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# The existence problem of $S$-plurisubharmonic currents

## *Le problème de l'existence de courants $S$-plurisousharmoniques*

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A B S T R A C T

In this paper, we prove the existence of the trivial extension of $S$-plurisubharmonic currents of bidimension $(p, p)$ defined outside an obstacle $A$ of Hausdorff measure $\mathcal{H}_{2p}(A) = 0$. Furthermore, a valid definition of the current $\mathrm{d}g \wedge \mathrm{d}^c g \wedge T$ is achieved for every positive closed current $T$ and plurisubharmonic function $g$. The above results rely on an improvement of a classical result due to Demailly on the Monge–Ampère operator with a sharp condition on the Hausdorff measure.



R É S U M É

Dans cette Note, nous montrons l'existence de l'extension triviale des courants $S$-plurisousharmoniques de bi-dimension $(p, p)$, définis en-dehors d'un obstacle $A$ de mesure de Hausdorff $\mathcal{H}_{2p}(A) = 0$. De plus, nous montrons que le courant $\mathrm{d}g \wedge \mathrm{d}^c g \wedge T$ est bien défini, pour tout courant positif fermé $T$ et toute fonction plurisousharmonique $g$. Ces résultats reposent sur un relâchement de la condition de nullité d'une mesure de Hausdorff, dans un résultat classique de Demailly sur l'opérateur de Monge–Ampère.



## 1. Introduction

Let $\Omega$ be an open subset of $\mathbb{C}^n$ and $T$ be a positive current of bidimension $(p, p)$ on $\Omega$. Recall that $T$ is said to be closed if $\mathrm{d}T = 0$, and is said to be $S$-plurisubharmonic if there exists a positive current $S$ on $\Omega$ such that $\mathrm{dd}^c T \geq -S$. In the first part of this paper, which treats the existence problem, we give the sufficient conditions that guarantee the existence of the trivial extension of negative $S$-plurisubharmonic currents.

**Theorem 1.1.** *Let $A$ be a closed complete pluripolar subset of $\Omega$ and $T$ be a negative $S$-plurisubharmonic current on $\Omega \setminus A$. If the Hausdorff measure $\mathcal{H}_{2p}(A \cap \overline{\operatorname{Supp} T})$ vanishes, then $\widetilde{T}$ exists. Moreover, the residual current $R = \widetilde{\mathrm{dd}^c T} - \mathrm{dd}^c \widetilde{T}$ is negative and supported in $A$.*

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This result generalizes a situation considered by Dabbek–Elkhadhra–El Mir result [4] where the authors studied the case $S = 0$. Both cases when the current $S$ is closed and when $S$ is plurisubharmonic were considered in [9] and [2], respectively. In [1], Al Abdulaali succeeded to obtain the extension $\widetilde{T}$, under the same hypotheses above, but when $\mathcal{H}_{2p-1}(A \cap \overline{\operatorname{Supp} T}) = 0$.

The second part deals with the wedge product of currents. Namely, our interest is to define the current $\mathrm{d}g \wedge \mathrm{d}^c g \wedge T$.

**Theorem 1.2.** *Let $A$ be a closed subset of $\Omega$ and $T$ be a positive closed current on $\Omega$, and let $g \in Psh(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus A)$. If $\mathcal{H}_{2p-2}(A \cap \operatorname{Supp} T) = 0$, then the current $\mathrm{d}g \wedge \mathrm{d}^c g \wedge T$ is well defined.*

As an application of Theorem 1.2, one can show that $\frac{\partial g}{\partial z_j}$ is of class $L^2_{\mathrm{loc}}$ when $\mathcal{H}_{2n-2}(A) = 0$. Notice that, this result is not true for subharmonic function, and the Newton kernel $\frac{-1}{||z||^{2n-2}}$ is a counterexample. In this part, we also give a new proof which improves a classical result due to Demailly [5], dealing with Monge–Ampère operators of certain unbounded plurisubharmonic functions.

**Theorem 1.3.** *Let $A$ be a closed subset of $\Omega$ and $T$ be a positive closed current on $\Omega$. Assume that $g \in Psh(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus A)$ such that $\mathcal{H}_{2p}(A \cap \operatorname{Supp} T) = 0$. Then the currents $gT$ and $\mathrm{dd}^c g \wedge T$ are well defined.*

## 2. Existence problem

In order to prove the existence of an extension $\widetilde{T}$ of $T$, we follow here the strategy previously employed by Ben Messaoud and El Mir, which consists in establishing a capacity inequality.

