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Probability theory

The central limit theorem for complex Riesz–Raikov sums

Le théorème limite central pour des sommes de Riesz–Raikov complexes

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ABSTRACT

For complex Riesz–Raikov sums, the central limit theorem is proved. As a byproduct, metric discrepancy results are proved for complex geometric progressions.

RÉSUMÉ

Nous démontrons un théorème limite central pour les sommes de Riesz–Raikov complexes. En application de nos méthodes, nous établissons aussi des résultats de discrépance pour les progressions géométriques complexes.

Let $\mathbf{I} = \mathbf{Z}[\sqrt{-1}]$, $\mathbf{K} = \mathbf{Q}[\sqrt{-1}]$, and $D = \{z \in \mathbf{C} \mid \Re z, \Im z \in [0, 1)\}$. We denote by μ the Lebesgue measure on $\mathbf{C}$. Let f be a locally square integrable real valued function on $\mathbf{C}$ satisfying

$$f(z + \sqrt{-1}) = f(z + 1) = f(z), \qquad \int_D f(z)\,\mu(\mathrm{d}z) = 0, \qquad \int_D f^2(z)\,\mu(\mathrm{d}z) < \infty. \tag{1}$$

We denote the Fourier series of f by $\sum_{n\in\mathbf{I}^\times} \widehat{f}(n)\exp(2\pi\sqrt{-1}\,\Re(\bar{n}z))$. For a positive integer d, we put $f_d(z) = \sum_{n\in\mathbf{I}:\,|n|<d} \widehat{f}(n)\exp(2\pi\sqrt{-1}\,\Re(\bar{n}z))$ and $R(f,d) = \|f - f_d\|_2$. We assume the condition

$$R(f,d) = O\big((\log d)^{-1-\varepsilon}\big) \quad \text{for some} \quad \varepsilon > 0. \tag{2}$$

It is known that a function f of bounded variation in the sense of Hardy–Krause satisfies the condition $\big|\widehat{f}(n)\big| = O\big((|\Re n| \vee 1)^{-1}(|\Im n| \vee 1)^{-1}\big)$ (cf. Zaremba [16]), which implies $R(f,d) = O(d^{-1})$ and (2).

Theorem 1. *Assume that a real valued function f on $\mathbf{C}$ satisfies (1) and (2), and that $\theta \in \mathbf{C}$ satisfies $|\theta| > 1$. Regarding $\sum f(\theta^k z)$ as a random variable on the probability space $(D, \mathcal{B}_D, \mu)$, we have:*

$$\frac{1}{\sqrt{N}}\sum_{k=1}^{N} f(\theta^k z) \xrightarrow{\mathcal{D}} N(0, \sigma^2(\theta, f)). \tag{3}$$

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Here the limiting variance $\sigma^2(\theta, f)$ is given by

$$\sigma^2(\theta, f) = \int_D f^2(z)\,\mu(\mathrm{d}z), \quad \text{or} \quad \sigma^2(\theta, f) = \int_D f^2(z)\,\mu(\mathrm{d}z) + 2\sum_{k=1}^{\infty}\int_D f(p^k z) f(q^k z)\,\mu(\mathrm{d}z), \tag{4}$$

depending on whether $\theta^r \notin \mathbf{K}$ $(r = 1, 2, \ldots)$ holds or not. In the second case, p and $q \in \mathbf{I}$ are relatively irreducible and satisfy $\theta^r = p/q$, where r is the minimal positive integer satisfying $\theta^r \in \mathbf{K}$. In this case, $\sigma^2(\theta, f) = 0$ holds if and only if there exists a function g satisfying (1) and

$$f(z) = g(pz) - g(qz). \tag{5}$$

For real Riesz–Raikov sums $\sum f(\theta^k x)$ with $\theta > 1$ and $x \in [0, 1)$, the results were proved by Kac [8], Petit [14], and [4]. The complex case with $\theta \in \mathbf{I}$ was studied by Leonov [10] and Conze, Le Borgne and Roger [3].

