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Quasi-periodic solutions for nonlinear wave equations

Solutions quasi périodiques pour l'équation des ondes non linéaire

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A R T I C L E   I N F O

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A B S T R A C T

We construct time quasi-periodic solutions to nonlinear wave equations on the torus in arbitrary dimensions. All previously known results (in the case of zero or a multiplicative potential) seem to be limited to the circle. This extends the method developed in the limit-elliptic setting in [12] to the hyperbolic setting. The additional ingredient is a Diophantine property of algebraic numbers.

R É S U M É

On construit des solutions quasi-périodiques en temps pour l'équation des ondes non linéaire sur le tore en dimension quelconque. Tous les résultats précédents se limitent au cercle. Cet article étend la méthode développée pour le cas limite elliptique dans [12] au cas hyperbolique. Le nouvel ingrédient est une propriété diophantienne des nombres algébriques.

1. Introduction and statement of the Theorem

We consider *real* valued solutions to the nonlinear wave equation (NLW) on the d-torus $\mathbb{T}^d = [0, 2\pi)^d$:

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + v + v^{p+1} + H(x, v) = 0, \tag{1}$$

where $p \in \mathbb{N}$ and $p \geq 1$; considered as a function on $\mathbb{R}^d$, v satisfies: $v(\cdot, x) = v(\cdot, x + 2j\pi)$, $x \in [0, 2\pi)^d$ for all $j \in \mathbb{Z}^d$; $H(x, v)$ is analytic in x and v and has the expansion:

$$H(x, v) = \sum_{m=p+2}^{\infty} \alpha_m(x) v^m,$$

where α_m as a function on $\mathbb{R}^d$ is $(2\pi)^d$ periodic and real and analytic in a strip of width $\mathcal{O}(1)$ for all m. The integer p in (1) is *arbitrary*.

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Eq. (1) can be rewritten as a first-order equation in t. Let

$$D=\sqrt{-\Delta+1} \tag{2}$$

and

$$u=(v, D^{-1}\frac{\partial v}{\partial t})\in\mathbb{R}^2.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$, one obtains the corresponding first-order equation

$$\mathrm{i}\frac{\partial u}{\partial t}=Du+D^{-1}[(\frac{u+\bar{u}}{2})^{p+1}+H(x,\frac{u+\bar{u}}{2})].$$

Using Fourier series, the solutions to the linear equation:

$$\mathrm{i}\frac{\partial u}{\partial t}=Du \tag{3}$$

are linear combinations of eigenfunction solutions of the form:

$$\mathrm{e}^{-\mathrm{i}(\sqrt{j^2+1})t}\mathrm{e}^{\mathrm{i}j\cdot x},\quad j\in\mathbb{Z}^d,$$

where $j^2=|j|^2$ and $\cdot$ is the usual inner product. These solutions are either periodic or quasi-periodic in time.

The main purpose of this note is to announce the following new result, namely, under appropriate genericity conditions on the Fourier frequencies, to be detailed in sect. 2, a class of quasi-periodic solutions to the linear wave equation (3) bifurcates to quasi-periodic solutions of the NLW in (1). We note that when $p\geq\frac{4}{d-2}$ ($d\geq 3$, $H=0$), global solutions to (1) do not seem to be known in general.

Under the assumption that H is a polynomial in u, $\bar{u}$, $\mathrm{e}^{\mathrm{i}x_k}$ and $\mathrm{e}^{-\mathrm{i}x_k}$, $k=1,2,\dots,b$, $x_k\in[0,2\pi)$, below is the precise statement.

Theorem. *Assume that*

$$v^{(0)}(t,x)=\mathrm{Re}\sum_{k=1}^{b}a_k\mathrm{e}^{-\mathrm{i}(\sqrt{j_k^2+1})t}\mathrm{e}^{\mathrm{i}j_k\cdot x}$$

is generic, satisfying the genericity conditions (i–iii), $a=\{a_k\}\in(0,\delta]^b=\mathcal{B}(0,\delta)$ and p even. Assume that $b>C_pd$. Then for all $0<\epsilon<1$, there exists $\delta_0>0$, such that for all $\delta\in(0,\delta_0)$, there is a Cantor set $\mathcal{G}\subset\mathcal{B}(0,\delta)$ with

