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Nonnegative measures belonging to $H^{-1}(\mathbb{R}^2)$

Les mesures positives appartenantes à $H^{-1}(\mathbb{R}^2)$

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ABSTRACT

Motivated by applications in fluid dynamics, we show elementarily that a nonnegative compactly supported Radon measure μ belongs to the negative Sobolev space $H^{-1}(\mathbb{R}^2)$ provided that function $r \mapsto \mu(B(0,r))$ is Hölder continuous. In passing we obtain embedding of the space of nondecreasing Hölder continuous functions on $\mathbb{R}$ into the fractional Sobolev space $H^{1/2}(\mathbb{R})$. We comment on possible generalizations and numerical applications.

RÉSUMÉ

En vue d'applications en mécanique des fluides, on démontre qu'une mesure positive de Radon à support compact appartient à l'espace négatif de Sobolev $H^{-1}(\mathbb{R}^2)$ à condition que la fonction $r \mapsto \mu(B(0,r))$ soit hölderienne. En passant, on obtient un plongement d'espace des fonctions croissantes hölderiennes sur $\mathbb{R}$ dans l'espace de Sobolev fractionnaire $H^{1/2}(\mathbb{R})$. On discute des généralisations et des applications numériques.

1. Introduction

Let $u : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity of an incompressible flow in 2D, satisfying, for some $p : \mathbb{R}^2 \to [0,\infty)$, the incompressible Euler equations,

$$\partial_t u + u\nabla u + \nabla p = 0, \tag{1}$$

$$\operatorname{div}(u) = 0. \tag{2}$$

Let $\omega : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ be the corresponding vorticity, defined by

$$\omega(t, x_1, x_2) = \operatorname{curl}(u) := \partial_{x_1} u_2(t, x_1, x_2) - \partial_{x_2} u_1(t, x_1, x_2).$$

We will say that u is a *nonnegative vortex sheet solution* to (1)–(2) if (1)–(2) hold in the sense of distributions, $\omega(t,\cdot,\cdot) \in \mathcal{M}^+(\mathbb{R}^2)$ and $u(t,\cdot,\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ for every $t \geq 0$, and u belongs locally to $\mathrm{Lip}([0,\infty); H^{-L})$ for some $L > 0$. Here, $\mathcal{M}^+(\mathbb{R}^2)$

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denotes the space of bounded nonnegative Radon measures on $\mathbb{R}^2$ (see [7]), $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ is the space of locally square integrable functions on $\mathbb{R}^2$ and $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, is the Sobolev space of all tempered distributions f on $\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} (1+|y|^2)^s |\hat{f}(y)|^2 \, \mathrm{d}y < \infty,$$

where $\hat{f}$ denotes the Fourier transform of f. Space $H^{-1}(\mathbb{R}^2)$ can also be viewed as the space of all continuous functionals on the Sobolev space $W^{1,2}(\mathbb{R}^2)$ (see, e.g., [1]).

In [5], see also [12], Delort proved a basic existence theorem for nonnegative vortex sheets, which states that for initial data $u_0(x)$ such that $\mathrm{curl}(u_0)$ belongs to $\mathcal{M}^+_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$, where $\mathcal{M}^+_c(\mathbb{R}^2)$ is the space of compactly supported measures belonging to $\mathcal{M}^+(\mathbb{R}^2)$, there exists a nonnegative vortex sheet solution $u(t,x)$ of (1)–(2) such that $u(0,\cdot) = u_0$. Uniqueness of such solutions is an outstanding open problem. Our aim in this paper is to approach this problem by characterizing the space $\mathcal{M}^+_c(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ in geometric terms. To this end, we define the radial cumulative distribution function of a measure $\omega \in \mathcal{M}^+_c(\mathbb{R}^2)$ by

$$G^\omega(r) := \begin{cases} \omega(B(0,r)) & \text{for } r > 0, \\ 0 & \text{for } r \le 0, \end{cases} \tag{3}$$

where $B(0,r)$ is the closed ball centered at 0 and with radius r. Denote by $C^{0,\gamma}(\mathbb{R})$ the space of all Hölder continuous functions on $\mathbb{R}$ (i.e. functions f such that $|f(x_1) - f(x_2)| \le K|x_1 - x_2|^\gamma$ for some constant $K > 0$). Our results are the following.