We can also derive metric discrepancy results for complex geometric progressions $\{\theta^k z\}$ with $|\theta| > 1$. Relating results for real θ are proved in [5]. Denote $\langle z\rangle = \Re z - \lfloor \Re z\rfloor + (\Im z - \lfloor \Im z\rfloor)\sqrt{-1} \in D$. For $0 \le a < a' < 1$ and $0 \le b < b' < 1$, denote $\widetilde{\mathbf{1}}_{a,a',b,b'}(z) = \mathbf{1}_{[a,a')+\sqrt{-1}\,[b,b')}(\langle z\rangle) - (a'-a)(b'-b)$ and

$$D_N\{z_k\} = \sup_{a,a',b,b':0\le a<a'<1,0\le b<b'<1} \frac{1}{N}\left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{a,a',b,b'}(z_k)\right|.$$

Theorem 2. *For any $\theta \in \mathbf{C}$ with $|\theta| > 1$, we have the law of the iterated logarithm:*

$$\varlimsup_{N\to\infty} \frac{N D_N\{\theta^k z\}}{\sqrt{2N\log\log N}} = \Sigma_\theta = \sup_{a,a',b,b':0\le a<a'<1,0\le b<b'<1} \sigma(\theta, \widetilde{\mathbf{1}}_{a,a',b,b'}) \quad \mu\text{-a.e. } z. \tag{6}$$

Especially when $\theta^r \notin \mathbf{K}$ for all $r \in \mathbf{N}$, we have $\Sigma_\theta = \frac{1}{2}$.

1. Preliminary

Put $\mathbf{I}^\times = \mathbf{I} \setminus \{0\}$ and $\mathbf{I}^+ = \{p \in \mathbf{I} \mid \Im p > 0\} \cup \{p \in \mathbf{I} \mid \Im p = 0, \Re p > 0\}$. Let μ_0 be the probability measure on $\mathbf{C}$ given by $\mu_0(\mathrm{d}z) = \frac{1}{\pi^2}\left(\frac{\sin \Re z}{\Re z}\right)^2\left(\frac{\sin \Im z}{\Im z}\right)^2 \mu(\mathrm{d}z)$. There exists a $C_0 > 0$ such that

$$\mathbf{1}_D\,\mu(\mathrm{d}z) \le (C_0/2)\mu_0(\mathrm{d}z) \qquad (z \in \mathbf{C}) \tag{7}$$

Note that $\int_{\mathbf{C}} \exp\big(2\pi\sqrt{-1}\,\Re(\overline{w}z)\big)\,\mu_0(\mathrm{d}z) = \big(1 - \pi|\Re w|\big)^+\big(1 - \pi|\Im w|\big)^+$, and it equals to zero if $|w| \ge 1/2$. Hence we have

$$\left|\int_{\mathbf{C}} \left(\sum_{k=1}^{N} f(\theta^k z)\right)^2 \mu_0(\mathrm{d}z)\right| \le \sum_{k,l=1}^{N} \sum_{n,m\in\mathbf{I}^\times} \big|\widehat{f}(n)\widehat{f}(m)\big|\mathbf{1}(|\bar{n}\theta^k + \bar{m}\theta^l| < 1/2).$$

For $\zeta \in \mathbf{C}$, denote by $\varphi(\zeta)$ the $n \in \mathbf{I}$ such that $|\zeta + \bar{n}| < 1/2$ if it exists, and put $\varphi(\zeta) = \infty$ if such n does not exist. Put $\widehat{f}(\infty) = 0$. Suppose that $l \le k$ and $|\bar{n}\theta^k + \bar{m}\theta^l| < 1/2$ for some n and $m \in \mathbf{I}^\times$. Then $|\bar{n}\theta^{k-l} + \bar{m}| < 1/2$, and hence $m = \varphi(\bar{n}\theta^{k-l})$. Hence (7) implies $\big|\int_{\mathbf{D}} \big(\sum_{k=1}^{N} f(\theta^k z)\big)^2 \mu(\mathrm{d}z)\big| \le C_0 \sum_{k=1}^{N}\sum_{l=1}^{k}\sum_{n\in\mathbf{I}^\times} \big|\widehat{f}(n)\widehat{f}(\varphi(\bar{n}\theta^{k-l}))\big|$. If $n \in \mathbf{I}_0^h = \{n \in \mathbf{I}^\times \mid |\theta|^h \le |n| < |\theta|^{h+1}\}$ then $|\theta|^{h+k-l} \le |\bar{n}\theta^{k-l}|$ and $|\theta|^{h+k-l-1} \le |\varphi(\bar{n}\theta^{k-l})|$. By $\sum_{n\in\mathbf{I}_0^\times} = \sum_{h=0}^{\infty}\sum_{n\in\mathbf{I}_0^h}$, we see that