**Theorem 2.1.** *Let $A$ be a closed complete pluripolar subset of $\Omega$ and $T$ be a positive $\mathrm{dd}^c S$-negative current on $\Omega \setminus A$. Let $v$ be a plurisubharmonic function of class $\mathcal{C}^2$, $v \geq -1$ on $\Omega$ such that $\Omega' = \{z \in \Omega : v(z) < 0\}$ is relatively compact in $\Omega$. Let $K$ be a compact subset of $\Omega'$ and set $c_K = -\sup_{z \in K} v(z)$. Then for every plurisubharmonic function $u$ on $\Omega'$ of class $\mathcal{C}^2$ satisfying $-1 \leq u < 0$, we have:*

$$\int_{K \setminus A} T \wedge \mathrm{dd}^c u \wedge (\mathrm{dd}^c v)^{p-1} \leq c_K^{-1} \int_{\Omega' \setminus A} T \wedge (\mathrm{dd}^c v)^p + \|S \wedge \mathrm{dd}^c v\|_{\Omega'} \tag{2.1}$$

**Proof.** We apply the same technique as in [4]. By ([7], Proposition II.2) there exists a negative plurisubharmonic function $f$ on $\Omega'$ which is smooth on $\Omega' \setminus A$, such that

$$A \cap \Omega' = \left\{ z \in \Omega' : f(z) = -\infty \right\} \tag{2.2}$$

Choose $\lambda, \mu$ such that $0 < \mu < \lambda < c_K$. For $m \in \mathbb{N}$ and $\varepsilon$ small enough, we set

$$\varphi_m(z) = \mu u(z) + \frac{f(z) + m}{m+1} \text{ and } \varphi_{m,\varepsilon}(z) = \max_{\varepsilon} (v(z) + 1, \varphi_m(z)) \tag{2.3}$$

Thus, $\varphi_{m,\varepsilon} \in Psh(\Omega') \cap C^\infty(\Omega')$. Furthermore, $\varphi_{m,\varepsilon}(z) = v(z) + 1$ on a neighborhood of $\partial\Omega' \cup \left(\Omega' \cap \{f \leq -m\}\right)$. Consider the open subset

$$\Omega'_m = \Omega' \cap \{f > -m\} \tag{2.4}$$

By Stokes' formula,

$$\int_{\Omega'_m} T \wedge \mathrm{dd}^c(\varphi_{m,\varepsilon} - v - 1) \wedge (\mathrm{dd}^c v)^{p-1} \leq \int_{\Omega'_m} (\varphi_{m,\varepsilon} - v - 1) S \wedge (\mathrm{dd}^c v)^{p-1}$$

Hence, there exists a constant $\eta_v > 0$ depending on $\partial\bar{\partial} v$ so that

$$\int_{\Omega'_m} T \wedge \mathrm{dd}^c \varphi_{m,\varepsilon} \wedge (\mathrm{dd}^c v)^{p-1} \leq \int_{\Omega'_m} T \wedge (\mathrm{dd}^c v)^p + \eta_v \|S\|_{\Omega'} \tag{2.5}$$

Let $R > 0$ and $K_R = \{z \in K : f(z) \geq -R\}$. For $m$ sufficiently large, we have $K_R \subset \Omega'_m$ and $\varphi_m(z) \geq -\mu + \frac{m-R}{m+1} > 1 - \lambda$ for any $z \in K_R$. Since $v \leq -c_K$ on $K_R$, then $v + 1 \leq 1 - c_K \leq 1 - \lambda$, and because of this we find that $\varphi_{m,\varepsilon} = \varphi_m$ on a neighborhood of $K_R$. Therefore, by (2.5) we get

$$\int_{K_R} T \wedge \mathrm{dd}^c \varphi_m \wedge (\mathrm{dd}^c v)^{p-1} \leq \int_{\Omega'_m} T \wedge (\mathrm{dd}^c v)^p + \eta_v \|S\|_{\Omega'}. \tag{2.6}$$

As $dd^c f \geq 0$, we see that we have an inequality of $(1,1)$-forms $dd^c\varphi_m \geq \mu dd^c u$, thus

$$\mu \int_{K_R} T \wedge dd^c u \wedge (dd^c v)^{p-1} \leq \int_{\Omega'\setminus A} T \wedge (dd^c v)^p + \eta_v \|S\|_{\Omega'}. \tag{2.7}$$

The relation in the theorem follows by letting $R \to \infty$ and $\mu \to c_K$.