Theorem 1.1. *Let $\omega \in \mathcal{M}^+_c(\mathbb{R}^2)$ and suppose G^ω, given by (3), is Hölder continuous with some exponent $0 < \gamma \le 1$. Then $\omega \in H^{-1}(\mathbb{R}^2)$.*

On the other hand, the mere continuity of G^ω is insufficient.

Proposition 1.2. *There exists $\omega \in \mathcal{M}^+_c(\mathbb{R}^2)$ such that $\omega \notin H^{-1}(\mathbb{R}^2)$ and G^ω, given by (3), is absolutely continuous and bounded.*

The reason for considering the radial cumulative distribution functions stems from the fact that spirals of vorticity can be conveniently expressed in terms of G^ω, see [4]. In particular, for G^ω being a monomial, Cieślak and Szumańska obtained the following result.

Theorem 1.3. *(See Theorem 1.1 from [4].) Let μ be a positive Radon measure supported in a ball $B(0,R_0) \subset \mathbb{R}^2$. Assume that there exists a positive constant c_1 such that for any $r \le R_0$*

$$\mu(B(0,r)) = c_1 r^\alpha, \text{ where } \alpha > 0.$$

Then $\mu \in H^{-1}(\mathbb{R}^2)$.

As an application, they showed that the Prandtl spiral, whose both positive and negative branches satisfy $\mu(B(0,r)) \propto r^2$ belongs locally to $H^{-1}(\mathbb{R}^2)$.

Theorem 1.1 is a generalization of Theorem 1.3 and thus it covers more complex spirals of vorticity. Moreover, it allows simple handling of Cantor-type measures and may be useful for numerical purposes, see Section 4. Finally, as a side application of Theorem 1.1, we obtain the following embedding.

Proposition 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be nondecreasing and Hölder continuous. Then f belongs locally to the fractional Sobolev space $H^{1/2}(\mathbb{R})$.*

The problem of compact embedding of spaces of measures into $H^{-1}(\mathbb{R}^2)$ dates back to the original works of DiPerna and Majda, who proved certain estimates on the vorticity maximal function, see [6, Theorem 3.1]. In the same vein, various other spaces were studied in [11], see also [14]. This regards for example the so-called Morrey spaces of measures. One possible strategy to prove Theorem 1.1 is to reinterpret the condition $G^\omega \in C^{0,\gamma}$ as a condition for ω to belong to a certain Morrey space, see Section 3. Nevertheless, our aim in this paper is to prove Theorem 1.1 elementarily, without resorting to the more complicated language of Morrey spaces. The proof is based on an explicit characterization of $\mathcal{M}^+_c \cap H^{-1}$, see Lemma 2.1, proved originally in [13], and provides more general sufficient conditions for a measure to belong to $\mathcal{M}^+_c \cap H^{-1}$, see Lemma 2.2, than the proof involving Morrey spaces. The paper is organized as follows. In Section 2, we prove Theorem 1.1. Section 3 is devoted to proofs of Proposition 1.2 and Proposition 1.4 as well as an alternative proof of Theorem 1.1. Finally, in Section 4, we discuss two applications.

2. Proof of Theorem 1.1

Define the positive logarithmic energy of a measure $\omega \in \mathcal{M}^+(\mathbb{R}^2)$ by

$$\mathcal{H}^+(\omega) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log^+ \frac{1}{|x-y|}\,\omega(\mathrm{d}x)\omega(\mathrm{d}y), \tag{4}$$

where $\log^+(x) = \max(\log(x), 0)$. The main tool for demonstrating Theorem 1.1 is the following crucial characterization proved by Schochet in [13], which builds upon previous ideas of Delort [5], see also [9].