$$\sum_{n\in\mathbf{I}^\times} \big|\widehat{f}(n)\widehat{f}(\varphi(\bar{n}\theta^{k-l}))\big| \le \sum_h \left(\sum_{n\in\mathbf{I}_0^h} \big|\widehat{f}(n)\big|^2\right)^{1/2} \left(\sum_{n\in\mathbf{I}_0^h} \big|\widehat{f}(\varphi(\bar{n}\theta^{k-l}))\big|^2\right)^{1/2} \le \sum_h R(f, |\theta|^h) R(f, |\theta|^{h+k-l-1}),$$

since $n \mapsto \varphi(\bar{n}\theta^{k-l})$ is injective on $\mathbf{I}_0^h \cap \{n \in \mathbf{I}^\times \mid \varphi(\bar{n}\theta^{k-l}) \ne \infty\}$. Hence we have:

$$\left|\int_{\mathbf{D}} \left(\sum_{k=1}^{N} f(\theta^k z)\right)^2 \mu(\mathrm{d}z)\right| \le C_0 \sum_{k=1}^{N}\sum_{l=1}^{k}\sum_{h=0}^{\infty} R(f, |\theta|^h) R(f, |\theta|^{h+k-l-1}) \le C_0 N \left(\sum_{h=0}^{\infty} R(f, |\theta|^{h-1})\right)^2. \tag{8}$$

By $(|\Re z| \vee 1)(|\Im z| \vee 1) \ge |z|/\sqrt{2}$, for any $w \in \mathbf{C}$, we have $\big|\int_{D+w} \exp\big(2\pi\sqrt{-1}\,\Re(\bar{n}\lambda z)\big)\,\mu(\mathrm{d}z)\big| \le 4/\big(\Re(n\bar{\lambda}) \vee 1\big)\big(\Im(n\bar{\lambda}) \vee 1\big) \le 4\sqrt{2}/|n\lambda|$. Hence we have the following two lemmas for a trigonometric polynomial $h(z) = \sum_{n\in\mathbf{I}^\times:|n|\le d} c_n \exp(2\pi\sqrt{-1}\,\Re(\bar{n}z))$.

Lemma 1.1. *For $|\lambda| \ge 1$ and $w \in \mathbf{C}$, $\big|\int_{D+w} h(\lambda z)\,\mu(\mathrm{d}z)\big| \le (4\sqrt{2}/|\lambda|)\sum_{n\in\mathbf{I}^\times:|n|\le d} |c_n|/|n| = O\big(1/|\lambda|\big)$.*

Lemma 1.2. *Suppose that a sequence $\{\lambda_k\}$ of complex numbers satisfies the modulus Hadamard gap condition $|\lambda_{k+1}/\lambda_k| \geq Q > 1$. Then there exists a constant C_Q depending only on Q, such that*

$$\int_D \left(\max_{l=1}^{N} \sum_{k=M+1}^{M+l} h(\lambda_k z) \right)^4 \mu(\mathrm{d}z) \leq C_Q \left(\sum_{n \in \mathbf{I}^\times : |n| \leq d} |c_n| \right)^4 N^2$$

Proof. By applying the triangle inequality for an L^4-norm, we see that it is sufficient to prove for $h(z) = \cos(2\pi \Re(\bar{j}z) + \gamma_j)$. By dividing into subsequences, we see that it is sufficient to prove under the condition $Q \geq 3$. In this case, $\{\cos(2\pi\Re(\bar{j}\lambda_k z) + \gamma_j)\}_k$ forms a multiplicative system under the measure μ_0, and the above inequality with respect to measure μ_0 is already proved as a combination of the main lemma of Komlós–Révész [9] and of the Erdős–Stečkin result [13]. Thanks to (7), we can conclude the proof. □

