C. R. Acad. Sci. Paris, Ser. I 353 (2015) 517–522

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Harmonic analysis

# Fractional Laplacians, extension problems and Lie groups

## *Laplaciens fractionnaires, problèmes d'extension et groupes de Lie*

Diego Chamorro, Oscar Jarrín

*Laboratoire de mathématiques et modélisation d'Évry (LaMME), UMR 8071, Université d'Évry-Val-d'Essonne, 23, boulevard de France, 91037 Évry cedex, France*

ARTICLE INFO

*Article history:*
Received 13 January 2015
Accepted after revision 13 April 2015
Available online 27 April 2015

Presented by Haïm Brézis

ABSTRACT

We generalize some results concerning the fractional powers of the Laplace operator to the setting of nilpotent Lie Groups and we study its relationship with the solutions to a partial differential equation in the spirit of the articles of Caffarelli & Silvestre [1] and Stinga & Torrea [7].



RÉSUMÉ

Nous généralisons aux groupes de Lie nilpotents les travaux de Caffarelli & Silvestre [1] et Stinga & Torrea [7] concernant la relation existant entre les puissances fractionnaires de l'opérateur laplacien et les solutions d'une équation aux dérivées partielles.



**Version française abrégée**

Nous présentons ici une extension au cas des groupes de Lie à croissance polynômiale du volume des relations existant entre les solutions $u(t,x)$ de l'équation aux dérivées partielles

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) + \Delta_x u(t,x) = 0, \text{ où } s \in ]0,1[, t > 0, \text{ et } u(0,x) = \varphi(x), \text{ avec } \varphi \in \mathcal{S}(\mathbb{R}^n),$$

et les puissances fractionnaires du laplacien de la donnée initiale $\varphi$. Cette relation est donnée par l'expression

$$\lim_{t\to 0^+} t^{1-2s}\partial_t u(t,x) = -C(s)(-\Delta)^s \varphi(x) \qquad (0 < s < 1).$$

Étudiée tout d'abord par Caffarelli & Silvestre dans [1], cette relation a été généralisée très rapidement, soit en considérant différentes familles d'opérateurs, soit en travaillant sur des cadres plus généraux (voir [7], [2], [5], [6] et [3]). Nous adoptons ici un point de vue intermédiaire en travaillant sur les groupes de Lie nilpotents : en effet, dans ce cadre, il est intéressant de remarquer qu'il n'existe pas une manière canonique de définir un opérateur laplacien. Nous considérerons alors une famille d'opérateurs laplaciens du type $\mathcal{J} = -\sum_{j=1}^k X_j^2$, où les champs de vecteurs invariants à gauche $(X_j)_{1\le j\le k}$ vérifient la condition de Hörmander. Pour ce type de laplaciens, nous démontrons dans cette note le théorème suivant.

*E-mail address:* diego.chamorro@univ-evry.fr (D. Chamorro).

http://dx.doi.org/10.1016/j.crma.2015.04.007

**Théorème 0.1.** *Soit $\mathbb{G}$ un groupe de Lie nilpotent et soit $\mathcal{J}$ un laplacien qui vérifie la condition de Hörmander. Si $\varphi \in \mathcal{S}(\mathbb{G})$ est une fonction dans la classe de Schwartz, on considère l'équation*

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) - \mathcal{J}u(t,x) = 0, \text{ pour } x \in \mathbb{G} \text{ avec } s \in ]0,1[, t > 0 \text{ et } u(0,x) = \varphi(x).$$

*Alors la fonction* $u(t,x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} H_\tau \mathcal{J}^s \varphi(x) \, e^{\frac{-t^2}{4\tau}} \frac{d\tau}{\tau^{1-s}}$ *est une solution au sens $L^p$, avec $1 < p < +\infty$, de l'équation ci-dessus. De plus, on a dans $L^p$ la relation $\mathcal{J}^s\varphi(x) = -C(s)\lim_{t\to 0^+} t^{1-2s}\partial_t u(t,x)$.*

## 1. Introduction

We are interested in a generalization of the relationship between the solutions $u(t,x)$ of the partial differential equation (also called *extension problem*):

