ will hold. If $\Gamma_1(\Omega) = n(\lambda_1 - C)$, we have

$$-\Gamma_1(\Omega)\Delta u_1 = \Delta^2 u_1,$$

then $\Gamma_1 = \lambda_1$; this is a contradiction, since $\Gamma_1 > \lambda_1$.

If $\Gamma_1(\Omega) = n\lambda_1(\lambda_1 - C)$, we must have:

$$|\nabla^2 u_1|^2 = \frac{1}{n}(\Delta u_1)^2, \ Ric(\nabla u, \nabla u) = C|\nabla u|^2.$$

It is obvious that

$$n\lambda_1(\lambda_1 - C)u_1 = \Gamma_1 u_1 = \Delta^2 u_1 = \lambda_1^2 u_1,$$

which implies

$$\lambda_1 = \frac{n}{n-1}C.$$

Using a similar argument as that in [2], we can prove that this is a contradiction. The details are as follows. By the Bochner–Weitzenböck formula, we have:

$$\begin{aligned}
&\frac{1}{2}\Delta\left(|\nabla u|^2 + \frac{1}{n-1}Cu^2\right) \\
&= \langle \nabla(\Delta u), \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u) + \frac{(\Delta u)^2}{n} + \frac{1}{n-1}C|\nabla u|^2 + \frac{1}{n-1}Cu\Delta u \\
&= -\frac{n}{n-1}C|\nabla u|^2 + C|\nabla u|^2 + \frac{n}{(n-1)^2}C^2u^2 + \frac{1}{n-1}C|\nabla u|^2 - \frac{n}{(n-1)^2}C^2u^2 \\
&= 0.
\end{aligned}$$

Since $|\nabla u|^2 + \frac{1}{n-1}Cu^2$ is continuous on $\Omega$ and $\left(|\nabla u|^2 + \frac{1}{n-1}Cu^2\right)|_{\partial\Omega} = 0$, we conclude from the maximum principle that $|\nabla u|^2 + \frac{1}{n-1}Cu^2 = 0$ on $M$; this is a contradiction. By the above discussion, we get:

$$\Lambda_1(\Omega) < n(\lambda_1(\Omega) - C), \ \Gamma_1(\Omega) < n\lambda_1(\Omega)(\lambda_1(\Omega) - C). \qquad \square$$

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