Lemma 2.1. *(See Lemma 3.1 in [13].) Let ω be a nonnegative measure of finite mass and compact support, and let $u = K * \omega = \frac{1}{2\pi}\iint \frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)$ be the velocity corresponding to the vorticity ω. Then the following are equivalent:*

(i) *ω is in H^{-1}.*
(ii) *u is in L^2_{loc}.*
(iii) *$\mathcal{H}^+(\omega) < \infty$.*

To prove Theorem 1.1, it suffices then to show that the right-hand side of (4) is finite. Using inequality $\log^+ \frac{1}{|x-y|} \le \log^+ \frac{1}{||x|-|y||}$ and the fact that for every Borel function $h : [0,\infty) \to [0,\infty)$ equality

$$\int_{\mathbb{R}^2} h(|x|)\,\omega(\mathrm{d}x) = \int_{[0,\infty)} h(r)\,\mathrm{d}G^\omega(r) \tag{5}$$

holds (the integral on the right-hand side is understood in the Lebesgue–Stjeltjes sense), we calculate:

$$\begin{aligned}
\mathcal{H}^+(\omega) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log^+ \frac{1}{|x-y|}\,\omega(\mathrm{d}x)\,\omega(\mathrm{d}y) \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log^+ \frac{1}{||x|-|y||}\,\omega(\mathrm{d}x)\,\omega(\mathrm{d}y) \\
&= \int_{\mathbb{R}^2}\left[\int_{[0,\infty)} \log^+ \frac{1}{|r_x-|y||}\,\mathrm{d}G^\omega(r_x)\right]\omega(\mathrm{d}y) = \int_{[0,\infty)}\int_{[0,\infty)} \log^+ \frac{1}{|r_x-r_y|}\,\mathrm{d}G^\omega(r_x)\,\mathrm{d}G^\omega(r_y) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \log^+ \frac{1}{|r_x-r_y|}\,\mathrm{d}G^\omega(r_x)\,\mathrm{d}G^\omega(r_y) = \mathcal{H}^+(\mathrm{d}G^\omega),
\end{aligned}$$

where we denote $\mathcal{H}^+(\mathrm{d}F) := \int_{\mathbb{R}}\int_{\mathbb{R}} \log^+ \frac{1}{|x-y|}\,\mathrm{d}F(x)\,\mathrm{d}F(y)$. Theorem 1.1 follows now from the following one-dimensional result.

Lemma 2.2. *Suppose a bounded continuous nondecreasing $F : \mathbb{R} \to [0,\infty)$ satisfies:*

i) *$(F(x+\varepsilon) - F(x))\log\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in x,*
ii) *$(F(x-\varepsilon) - F(x))\log\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in x,*
iii) *$\int_0^1 \frac{F(x+y)-F(x)}{y}\,\mathrm{d}y \le C$ uniformly in x,*
iv) *$\int_0^1 \frac{F(x)-F(x-y)}{y}\,\mathrm{d}y \le C$ uniformly in x.*

Then

$$\mathcal{H}^+(\mathrm{d}F) = \int_{\mathbb{R}}\left(\int_0^1 \frac{1}{y}(F(x+y)-F(x-y))\mathrm{d}y\right)\mathrm{d}F(x)$$

and in particular, $\mathcal{H}^+(\mathrm{d}F) < +\infty$.