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) + \Delta_x u(t,x) = 0 \text{ with } s \in ]0,1[, t > 0 \text{ and } u(0,x) = \varphi(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{1}$$

and the fractional powers of the Laplacian of the initial data $\varphi$. This relationship is given by the expression:

$$\lim_{t\to 0^+} t^{1-2s}\partial_t u(t,x) = -C(s)(-\Delta)^s\varphi(x) \qquad (0 < s < 1). \tag{2}$$

This formula was first studied by Caffarelli & Silvestre [1] in 2007 and since then this work has been generalized to many different frameworks (see [7], [2] and [3]). Our aim here is to generalize the relationship between (1) and (2) to the setting of nilpotent Lie groups and in the framework of $L^p$ spaces. In the recent article [2], this relationship is studied in the setting of the Carnot groups and we want to give here a different point of view that is based on the fact that there is not a *unique* way to define a Laplace operator in this framework. Here is an example: the Heisenberg group $\mathbb{H}$ is given by $\mathbb{R}^3$ with the non-commutative group law $x \cdot y = \big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - y_1x_2)\big)$. Associated with this group, we have a Lie algebra $\mathfrak{h}$ given by the left-invariant vector fields $X_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_3}$, $X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3}$ and $T = \frac{\partial}{\partial x_3}$ and we have the identities $[X_1, X_2] = X_1X_2 - X_2X_1 = T$, $[X_i, T] = [T, X_i] = 0$ where $i = 1, 2$. We say that $X_1$ and $X_2$ form the *first layer* of the stratification of the Lie algebra $\mathfrak{h}$ while $T$ lies in the *second layer* of the stratification of $\mathfrak{h}$. Now if we want to build from these vector fields an equivalent of the Laplace operator in $\mathbb{H}$, we have several choices:

$$\mathcal{J}_1 = -(X_1^2 + X_2^2), \quad \mathcal{J}_2 = -(X_1^2 + X_2^2 + T), \quad \text{or} \quad \mathcal{J}_3 = -(X_1^2 + X_2^2 + T^2). \tag{3}$$

In [2] the Laplacian was built solely from the vector fields of the *first* layer of the stratification (*i.e.* $\mathcal{J}_1$). In this article we will study this relationship taking into account other types of Laplace operators.

## 2. Presentation of the framework and statement of the results

Let $\mathbb{G}$ be a connected unimodular Lie group endowed with its Haar measure $dx$. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$ and consider a family (that will be fixed from now on) of left-invariant vector fields on $\mathbb{G}$

$$\mathbf{X} = \{X_1, \ldots, X_k\}, \tag{4}$$

satisfying the *Hörmander condition*: the Lie algebra generated by the $X_j$ for $1 \le j \le k$ is $\mathfrak{g}$. We also have at our disposal the Carnot–Carathéodory metric associated with $\mathbf{X}$, see [8] for a definition. We will denote $\|x\|$ the distance between the origin $e$ and $x$ and $\|y^{-1}\cdot x\|$ the distance between $x$ and $y$.

For $r > 0$ and $x \in \mathbb{G}$, denote by $B(x,r)$ the open ball with respect to the Carnot–Carathéodory metric centered in $x$ and of radius $r$, and by $V(r)$ the Haar measure of any ball of radius $r$. When $0 < r < 1$, there exists $d \in \mathbb{N}^*$, $c_l$ and $C_l > 0$ such that, for all $0 < r < 1$, we have $c_l r^d \le V(r) \le C_l r^d$. The integer $d$ is the *local* dimension of $(\mathbb{G}, \mathbf{X})$. When $r \ge 1$, we will assume that $\mathbb{G}$ is of *polynomial volume growth*, *i.e.* there exist $D \in \mathbb{N}^*$, $c_\infty$ and $C_\infty > 0$ such that, for all $r \ge 1$ we have $c_\infty r^D \le V(r) \le C_\infty r^D$. When $\mathbb{G}$ has polynomial volume growth, the integer $D$ is called the dimension at infinity of $\mathbb{G}$. Recall that nilpotent groups have polynomial volume growth and that a *strict* subclass of the nilpotent groups consists of stratified Lie groups where $d = D$. For example, in the case of the Heisenberg group, we have $d = D = 4$.