Similarly, one can also show that

$$\int_{K\setminus A} T \wedge (dd^c u)^p \leq \frac{1}{c_K} \int_{\Omega'\setminus A} T \wedge (dd^c u)^{p-1} \wedge dd^c v + \|S \wedge (dd^c u)^{p-1}\|_{\Omega'} \qquad \square \tag{2.8}$$

Now, under the preceding hypotheses, our goal is to prove that the boundedness of $\int_{K\setminus A} T \wedge (dd^c u)^p$ can be achieved by estimating from above the mass of $T \wedge (dd^c v)^p$ near $A$. This fact will imply our claims. We proceed as follows.

For $n \in \mathbb{N}$, set $v_n = v + \frac{1}{n}$ and $\Omega'_n = \{v_n < 0\}$. Take $n_1 < n_2 < \ldots < n_{p-1}$ so that $K \subset \Omega'_{n_1} \Subset \ldots \Subset \Omega'_{n_{p-1}} \Subset \Omega'$ and put $c_{n_j} = -\sup_{\Omega'_j} v_j$. Now by applying (2.8) to $v_{n_1}$, we find:

$$\int_{K\setminus A} T \wedge (dd^c u)^p \leq \frac{1}{c_K} \int_{\Omega'_{n_1}\setminus A} T \wedge (dd^c u)^{p-1} \wedge dd^c v_{n_1} + \|S \wedge (dd^c u)^{p-1}\|_{\Omega'} \tag{2.9}$$

But $dd^c v = dd^c v_{n_1} = \ldots = dd^c v_{n_{p-1}}$, and as $\Omega'_{n_1} \Subset \Omega'_{n_2}$, then (1) implies that

$$\int_{\overline{\Omega}'_{n_1}\setminus A} T \wedge (dd^c u)^{p-1} \wedge dd^c v_{n_1} \leq \frac{1}{c_{n_1}} \int_{\Omega'_{n_2}\setminus A} T \wedge (dd^c u)^{p-2} \wedge (dd^c v_{n_2})^2 + \eta_v \|S \wedge (dd^c u)^{p-2}\|_{\Omega'} \tag{2.10}$$

Hence,

$$\int_{K\setminus A} T \wedge (dd^c u)^p \leq \frac{1}{c_K c_{n_1}} \int_{\Omega'_{n_2}\setminus A} T \wedge (dd^c u)^{p-2} \wedge (dd^c v_{n_2})^2 + \frac{\eta^{(p-2)}_{v,u}}{c_K} \|S\|_{\Omega'} \tag{2.11}$$

By iterating the above process $p-1$ times, we obtain that

$$\begin{aligned}
\int_{K\setminus A} T \wedge (dd^c u)^p &\leq \frac{1}{c_K c_{n_1}} \int_{\Omega'_{n_2}\setminus A} T \wedge (dd^c u)^{p-2} \wedge (dd^c v_{n_2})^2 \\
&\quad + \frac{\eta^{(p-2)}_{v,u}}{c_K} \|S\|_{\Omega'} \\
&\leq \\
&\quad \vdots \\
&\leq \frac{1}{c_K c_{n_1} \ldots c_{n_{p-1}}} \int_{\Omega'_{n_{p-1}}\setminus A} T \wedge (dd^c u) \wedge (dd^c v_{n_{p-1}})^{p-1} \\
&\quad + \frac{\eta'_{v,u}}{c_K c_{n_1} \ldots c_{n_{p-2}}} \|S\|_{\Omega'} \\
&\leq D_v \int_{\Omega'\setminus A} T \wedge (dd^c v)^p + \eta_{v,u} \|S\|_{\Omega'},
\end{aligned}$$

for some positive constant $D_v$. Clearly, the argument above leads to the following result.

**Theorem 2.2.** *Let $A$ be a closed complete pluripolar subset of $\Omega$ and $T$ be a positive $dd^c S$-negative current on $\Omega \setminus A$ of dimension $p$. Let $v$ be a plurisubharmonic function of class $\mathcal{C}^2$, $v \geq -1$ on $\Omega$ such that $\Omega' = \{z \in \Omega : v(z) < 0\}$ is relatively compact in $\Omega$ and $\int_{\Omega'\setminus A} T \wedge (dd^c v)^p \neq 0$. Let $K$ be a compact subset of $\Omega'$. Then there exists a positive constant $D_v$ depending only on $v$ such that for every plurisubharmonic function $u$ on $\Omega'$ of class $\mathcal{C}^2$ satisfying we have*

$$\int_{K\setminus A} T\wedge(\mathrm{dd}^c u)^p \le D_v N_2^p(u)\int_{\Omega'\setminus A} T\wedge(\mathrm{dd}^c v)^p, \tag{2.12}$$

*where* $N_2(u)=\sup\{|\partial^\alpha u(z)|, |\alpha|\le 2, z\in\Omega\}$.