Lemma 1.3. *There exists a positive constant $C_{\theta,d}$ such that $|\bar{n}\theta^k + \bar{m}\theta^l| \geq C_{\theta,d}|\theta^{k\wedge l}|$ for any $k, l \in \mathbf{N}$, $n, m \in \mathbf{I}^\times$ with $|n|, |m| \leq d$ and $\bar{n}\theta^k + \bar{m}\theta^l \neq 0$.*

Proof. Assume $k \geq l$. Because of $\lim_{j\to\infty} |\bar{n}\theta^j + \bar{m}| = \infty$, $D_{m,n} = \inf\{|\bar{n}\theta^j + \bar{m}| \mid j \geq 0, |\bar{n}\theta^j + \bar{m}| \neq 0\}$ is positive. Hence $|\bar{n}\theta^k + \bar{m}\theta^l| \geq D_{m,n}|\theta^l|$ provided that the left-hand side is not zero. □

We denote $\int_{\mathbf{C}} g(z)\, \mu_R(\mathrm{d}z) = \lim_{T\to\infty} \frac{1}{4T^2} \int_{[-T,T]+\sqrt{-1}[-T,T]} g(z)\, \mu(\mathrm{d}z)$ if the limit exists. Now we verify (9) and (10) below. Since they are clear when $\theta^r \notin \mathbf{K}$ ($r = 1, 2, \ldots$) we consider the other case. Note that $\left|\int_D f(p^k z) f(q^k z)\, \mu(\mathrm{d}z)\right| = \left|\sum_{l\in\mathbf{I}^\times} \widehat{f}(l(\bar{q})^k)\widehat{f}(-l(\bar{p})^k)\right| \leq R(f, |q|^k)R(f, |p|^k)$, which is summable in k. Hence,

$$\sigma^2(\theta, f_d) \to \sigma^2(\theta, f) \quad (d \to \infty). \tag{9}$$

For a trigonometric polynomial h satisfying (1) and $\lambda \in \mathbf{C}^\times$, we have $\int_{\mathbf{C}} h(\Theta\lambda z)h(\lambda z)\, \mu_R(\mathrm{d}z) = 0$ if $\Theta \notin \mathbf{K}$, and $\int_{\mathbf{C}} h((P/Q)\lambda z)h(\lambda z)\, \mu_R(\mathrm{d}z) = \int_D h(Pz)h(Qz)\, \mu(\mathrm{d}z)$ if $P, Q \in \mathbf{I}^\times$. Hence we have:

$$\int_{\mathbf{C}} \left(\sum_{k=M+1}^{M+N} h(\theta^k z) \right)^2 \mu_R(\mathrm{d}z) = N \int_D h(z)^2\, \mu(\mathrm{d}z) + 2\sum_{l=1}^{\infty} (N - lr)^+ \int_D h(p^l z)h(q^l z)\, \mu(\mathrm{d}z).$$

By noting that $\int_D h(p^l z)h(q^l z)\, \mu(\mathrm{d}z) = 0$ for large l and that $0 \leq N - (N - lr)^+ \leq lr$, we have:

$$\int_{\mathbf{C}} \left(\sum_{k=M+1}^{M+N} h(\theta^k z) \right)^2 \mu_R(\mathrm{d}z) = N\sigma^2(\theta, h) + O(1). \tag{10}$$