Once we have fixed the family $\mathbf{X}$, we define the gradient on $\mathbb{G}$ by $\nabla = (X_1, \ldots, X_k)$ and we consider a Laplacian $\mathcal{J}$ on $\mathbb{G}$ defined by

$$\mathcal{J} = -\sum_{j=1}^{k} X_j^2, \tag{5}$$

which is a positive self-adjoint, hypo-elliptic operator since $\mathbf{X}$ satisfies Hörmander's condition, see [8]. Its associated heat operator on $]0, +\infty[ \times \mathbb{G}$ is given by $\partial_t + \mathcal{J}$ and we will denote by $(H_t)_{t>0}$ the semi-group obtained from the Laplacian $\mathcal{J}$.

Using the spectral theory associated with this Laplacian $\mathcal{J}$, we can define for $0 < s < 1$ the operator $\mathcal{J}^s$ (see the details in Section 3).

From now on we will work with a unimodular nilpotent Lie group $\mathbb{G}$ with a Haar measure d$x$ and with a family of left-invariant vector fields $\mathbf{X}$ that satisfy the Hörmander condition. Here $d$ and $D$ are the local dimension and the dimension at infinity respectively and we will fix the Laplace operator $\mathcal{J}$ given by the formula (5).

**Theorem 2.1.** *Let $\mathbb{G}$ be a nilpotent Lie group and let $\mathcal{J}$ be a Laplace operator. If $\varphi \in \mathcal{S}(\mathbb{G})$ is a smooth function and if we consider the following problem*

$$\partial_t^2 u(t,x) + \frac{1-2s}{t}\partial_t u(t,x) - \mathcal{J}u(t,x) = 0, \text{ for } x \in \mathbb{G} \text{ with } s \in ]0,1[, t > 0 \text{ and } u(0,x) = \varphi(x). \tag{6}$$

*Then the function $u : ]0,+\infty[ \times \mathbb{G} \longrightarrow \mathbb{R}$ given by the formula*

$$u(t,x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} H_\tau \mathcal{J}^s \varphi(x)\, e^{\frac{-t^2}{4\tau}} \frac{d\tau}{\tau^{1-s}}, \tag{7}$$

*is a $L^p$-solution, with $1 < p < +\infty$, of the extension problem (6). Moreover, we have in a $L^p$-sense the following relationship between the solution to this problem and the fractional powers of the Laplacian:*

$$\mathcal{J}^s \varphi(x) = -C(s) \lim_{t \to 0^+} t^{1-2s} \partial_t u(t,x).$$

## 3. Some tools and related properties

Since the Laplacian given by (5) is a positive self-adjoint operator, it admits a spectral decomposition of the form $\mathcal{J} = \int_0^{+\infty} \lambda\, dE(\lambda)$. This spectral decomposition allows us to define the fractional powers of the Laplacian by the expression $\mathcal{J}^s = \int_0^{+\infty} \lambda^s\, dE(\lambda)$, with $0 < s$. This formula is very useful to deduce some properties of the fractional Laplacian like $\mathcal{J}^{s_1}[\mathcal{J}^{s_2} f] = \mathcal{J}^{s_1+s_2} f$ whenever these quantities are well defined. But it will also help us to build a family of operators $m(\mathcal{J})$ associated with a Borel function $m$.

**Proposition 3.1.** *(See [4].) Let $k \in \mathbb{N}$ and $m \in \mathcal{C}^k(\mathbb{R}^+)$, we write $\|m\|_{(k)} = \sup_{0 \le r \le k, 0 < \lambda} (1+\lambda)^k |m^{(r)}(\lambda)|$. We define the operator $m(t\mathcal{J})$ for $t > 0$ by the expression $m(t\mathcal{J}) = \int_0^{+\infty} m(t\lambda)\, dE(\lambda)$. Then this operator admits a convolution kernel $M_t$ and moreover, for $\alpha \in \mathbb{R}$ and $I$ a multi-index, there exists $C > 0$ and $k \in \mathbb{N}$ such that $\left\|(1 + \|\cdot\|)^\alpha X^I M_t(\cdot)\right\|_{L^p} \le C\, t^{-|I|/2}(1+\sqrt{t})^\alpha [V(\sqrt{t})]^{-1/p'} \|m\|_{(k)}$ with $1/p + 1/p' = 1$.*