Indeed, $F \in C^{0,\gamma}$ satisfies i)–iv) since $|F(x\pm\varepsilon) - F(x)|\log(\varepsilon) \le K\varepsilon^\gamma \log(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\int_0^1 \frac{|F(x\pm y)-F(x)|}{y}\,\mathrm{d}y \le K\int_0^1 y^{\gamma-1}\,\mathrm{d}y = K/\gamma$. This proves Theorem 1.1. $\square$

Proof of Lemma 2.2. Using the properties of Lebesgue–Stieltjes integrals (see [2]), we obtain:

$$
\begin{aligned}
\mathcal{H}^+(\mathrm{d}F) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \log^+ \frac{1}{|x-y|}\,\mathrm{d}F(x)\mathrm{d}F(y) = \int_{\mathbb{R}}\left[\int_{x-1}^{x+1} \log\frac{1}{|x-y|}\,\mathrm{d}F(y)\right]\mathrm{d}F(x)\\
&= \int_{\mathbb{R}}\left[\int_{-1}^{1} \log\frac{1}{|y|}\,\mathrm{d}F(x+y)\right]\mathrm{d}F(x) = \int_{\mathbb{R}}\left[\int_{0}^{1} \log\left(\frac{1}{y}\right)\mathrm{d}(F(x+y)-F(x-y))\right]\mathrm{d}F(x)\\
&= \int_{\mathbb{R}}\int_{0}^{1} \log\left(\frac{1}{y}\right)\mathrm{d}(F(x+y)-F(x))\,\mathrm{d}F(x) + \int_{\mathbb{R}}\int_{0}^{1} \log\left(\frac{1}{y}\right)\mathrm{d}(F(x)-F(x-y))\,\mathrm{d}F(x)\\
&= \int_{\mathbb{R}}\lim_{\varepsilon\to0}\left[\int_{\varepsilon}^{1} \log\left(\frac{1}{y}\right)\mathrm{d}(F(x+y)-F(x))\right]\mathrm{d}F(x) + \int_{\mathbb{R}}\lim_{\varepsilon\to0}\left[\int_{\varepsilon}^{1} \log\left(\frac{1}{y}\right)\mathrm{d}(F(x)-F(x-y))\right]\mathrm{d}F(x)\\
&= \int_{\mathbb{R}}\lim_{\varepsilon\to0}\left[\left[\log\left(\frac{1}{y}\right)(F(x+y)-F(x))\right]_{\varepsilon}^{1} + \int_{\varepsilon}^{1}\frac{1}{y}(F(x+y)-F(x))\,\mathrm{d}y\right]\mathrm{d}F(x)\\
&\quad + \int_{\mathbb{R}}\lim_{\varepsilon\to0}\left[\left[\log\left(\frac{1}{y}\right)(F(x)-F(x-y))\right]_{\varepsilon}^{1} + \int_{\varepsilon}^{1}\frac{1}{y}(F(x)-F(x-y))\,\mathrm{d}y\right]\mathrm{d}F(x)\\
&= \int_{\mathbb{R}}\left(\int_{0}^{1}\frac{1}{y}(F(x+y)-F(x)+F(x)-F(x-y))\,\mathrm{d}y\right)\mathrm{d}F(x) \le 2C\int_{\mathbb{R}}\mathrm{d}F(x),
\end{aligned}
$$

where in the last equality we used the Lebesgue-dominated convergence theorem and the fact that measure $\mathrm{d}F$ is bounded. □

Remark 1. Proofs of Lemma 2.2 and Theorem 1.1 show that if F satisfies $|F(x+y)-F(x)| \le K|y|^{\gamma}$, then

$$\mathcal{H}^+(\mathrm{d}F) \le 2(K/\gamma)\omega(\mathbb{R}^2).$$

Remark 2. Conditions i)–iv) from Lemma 2.2 encompass a larger class of functions than functions that are Hölder continuous. For instance, it suffices to assume that $|F(x+y)-F(x)| \le 1/|\log(|y|)|^{\beta}$ for $|y| \le \varepsilon$, $x\in\mathbb{R}$ and fixed $\beta > 1$ and $\varepsilon > 0$. Such a regularity can be interpreted in terms of Morrey spaces $M^{(1;\alpha)}$ studied in [11], see also [6, Theorem 3.1]. A result going beyond the results from [11] can be obtained by assuming, e.g.,

$$|F(x+y)-F(x)| \le 1/|\log(|y|)|(|\log(|\log(|y|)|)|)^{\beta}$$

for $|y| \le \varepsilon$, $x\in\mathbb{R}$ and fixed $\beta > 1$ and $\varepsilon > 0$.