2. Coboundary

When $\theta^r = p/q \in \mathbf{I}$, we may assume $q = 1$. Put $\Pi(p, q) = \{w \in \mathbf{I} \mid \bar{p} \nmid w,\ \bar{q} \nmid w\}$ when $q \neq 1$, $\Pi(p, 1) = \{w \in \mathbf{I} \mid \bar{p} \nmid w\}$, and $\Pi(p, q)^+ = \Pi(p, q) \cap \mathbf{I}^+$. We have $\sigma^2(\theta, f) = \sum_{n\in\mathbf{I}^\times} |\widehat{f}(n)|^2 + 2\sum_{k=1}^{\infty} \sum_{n\in\mathbf{I}^\times} \sum_{m\in\mathbf{I}^\times} \widehat{f}(n)\widehat{f}(m)\mathbf{1}(\bar{n}p^k + \bar{m}q^k = 0)$, and by the derivation of (9) we see that this series is absolutely convergent. If we write $n = \bar{p}^s\bar{q}^t w$ and $m = \bar{p}^u\bar{q}^v w'$ by using $s, t, u, v = 0, 1, 2, \ldots$, and $w, w' \in \Pi(p, q)$, we have:

$$\sigma^2(\theta, f) = 2\sum_{w\in\Pi(p,q)^+} \sum_{l=0}^{\infty} \left| \sum_{s=0}^{l} \widehat{f}(\bar{p}^s\bar{q}^{l-s}w) \right|^2 \quad (q \neq 1), \quad \sigma^2(\theta, f) = 2\sum_{w\in\Pi(p,1)^+} \left| \sum_{s=0}^{\infty} \widehat{f}(\bar{p}^s w) \right|^2 \quad (q = 1)$$

If $\sigma^2(\theta, f) = 0$ and $q \neq 1$, then $\sum_{s=0}^{l} \widehat{f}(\bar{p}^s\bar{q}^{l-s}w) = 0$ for $l = 0, 1, 2, \ldots$ and $w \in \Pi(p, q)^+$, and hence for $w \in \Pi(p, q)$. Put $\widehat{g}(\bar{p}^s\bar{q}^{l-s}w) = -\sum_{j=0}^{s} \widehat{f}(\bar{p}^j\bar{q}^{l+1-j}w) = \sum_{j=0}^{\infty} \widehat{f}(\bar{p}^{s+1+j}\bar{q}^{l-s-j}w)$, where we use by convention $\widehat{f}(n) = 0$ for $n \notin \mathbf{I}$. Thanks to the Schwarz inequality, we have

$$\left(\sum_{w\in\Pi(p,q)} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \left| \sum_{j=0}^{\infty} \widehat{f}(\bar{p}^{s+1+j}\bar{q}^{l-s-j}w) \right|^2 \right)^{1/2} \leq \sum_{j=0}^{\infty} \left(\sum_{w\in\Pi(p,q)} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \left| \widehat{f}(\bar{p}^{s+1+j}\bar{q}^{l-s-j}w) \right|^2 \right)^{1/2},$$

which implies $\left(\sum_{w\in\Pi(p,q)} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \left| \widehat{g}(\bar{p}^s\bar{q}^{l-s}w) \right|^2 \right)^{1/2} \leq \sum_{j=0}^{\infty} R(f, |p|^j) < \infty$. Hence $\{\widehat{g}(n)\}_{n\in\mathbf{I}} \in \ell^2$ and $g(z) = \sum_{n\in\mathbf{I}^\times} \widehat{g}(n) \exp\left(2\pi\sqrt{-1}\,\Re(\bar{n}z)\right) \in L^2$. We can verify (5) by comparing the Fourier coefficients.

If $\sigma^2(\theta, f) = 0$ and $q = 1$, then $\sum_{s=0}^{\infty} \widehat{f}(\bar{p}^s w) = 0$ for $w \in \Pi(p, 1)$. By putting $\widehat{g}(\bar{p}^s w) = -\sum_{j=0}^{s} \widehat{f}(\bar{p}^j w) = \sum_{j=0}^{\infty} \widehat{f}(\bar{p}^{s+1+j} w)$, we see $\{\widehat{g}(n)\} \in \ell^2$ and $g(z) = \sum_{n\in\mathbf{I}} \widehat{g}(n) \exp\left(2\pi\sqrt{-1}\,\Re(\bar{n}z)\right) \in L^2$.

3. Almost sure invariance principle

We follow the method of Aistleitner [1], which originated from the works of Berkes [2] and Philipp [15].