**Proof of Theorem 1.1.** Let us first assume that $\widetilde{T}$ exists. Then by ([6], Theorem 1.3), the extension $\widetilde{\mathrm{dd}^c T}$ exists and $R$ is negative current.

Now, we show the existence of $\widetilde{T}$. Since the problem is local, it is enough to show that $T$ is of locally finite mass near every point $z_0$ in $A$. Without loss of generality, one can assume that $z_0$ is the origin. Since $\mathcal{H}_{2p}(A\cap\overline{\text{Supp }T})=0$, then by ([3] and [10]) there exist a system of coordinates $(z',z'')$ of $\mathbb{C}^p\times\mathbb{C}^{n-p}$ and a polydisk $\triangle^p\times\triangle^{n-p}\subset\mathbb{C}^p\times\mathbb{C}^{n-p}$ such that $(A\cap\overline{\text{Supp }T})\cap(\triangle^p\times\partial\triangle^{n-p})=\emptyset$. Moreover, the projection map $\pi:(A\cap\overline{\text{Supp }T})\cap(\triangle^p\times\triangle^{n-p})\to\triangle^p$ is proper, and as $\pi(A\cap\overline{\text{Supp }T})$ is closed with a zero Lebesgue measure in $\triangle^p$, one can find an open subset $O\subset\triangle^p\setminus\pi(A\cap\overline{\text{Supp }T})$. Therefore the current has locally finite mass on $O\times\triangle^{n-p}$. Let $0<\delta<1$ such that $(A\cap\overline{\text{Supp }T})\cap(\triangle^p\times\{z'',\ \delta<|z''|<1\})=\emptyset$, and fix $a$ and $t$ two real numbers such that $\delta<a<t<1$. Set

$$\rho_\varepsilon=\max_\varepsilon\left(\pi^*\rho,\frac{1}{t^2-a^2}(|z''|^2-t^2)\right) \tag{2.13}$$

where $\rho$ is a smooth plurisubharmonic function on $\triangle^p$ such that $(\mathrm{dd}^c\rho)^p$ supported in $O$. We have $-1\le\rho_\varepsilon<0$ in $t\triangle^n$ and $\rho_\varepsilon=\pi^*\rho$ on $\{|z''|\le a\}$, and we obtain

$$\int_{(t\triangle^n)\setminus A} T\wedge(\mathrm{dd}^c\rho_\varepsilon)^p=\int_{(t\triangle^p)\times\{|z''|<a\}\setminus A} T\wedge(\mathrm{dd}^c(\pi^*\rho))^p + \int_{(t\triangle^p)\times\{a\le|z''|<t\}} T\wedge(\mathrm{dd}^c\rho_\varepsilon)^p$$

since $(\mathrm{dd}^c\pi^*\rho)^p$ supported in $O\times\triangle^{n-p}$ then both integrals of the right hand side are finite. By applying the previous result on $-T$ for $v=\rho_\varepsilon$ and $u=\frac{|z|^2-nt^2}{nt^2}$, we infer that $\widetilde{T}$ exists. $\square$

## 3. Wedge product of currents

In this section we study the wedge product of positive currents. From now on, in this section, we assume that $A$ is a closed subset of $\Omega$ and $g\in Psh(\Omega)\cap L^\infty_{\text{loc}}(\Omega\setminus A)$. Here we start by giving a new proof of a classical result due to Fornæss and Sibony [8] which asserts Theorem 1.3.

**Theorem 1.3.** *Let $T$ be a positive closed current on $\Omega$ and let $\mathcal{H}_{2p}(A\cap\text{Supp }T)=0$. Then the currents $gT$ together with $\mathrm{dd}^c g\wedge T$ are well defined.*

The result was first considered by Demailly in [5] where he studied the case when $\mathcal{H}_{2p-1}(A\cap\text{Supp }T)=0$.