3. Proofs of Proposition 1.2, Proposition 1.4 and an alternative proof of Theorem 1.1

Proof of Proposition 1.2. Take $\omega = f(x_1)\mathrm{d}x_1\delta_0(\mathrm{d}x_2)$ and suppose f has the form

$$f(x) = \sum_{n=1}^{\infty} h_n \mathbb{1}_{[a_n,a_n+d_n]}(x),$$

where $a_n\in\mathbb{R}$, $h_n > 1$, $0 < d_n \le 1$, $a_n + d_n \le a_{n+1}$ for every $n = 1, 2, \ldots$, and $\mathbb{1}_A(x)$ is equal 1 if $x \in A$ and 0 otherwise. Observe that

$$\mathcal{H}^+(h\mathbb{1}_{[a,a+d]}(x_1)\,\mathrm{d}x_1\,\delta_0(\mathrm{d}x_2)) \ge \int_a^{a+d}\int_a^{a+d}\log^+\frac{1}{|x-y|}h^2\,\mathrm{d}x\,\mathrm{d}y \ge h^2d^2\log(1/d)$$

and hence

$$\mathcal{H}^+(\omega) \ge \sum_{n=1}^{\infty} h_n^2 d_n^2 \log(1/d_n). \tag{6}$$

Take $d_n = \exp(-2^{2n})$ and $h_n = 1/(2^n d_n)$. Then, on the one hand, $\|f\|_{L^1} = \sum_{n=1}^{\infty} h_n d_n = 1$. On the other hand, however, by (6):

$$\mathcal{H}^+(\omega) \geq \sum_{n=1}^{\infty} 2^{-2n} \log(1/d_n) = +\infty.$$

To conclude the proof, we observe that G^ω for ω defined above is absolutely continuous. □

Remark 3. It can be shown that the function f constructed in the proof of Proposition 1.2 belongs in fact to the Zygmund class $L(\log L)^\gamma$, for every $\gamma < 1/2$. Note that for $\gamma \geq 1/2$, space $L(\log L)^\gamma$ can be embedded into H^{-1}, see [3].

Proof of Proposition 1.4. The space of distributions belonging to $H^{-1}(\mathbb{R}^2)$, which are supported on the line $\{(x_1, 0) : x_1 \in \mathbb{R}\}$ can be identified with the fractional Sobolev space $H^{-1/2}(\mathbb{R})$. This follows by the fact that the trace operator $T : W^{1,2}(\mathbb{R}^2) \to H^{1/2}(\mathbb{R})$ is bounded and has a bounded right inverse, see [15, Section 16], and the identification operator $R : H^{-1}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R})$ is defined by requiring identity $R(S)(T(\phi)) = S(\phi)$ to hold for every test function $\phi \in W^{1,2}(\mathbb{R}^2)$.

Let now f be Hölder continuous and nondecreasing on $\mathbb{R}$ and suppose without loss of generality that f is constant for $x \leq 0$ and $x \geq 1$. Then measure $\omega := \mathrm{d}f(x_1)\,\delta_0(\mathrm{d}x_2)$ belongs to $H^{-1}(\mathbb{R}^2)$ by Theorem 1.1. Since ω is identified with $\mathrm{d}f(x_1)$, we obtain that $\mathrm{d}f \in H^{-1/2}(\mathbb{R})$. Consequently, $f \in H^{1/2}(\mathbb{R})$. □