Proposition 3. *Let h be a trigonometric polynomial satisfying (1). By enlarging the probability space, we can define a standard Gaussian i.i.d. $\{Z_i\}$ such that $\sum_{k=1}^{N} h(\theta^k z) = \sum_{k\le N\sigma^2(\theta,h)} Z_k + O(N^{124/250})$ a.s.*

Proof. If $\sigma^2(\theta,h)=0$, then h can be expressed as (5) and the sum is a telescoping sum. Thus we have $\sum_{k=1}^{N} h(\theta^k z) = O(1)$ and the above conclusion is clear. We assume that $\sigma^2(\theta,h)>0$.

Divide $\mathbf{N}$ into consecutive blocks Δ'_1, Δ_1, Δ'_2, $\Delta_2,\dots$ with $^{\#}\Delta'_i = \lfloor 1+9\log_{|\theta|} i\rfloor$ and $^{\#}\Delta_i = i$. Put $i^- = \min\Delta_i$ and $i^+ = \max\Delta_i$. We see $9\log_{|\theta|} i \le {}^{\#}\Delta'_i = i^- - (i-1)^+ - 1$ and $|\theta^{i^-}/\theta^{(i-1)^+}| > i^9$.

Denote $\rho(i) = \lceil \log_2 i^4|\theta^{i^+}|\rceil$. Let $\mathcal{F}_i$ be a σ-field on D generated by intervals $J_{i,j,j'} = \{z\in D \mid j2^{-\rho(i)} \le \Re z < (j+1)2^{-\rho(i)},\ j'2^{-\rho(i)} \le \Im z < (j'+1)2^{-\rho(i)}\}$ ($j,\ j'=0,\dots,2^{\rho(i)}-1$). Note $i^4|\theta^{i^+}| \le 2^{\rho(i)} \le 2i^4|\theta^{i^+}|$ and put $T_i(z) = \sum_{k\in\Delta_i} h(\theta^k z)$, $Y_i = E(T_i \mid \mathcal{F}_i) - E(T_i \mid \mathcal{F}_{i-1})$. Clearly $\{Y_i,\mathcal{F}_i\}$ forms a martingale difference sequence, i.e., $E(Y_i \mid \mathcal{F}_{i-1}) = 0$. We show:

$$\|Y_i - T_i\|_\infty = O(i^{-3}),\quad \|Y_i^2 - T_i^2\|_\infty = O(i^{-2}),\quad \|Y_i^4 - T_i^4\|_\infty = O(1),\quad EY_i^4 = O(i^2), \tag{11}$$

where the implied constants are depending only on h.

Let $k\in\Delta_i$ and $z\in J = J_{i,j,j'}$. We have $|h(\theta^k z) - E(h(\theta^k\,\cdot\,)\mid\mathcal{F}_i)| = \left|\frac{1}{\mu(J)}\int_J (h(\theta^k z) - h(\theta^k\zeta))\,\mu(\mathrm{d}\zeta)\right| \le \max_{\zeta\in J}|h(\theta^k z) - h(\theta^k\zeta)| = O(|\theta^k|2^{-\rho(i)}) = O(|\theta^k|/|\theta^{i^+}|i^4) = O(i^{-4})$. It implies $T_i - E(T_i\mid\mathcal{F}_i) = O(i^{-3})$. On $J = J_{i-1,j,j'}$ we have $E(h(\theta^k\,\cdot\,)\mid\mathcal{F}_{i-1}) = \frac{1}{\mu(J)}\int_J h(\theta^k z)\,\mu(\mathrm{d}z)$. Changing the variable for $2^{\rho(i-1)}z = \zeta$ and by applying Lemma 1.1, we have $E(h(\theta^k\,\cdot\,)\mid\mathcal{F}_{i-1}) = \int_{D+w} h(\theta^k 2^{-\rho(i-1)}\zeta)\,\mu(\mathrm{d}\zeta) = O(2^{\rho(i-1)}/|\theta^k|) = O(2(i-1)^4|\theta^{(i-1)^+}/\theta^{i^-}|) = O(i^{-5})$. Hence $E(T_i\mid\mathcal{F}_{i-1}) = O(i^{-4})$. By combining these we have the first estimate of (11). By $\|T_i\|_\infty = O(i)$, we have $\|Y_i\|_\infty = O(i)$. By $\|T_i^2 - Y_i^2\|_\infty \le \|T_i - Y_i\|_\infty\|T_i + Y_i\|_\infty$, we have the second. The third is proved similarly. The third and Lemma 1.2 implies the fourth.