**Proof.** Let us first assume that $g$ smooth and negative. Now, the current $-gT$ is positive $\mathrm{dd}^c$-negative current of bidimension $(p,p)$ on $\Omega$. Using the same notation as in the proof of Theorem 1.1, we find

$$\int_{(t\triangle^n)} -gT\wedge(\mathrm{dd}^c\rho_\varepsilon)^p=\int_{(t\triangle^p\times\{|z''|<a\}} -gT\wedge(\mathrm{dd}^c(\pi^*\rho))^p + \int_{(t\triangle^p)\times\{a\le|z''|<t\}} -gT\wedge(\mathrm{dd}^c\rho_\varepsilon)^p$$

Therefore, there exists a neighborhood $V$ of $t\,\triangle^n\cap A$ such that

$$\int_{(t\triangle^n)} -gT\wedge(\mathrm{dd}^c\rho_\varepsilon)^p\le\eta||gT||_{t\triangle^n\setminus V} \tag{3.1}$$

for some positive constant $\eta$. To conclude the proof, we let $K$ be a compact subset of $t\triangle^n$ and set $\Gamma_m=\{z\in K, d(z,A)<\frac{1}{m}\}$. Now, suppose that $g\in L^\infty_{\text{loc}}(\Omega\setminus A)$, we put

$$a_m=\inf_{z\in K\setminus\Gamma_m} g(z) \tag{3.2}$$

Since $g$ is smooth on $\Omega \setminus A$, then $a_m > -\infty$ for all $m$. For $\varepsilon_m > 0$ small enough, set

$$g_m = \max_{\varepsilon_m}(g, a_m - \frac{1}{2^m}) \tag{3.3}$$

Observe that, $g_m$ is smooth plurisubharmonic function and $g_m = g$ on $K \setminus \Gamma_m$. Then by the previous part, for $m$ sufficiently large, we have:

$$\begin{aligned}\int_{K\setminus\Gamma_m} -gT \wedge \beta^p &\le \int_K -g_m T \wedge \beta^p \\ &\le C\|g_m\|_{\mathcal{L}^\infty(t\triangle^n\setminus V)}\|T\|_{t\triangle^n\setminus V} \\ &= C\|g\|_{\mathcal{L}^\infty(t\triangle^n\setminus V)}\|T\|_{t\triangle^n\setminus V}\end{aligned}$$

By letting $m$ tends to $\infty$ we find that $\widetilde{gT}$ exists. Hence, by monotone convergence, we deduce that the current $gT$ is well defined, and the existence of $\mathrm{dd^c} g \wedge T$ is obtained from the continuity of the operators d and $\mathrm{d^c}$. □

Now, we show our second main theorem.

**Theorem 1.2.** *Let $T$ be a positive closed current on $\Omega$ and let $\mathcal{H}_{2p-2}(A \cap \operatorname{Supp} T) = 0$. Then the current* $\mathrm{d}g \wedge \mathrm{d^c} g \wedge T$ *is well defined.*

**Proof.** By regularizing $g$ and subtracting a constant, we may assume that $g$ is a smooth negative function. Since $\mathcal{H}_{2p-2}(A \cap \operatorname{Supp} T) = 0$, then $\mathcal{H}_{2p-1}(A \cap \operatorname{Supp} T) = 0$. Let us assume that $0 \in \operatorname{Supp} T \cap A$. Then by [3] and [10], there exist a system of coordinates $(z', z'') \in \mathbb{C}^s \times \mathbb{C}^{n-s}$, $s = p-1$ and a polydisk $\triangle^n = \triangle' \times \triangle''$ such that $\overline{\triangle'} \times \partial\triangle'' \cap (\operatorname{Supp} T \cap A) = \emptyset$. Now, take $0 < t < 1$ so that $\triangle' \times \{z'', t < |z''| < 1\} \cap (\operatorname{Supp} T \cap A) = \emptyset$. As $\overline{\triangle^n} \cap A$ is compact set, one can find a neighborhood $\omega$ of $\overline{\triangle^n} \cap A$ such that $\omega \cap (\triangle' \times \{z'', t < |z''| < 1\}) = \emptyset$. Let $a \in (t, 1)$ and choose $\rho(z') \in \mathcal{C}_0^\infty(a\triangle')$ such that $0 \le \rho \le 1$ and $\rho = 1$ on $\frac{1}{2}a\triangle'$. Take $\chi \in \mathcal{C}_0^\infty(\omega)$ such that $0 \le \chi \le 1$ and $\chi = 1$ on a neighborhood $\omega_0$ of $\overline{\triangle^n} \cap A$. Obviously, the function $\chi(z)\rho(z')$ is positive smooth and compactly supported in $a\triangle^n$. For convenience, we set $\beta' = \mathrm{dd^c}(|z'|^2)$, $\beta'' = \mathrm{dd^c}(|z''|^2)$ and $\alpha(z') = \rho(z')\beta'^s$.