Alternative proof of Theorem 1.1 using Morrey spaces. A (signed) measure μ on $\mathbb{R}^2$ belongs to the Morrey space $\widetilde{M}^p$ if

$$\|\mu\|_{\widetilde{M}^p} := \sup_{R>0} R^{-2/p'} \sup_{x\in\mathbb{R}^2} |\mu|(B(x,R)) < \infty,$$

where p' satisfies $1/p + 1/p' = 1$. Theorem 4.3 from [11] asserts that for bounded $\Omega \subset \mathbb{R}^2$ the space $\widetilde{M}^p(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$ for every $p > 1$. Let now $\omega \in \mathcal{M}_c^+(\mathbb{R}^2)$ and suppose G^ω, given by (3), is Hölder continuous with some exponent $0 < \gamma \leq 1$. Then, for $|x| > r$, we have:

$$\omega(B(x,r)) \leq G^\omega(|x|+r) - G^\omega(|x|-r) \leq 2Kr^\gamma$$

due to inclusion $B(x,r) \subset B(0,|x|+r) \backslash B(0,|x|-r)$. For $|x| \leq r$, we obtain similarly:

$$\omega(B(x,r)) \leq G^\omega(2r) = G^\omega(2r) - G^\omega(0) \leq 2Kr^\gamma.$$

Thus, $\omega \in \widetilde{M}^p$ for p satisfying $2/p' = \gamma$. Consequently, $p = 2/(2-\gamma) > 1$ and hence $\omega \in H^{-1}(\mathbb{R}^2)$. □

4. Applications

Proposition 4.1. *A nonnegative Radon measure belonging to $H^{-1}(\mathbb{R}^2)$ may be supported on a set of arbitrary small positive Hausdorff dimension.*

Sketch of proof. Consider a Cantor set $C_K \subset [0,1]$ obtained by removing in every step of the construction the middle $(K-2)/K$ portion of every interval (note that for $K = 3$ we obtain the standard ternary Cantor set). Let $\Gamma_K(r)$ be the corresponding Cantor function and consider the measure

$$\omega_K = \mathrm{d}\Gamma_K(x_1)\,\delta_0(\mathrm{d}x_2).$$

Then ω_K is supported on the closed set C_K of dimension $\alpha = \log(2)/\log(K)$. Moreover, $G^{\omega_K}(r) = \Gamma_K(r)$ is Hölder continuous with exponent $\alpha = \log(2)/\log(K)$, see, e.g., [8]. By Theorem 1.1, $\omega_K \in H^{-1}(\mathbb{R}^2)$. □

Adapting the above construction, we can prove that a measure belonging to $H^{-1}(\mathbb{R}^2)$ may be supported on a set of Hausdorff dimension 0. We postpone the rigorous proof to further work and close the paper with a brief discussion of numerical applications of Theorem 1.1.

Remark 4 *(Numerical applications).* From the point of view of proving the convergence of numerical schemes, it is important to know that ω^n, a sequence of approximations of a compactly supported measure

$$\omega \in \mathcal{M}_c^+(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2),$$

is such that $\mathcal{H}^+(\omega^n)$ remains bounded uniformly in n (see, e.g., [13] or [10]). Let, for instance, ω be the positive branch of the Kaden spiral (see [4]) at some point in time. Then function $r \mapsto \omega(B(0,r))$ is Hölder continuous with exponent $\gamma = 1/2$ (see [4]) and hence belongs locally to $H^{-1}(\mathbb{R}^2)$. Let ω_n be a smooth approximation of ω, e.g. a vortex blob approximation, see [10]. To prove that $\mathcal{H}^+(\omega^n)$ is bounded uniformly with respect to n it suffices, by Remark 1, to show that functions

$$r \mapsto \omega^n(B(0,r))$$

are uniformly Hölder continuous with constant K and exponent γ independent of n. Whether this is the case depends on a particular form of vortex blob approximation. The goal is then to construct an approximation that satisfies the uniform Hölder condition. This, however, is relatively simple, since $r \mapsto \omega(B(0,r))$ is Hölder continuous. For an alternative approach, we refer the reader, e.g., to [9, Theorem 3.1].

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