Put $v_i = \int_{\mathbf{C}} T_i^2(z)\,\mu_R(\mathrm{d}z)$, $l_M = \sum_{i=1}^{M} {}^{\#}\Delta_i$, $\beta_M = \sum_{i=1}^{M} v_i$, and $V_M = \sum_{i=1}^{M} E(Y_i^2\mid\mathcal{F}_{i-1})$. We show

$$\beta_M = \sigma^2(\theta,h)l_M + O(l_M^{1/2}),\qquad \beta_M \sim \sigma^2(\theta,h)l_M,\qquad \|V_M - \beta_M\|_\infty = O(1). \tag{12}$$

The first and the second is clear from (10), and we show the third. Express $T_i^2 - v_i$ as a linear combination of $\exp(2\pi\sqrt{-1}\,\Re((\bar{n}\theta^k + \bar{m}\theta^l)z))$ where $|n|,\ |m|\le d$ and $k,\ l\in\Delta_i$. Since $\bar{n}\theta^k + \bar{m}\theta^l \ne 0$ in this expression, we see by Lemma 1.3 that $|\bar{n}\theta^k + \bar{m}\theta^l| \ge C_{\theta,d}|\theta^{i^-}|$. Changing the variable for $2^{\rho(i-1)}z = \zeta$, and by applying Lemma 1.1, we have $E(T_i^2 - v_i\mid\mathcal{F}_{i-1}) = \int_{D+w}(T_i^2(2^{-\rho(i-1)}\zeta) - v_i)\mu(\mathrm{d}\zeta) = O(i^2 2^{\rho(i-1)}/|\theta^{i^-}|) = O(1/i^3)$ and $\|\sum_{i=1}^{M} E(T_i^2\mid\mathcal{F}_{i-1}) - \beta_M\|_\infty = O(1)$. By (11), we have $\|\sum_{i=1}^{M} E(T_i^2\mid\mathcal{F}_{i-1}) - V_M\|_\infty = O(1)$.

We use the following version of Strassen's theorem.

Theorem 4. *(See Monrad–Philipp [12].) Suppose that a square-integrable martingale difference sequence $\{Y_i,\mathcal{F}_i\}$ satisfies $V_M = \sum_{i=1}^{M} E(Y_i^2\mid\mathcal{F}_{i-1}) \to \infty$ a.s. and $\sum_{i=1}^{\infty} E\big(Y_i^2\mathbf{1}_{\{Y_i^2\ge\psi(V_i)\}}/\psi(V_i)\big) < \infty$ for some non-decreasing function ψ with $\psi(x)\to\infty$ ($x\to\infty$) such that $\psi(x)(\log x)^\alpha/x$ is non-increasing for some $\alpha > 50$. If there exists a uniformly distributed random variable U that is independent of $\{Y_n\}$, there exists a standard normal i.i.d. $\{Z_i\}$ such that $\sum_{i\ge1} Y_i\mathbf{1}_{\{V_i\le t\}} = \sum_{i\le t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big)$, a.s.*

Put $\psi(x) = x^{4/5}$. By $V_i \ge Ci^2$, we see $E\big(Y_i^2\mathbf{1}_{\{Y_i^2\ge\psi(V_i)\}}/\psi(V_i)\big) \le EY_i^4/\psi(Ci^2)^2 = O(i^{-6/5})$ is summable in i. Let us take a constant C_0 satisfying $\|V_M - \beta_M\|_\infty < C_0$. By $\beta_{M+1} - \beta_M = v_M \to \infty$, we see $V_M < \beta_M + C_0 < \beta_{M+1} - C_0 < V_{M+1}$. By putting $t = \beta_M + C_0$, Theorem 4 implies $\sum_{i=1}^{M} Y_i = \sum_{i\le\beta_M+C_0} Z_i + o(\beta_M^{124/250})$, a.s. By (11), the left-hand side can be replaced by $\sum_{i=1}^{M} T_i$.