$$\begin{aligned}\int_{\triangle'\times\triangle''} \chi g \mathrm{dd^c} g \wedge T \wedge \alpha(z') &= -\int_{\triangle'\times\triangle''} \mathrm{d}(\chi g) \wedge \mathrm{d^c} g \wedge T \wedge \alpha(z') \\ &= \Big|\int_{\triangle'\times\triangle''} (\chi \mathrm{d}g \wedge \mathrm{d^c} g \wedge T + g\mathrm{d}\chi \wedge \mathrm{d^c} g \wedge T)\alpha(z')\Big|\end{aligned}$$

Using the Cauchy–Schwarz inequality one can show that:

$$\begin{aligned}\int_{\triangle'\times\triangle''} \chi \mathrm{d}g \wedge \mathrm{d^c} g \wedge T \wedge \alpha(z') \le{}& \Big|\int_{\triangle'\times\triangle''} \chi g \mathrm{dd^c} g \wedge T \wedge \alpha(z')\Big| \\ &+ \left(\int_{\triangle'\times\triangle''} g^2 \mathrm{d}\chi \wedge \mathrm{d^c}\chi \wedge T \wedge \alpha(z')\right)^{\frac{1}{2}} \left(\int_{\triangle'\times\triangle''} \mathrm{d}g \wedge \mathrm{d^c} g \wedge T \wedge \alpha(z')\right)^{\frac{1}{2}}\end{aligned} \tag{3.4}$$

The forms $\mathrm{d}\chi$, $\mathrm{d^c}\chi$ vanish on some neighborhood $V'$ of $\triangle^n \cap A$, therefore we can change $\triangle' \times \triangle''$ by $\triangle' \times \triangle'' \setminus V'$ in the last line integral of (3.4), and hence the extension $\widetilde{\mathrm{d}g \wedge \mathrm{d^c} g} \wedge T$ exists. Now, a simple computation shows that

$$\mathrm{dd^c}(g^2 T) = 2g\mathrm{dd^c} g \wedge T + 2\mathrm{d}g \wedge \mathrm{d^c} g \wedge T \le 2\mathrm{d}g \wedge \mathrm{d^c} g \wedge T$$

Therefore, $\widetilde{g^2 T}$ exists thanks to [2], and monotone convergence implies that $g^2 T$ is well-defined. As the current $g\mathrm{dd^c} g \wedge T$ is a well-defined current by Theorem 1.3, our desired goal can be obtained by defining:

$$\mathrm{d}g \wedge \mathrm{d^c} g \wedge T = \frac{1}{2}\mathrm{dd^c}(g^2 T) - g\mathrm{dd^c} g \wedge T \qquad \square \tag{3.5}$$

**Remark 3.1.**

(1) The statement of Theorem 1.2 is true when $T$ is defined outside of $A$. Indeed, in this case the extension $\widetilde{T}$ exists by [4], and is closed.
(2) The diagonal coefficients of $\mathrm{d}g \wedge \mathrm{d^c} g$ are $2|\frac{\partial g}{\partial z_j}|^2$. Therefore, $\mathrm{d}g \wedge \mathrm{d^c} g$ is of locally finite mass if and only if $\frac{\partial g}{\partial z_j}$ is $L^2_{\mathrm{loc}}$.

**Application of Theorem 1.2.** Let $A$ be a closed subset of $\Omega$. Assume that $g \in Psh(\Omega) \cap L^{\infty}_{loc}(\Omega \setminus A)$ such that $\mathcal{H}_{2n-2}(A) = 0$, then by Theorem 1.2, one can prove that $\frac{\partial g}{\partial z_j}$ is $L^2_{loc}$.

Finally, we wish to point out that the condition on the Hausdorff measure in Theorem 1.2 is sharp as the following example shows.

**Example 3.2.** In $\mathbb{C}$, let $T = 1$ and $g = \log|z|^2$. We find that $dg \wedge d^c g \wedge T$ is not of finite mass near the origin. This shows that our result since $\mathcal{H}_{2(1)-2}(\{0\}) = 1$ is sharp, and likewise that point (2) in the last remark is also sharp.

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