$$\operatorname{meas}\mathcal{G}/\delta^b\geq 1-\epsilon.$$

For all $a\in\mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear wave equation (1):

$$v(t,x)=\mathrm{Re}[\sum a_k\mathrm{e}^{-\mathrm{i}\omega_k t}\mathrm{e}^{\mathrm{i}j_k\cdot x}]+o(\delta^{3/2}),$$

with basic frequencies $\omega=\omega(a)=\{\omega_k(a)\}_{k=1}^b$ satisfying

$$\omega_k=\sqrt{j_k^2+1}+\mathcal{O}(\delta^p),$$

and the amplitude–frequency map $a\mapsto\omega(a)$ is a diffeomorphism. The remainder $o(\delta^{3/2})$ is in a Gevrey norm on $\mathbb{T}^{b+d}$.

Remark. One views quasi-periodic solutions of b frequencies as periodic solutions on a b-dimensional torus in "time". Hence the use of Gevrey norms on $\mathbb{T}^{b+d}$. The condition of large b, namely $b>C_pd$, is imposed in order that certain determinants are not identically zero, as in [12]. It cannot be excluded that this condition could be improved after more technical work. Contrary to [12], however, aside from the genericity conditions, this is the only other condition needed to prove the Theorem. This is because the genericity condition (ii) below dictates that ω is Diophantine, see (4) below; moreover, the mass term 1 in the wave operator D introduces curvature, cf. [11]. The polynomial restriction on H is technical, the result most likely remains valid for analytic H.

This Theorem appears to be the first general existence results on quasi-periodic solutions to the NLW in (1) in arbitrary dimensions. Previously quasi-periodic solutions only seem to have been constructed in one dimension with positive mass m. In that case, the linear wave equation:

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}+mv=0,$$

gives rise to an eigenvalue set $\{\sqrt{j^2+m}, j\in\mathbb{Z}\}$ close to the set of integers, see [4,6,9] and [8,13] in a related context. For almost all m, this set is linearly independent over the integers. This property does not have higher dimensional analogues and seems to have been a serious obstacle. (The time periodic case is special and solutions have been constructed in higher dimensions in [3].)

To insert into a more general context, it is known, cf. [7], for example, that the spectrum of an elliptic first-order operator on generic compact manifold is *dense*. Since the spectrum of the wave operator D in (2) on the flat torus is the set

$$\{\sqrt{j^2+1}\,|\, j\in\mathbb{Z}^d\},$$

which becomes dense in $d\ge 2$, it can serve as a model example of the generic case.

2. The generic linear solutions

As in [12], the Fourier support:

$$\tilde\Gamma=\bigcup_{r=1}^{R}\operatorname{supp}[(u^{(0)}+\bar u^{(0)})^{*pr}]\backslash\{(0,0)\}\subset\mathbb{Z}^{b+d},$$

for some $R>0$, plays an essential role. Using (ν,η) to denote an element of $\tilde\Gamma$, $(\nu,\eta)\in\tilde\Gamma$, then as in Section 2 of [12], there is the following relation:

$$\eta=-\sum_{k=1}^{b}\nu_k j_k,$$

and consequently $\eta=\eta(j_1,j_2,\dots,j_b)$ is viewed as a function from $(\mathbb{Z}^d)^b$ to $\mathbb{Z}^d$.

For the purposes below, (ν,η) is, however, viewed as a point in $\mathbb{Z}^{b+d}$.

Definition. $u^{(0)}$ of b frequencies $j_1,j_2,\dots,j_b\in\mathbb{Z}^d$ is *generic* if the following three conditions are satisfied:

(i) any d vectors in $\{j_k\}_{k=1}^b$ are linearly independent.

(ii) the integers (j_k^2+1), $k=1,2,\dots,b$, are distinct:

$$1<j_1^2+1<j_2^2+1<\cdots<j_b^2+1,$$

and square free.