Put $\Delta''_M = \Delta'_1 \cup \ldots \cup \Delta'_M$. By Lemma 1.2, we have $E\big(l_M^{-2/5}\max_{l\in\Delta_M}\sum_{k\in\Delta_M:k\le l} h(\theta^k\,\cdot\,)\big)^4 = O(M^{-6/5})$ and $E\big(l_M^{-2/5}\max_{l\in\Delta''_M}\sum_{k\in\Delta''_M:k\le l} h(\theta^k\,\cdot\,)\big)^4 = O((\log M)^2 M^{-6/5})$. Since these are summable in M, by Beppo-Levi lemma, we have $\max_{l\in\Delta_M}\sum_{k\in\Delta_M:k\le l} h(\theta^k z) = o(l_M^{2/5})$ and $\max_{l\in\Delta''_M}\sum_{k\in\Delta''_M:k\le l} h(\theta^k z) = o(l_M^{2/5})$. Therefore, we have $\sum_{k=1}^{N} h(\theta^k z) = \sum_{i\le\beta_M+C_0} Z_i + o(N^{124/250})$, a.s. for $N\in\Delta'_M\cup\Delta_M$. By $\beta_M + C_0 = \sigma^2(\theta,h)l_M + O(l_M^{1/2})$, $^{\#}(\Delta'_M\cup\Delta_M) \le 2M$, $M^+ = l_M + O(M\log M)$, we see $|\beta_M + C_0 - \sigma^2(\theta,h)N| \le KM\log M$ for some $K>0$. Hence

$$P\left(\max_{N\in\Delta'_M\cup\Delta_M}\left|\sum_{i\le\beta_M+C_0} Z_i - \sum_{i\le\sigma^2(\theta,h)N} Z_i\right| \ge \sqrt{4KM}\log M\right) \le 2P(|N_{0,1}| \ge \sqrt{4\log M}) \le 4M^{-2}.$$

Since it is summable in M, we see $\max_{N\in\Delta'_M\cup\Delta_M}\left|\sum_{i\le\beta_M+C_0} Z_i - \sum_{i\le\sigma^2(\theta,h)N} Z_i\right| \le \sqrt{4KM}\log M = o(l_M^{2/5})$ for large M, a.s. By these, we have the conclusion. □

4. The central limit theorem and the metric discrepancy results

By Proposition 3, we see that the law of $\frac{1}{\sqrt{N}}\sum_{k=1}^{N} f_d(\theta^k z)$ converges weakly to $N(0,\sigma^2(\theta, f_d))$. By (9) and (8), we can take d such that $|\sigma^2(\theta, f_d) - \sigma^2(\theta, f)| < \varepsilon$ and $\int_D \left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(f - f_d)(\theta^k z)\right)^2 \mu(\mathrm{d}z) < \varepsilon$ for $N \ge 1$. By combining these, we can prove (3). By Proposition 3, we have

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} h(\theta^k z) = \sigma(\theta, h) \quad \mu\text{-a.e. } z. \tag{13}$$

for any trigonometric polynomial h satisfying (1). To prove (6), we use the following result. The one-dimensional version is proved in [5–7]. The following version can be proved in the same way as [11].

Proposition 4.1. *Let $\{\lambda_k\}$ be a sequence of complex numbers satisfying the gap condition $|\lambda_{k+1}/\lambda_k| > Q > 1$. Then for any dense countable set $S \subset [0,1)$, we have*

$$\varlimsup_{N\to\infty} \frac{ND_N\{\lambda_k z\}}{\sqrt{2N\log\log N}} = \sup_{a,a',b,b'\in S: 0\le a<a'\le 1, 0\le b<b'\le 1} \lim_{d\to\infty} \varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{a,a',b,b',d}(\lambda_k z)\right| \text{ a.e. } z.$$

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