(iii) For all $k=1,2,\dots,b$, $m\in[-p,p]\backslash\{0\}$, consider the set of η of the form

$$\begin{aligned}\eta&=mj_k-m_\ell j_\ell\\ &:=\eta(m_\ell)\neq 0,\end{aligned}$$

where $\ell=1,2,\dots,b$, $m_\ell\in\mathbb{Z}$, $|m_\ell|\le 2p(d+1)$. For each η, define L to be

$$L=2m\eta\cdot j_k+(m^2-m_\ell^2):=L(m_\ell).$$

Denote by $P(m_\ell)$ the corresponding d-dimensional hyperplane in $\mathbb{R}^d$:

$$2\eta\cdot j+L=0,$$

where $\eta=\eta(m_\ell)$ and $L=L(m_\ell)$.

Let σ be any set of $(d+1)m_\ell$, $\ell=1,2,\dots,b$, $m_\ell\in\mathbb{Z}$, $|m_\ell|\le 2p(d+1)$, such that there exists $m_\ell\in\sigma$, $m_\ell\neq\pm m$, then

$$\bigcap_\sigma P(m_\ell)=\emptyset.$$

Remark. In lieu of condition (ii), one may take (j_k^2+1) to be multiples of *distinct* square free integers—the proof of the Theorem is the same.

The following indicates that the above three conditions are viable.

Lemma. *There are infinite numbers of $j_1,j_2,\dots,j_b\in\mathbb{Z}^d$ that satisfy the genericity conditions (i–iii).*

Proof. The proof of (i, iii) for the Lemma is similar to the corresponding one in [12]. Clearly there are infinite numbers of integers satisfying (ii). In fact, given $N\in\mathbb{N}$, the number of square free integers less than or equal to N is asymptotically,

$$=\frac{6}{\pi^2}N+\mathcal{O}(\sqrt{N}).\qquad\square$$

It follows from basic algebra that condition (ii) implies the usual (linear) Diophantine property:

$$\left\| \sum_{k=1}^{b} n_k \omega_k \right\|_{\mathbb{T}} \neq 0,$$

where $n_k \in \mathbb{Z}$, $\sum_k |n_k| \neq 0$; as well as the quadratic Diophantine property:

$$\left\| \sum_{k,\ell;k<\ell} n_{k\ell} \omega_k \omega_\ell \right\|_{\mathbb{T}} \neq 0,$$

where $n_{k\ell} \in \mathbb{Z}$, $\sum |n_{k\ell}| \neq 0$. So

$$\left\| \sum_{k=1}^{b} n_k \omega_k \right\|_{\mathbb{T}} \geq c|n|^{-\alpha} \tag{4}$$

for some $\alpha > 0$, and $|n| = \sum |n_k| \neq 0$ and

$$\left\| \sum_{k,\ell,k<\ell} n_{k\ell} \omega_k \omega_\ell \right\|_{\mathbb{T}} \geq c'|n|^{-\beta}, \tag{5}$$

for some $\beta > 0$, and $|n| = \sum |n_{k\ell}| \neq 0$, cf. [10].

Consequently,

$$\left\| \sum_{k,\ell} n_{k\ell} \omega_k \omega_\ell \right\|_{\mathbb{T}} = 0 \Longleftrightarrow \sum_{k,\ell} n_{k\ell} \omega_k \omega_\ell = n_{kk} \omega_k^2, \tag{6}$$

for some $k \in \{1, 2, \ldots, b\}$. The expression above of linear combinations of products of pairs of eigenvalues appears as the principal symbol of an appropriate linearized operator, which is the "divisor" in the problem. The above "zero divisors" are dealt with as in [12], which essentially uses the sub-principal symbol of the linearized operator to control the small eigenvalues under the genericity conditions (i, iii). Combining with the small-divisor estimates in (5), one is then able to achieve amplitude–frequency modulation. One can then adapt the analysis construction of Bourgain in Chap. 20 of [5] to complete the proof of the Theorem as in [12]. □

Remark. In Chap. 20 of [5], when ω is a Fourier multiplier—an external parameter, nonlinear polynomial conditions on ω are imposed in oder to achieve separation properties. It can be shown that, in fact, quadratic polynomial conditions suffice by using antisymmetry property of the determinants, cf. [1,2,11]. When ω is *fixed*, in view of (6), the corresponding condition simply *cannot* be imposed—instead the genericity conditions (i)–(iii) are used to bypass this obstruction to invertibility.

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