**Lemma 3.2.** *Let $0 < s$ and $1 < p < +\infty$ then we have the inequalities $\|\mathcal{J}^s h_t\|_{L^1} \le Ct^{-s}$ and $\|\mathcal{J}^s h_t\|_{L^p} \le Ct^{-s}[V(\sqrt{t})]^{-1/p'}$ with $1/p + 1/p' = 1$.*

**Proof.** For the $L^1$ estimate we write

$$\mathcal{J}^s h_t(x) = \mathcal{J}^s(h_{t/2} * h_{t/2})(x) = Ct^{-s}\left(\int_0^{+\infty} m(t\lambda)\, dE(\lambda)\right)(h_{t/2})(x) = Ct^{-s} M_t * h_{t/2}(x),$$

where $m(\lambda) = \lambda^s e^{-\lambda}$. Then, taking the $L^1$-norm, using Young's inequality and applying Proposition 3.1, we obtain $\|\mathcal{J}^s h_t\|_{L^1} \le Ct^{-s}\|M_t\|_{L^1}\|h_{t/2}\|_{L^1} = Ct^{-s}\|m\|_{(0)} = Ct^{-s}$. The $L^p$ estimate is similar. □

## 4. Proof of the main theorem

We follow some of the ideas of [7] and we consider the general case of $L^p$ spaces with $1 < p < +\infty$. Our first task is to proof that each term of the expressions (6)–(7) is well defined in a $L^p$-sense. With Lemmas 4.1 and 4.2 below, we give more natural proofs adapted to the setting of nilpotent Lie groups.

**Lemma 4.1.** *Let $\varphi \in \mathcal{S}(\mathbb{G})$ and $t > 0$, then the function $u(t,x)$ given by (7) belongs to $L^p(\mathbb{G})$.*

**Proof.** By duality we consider $g \in L^{p'}(\mathbb{G})$ and we study the quantity

$$\langle u(t,\cdot); g\rangle_{L^p\times L^{p'}} = \int_{\mathbb{G}} \left(\frac{1}{\Gamma(s)}\int_0^{+\infty} H_\tau \mathcal{J}^s\varphi\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) g\, dx = \int_{\mathbb{G}} \left(\frac{1}{\Gamma(s)}\int_0^{1} H_\tau \mathcal{J}^s\varphi\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) g\, dx$$
$$+ \int_{\mathbb{G}} \left(\frac{1}{\Gamma(s)}\int_1^{+\infty} H_\tau \mathcal{J}^s\varphi\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) g\, dx$$
$$= \int_{\mathbb{G}} \left(\frac{1}{\Gamma(s)}\int_0^{1} \mathcal{J}^{\frac{s}{2}}\varphi * \mathcal{J}^{\frac{s}{2}}h_\tau\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) g\, dx + \int_{\mathbb{G}} \left(\frac{1}{\Gamma(s)}\int_1^{+\infty} \varphi * \mathcal{J}^{s}h_\tau\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) g\, dx.$$

Thus, by Young's inequality we obtain:

$$\left|\langle u(t,\cdot); g\rangle_{L^p\times L^{p'}}\right| \le \frac{1}{\Gamma(s)}\|g\|_{L^{p'}}\left(\int_0^1 \|\mathcal{J}^{\frac{s}{2}}\varphi\|_{L^p}\|\mathcal{J}^{\frac{s}{2}}h_\tau\|_{L^1}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}} + \int_1^{+\infty}\|\varphi\|_{L^1}\|\mathcal{J}^s h_\tau\|_{L^p}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right).$$

Now we use the estimates given by Lemma 3.2 for the heat kernel:

$$\left|\langle u(t,\cdot); g\rangle_{L^p\times L^{p'}}\right| \le \frac{1}{\Gamma(s)}\|g\|_{L^{p'}}\left(C\int_0^1 \|\mathcal{J}^{\frac{s}{2}}\varphi\|_{L^p}\tau^{-\frac{s}{2}}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}} + C'\int_1^{+\infty}\|\varphi\|_{L^1}\tau^{-s}[V(\sqrt{\tau})]^{-\frac{1}{p'}}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right)$$
$$\le \frac{C''}{\Gamma(s)}\|g\|_{L^{p'}}\left(\|\mathcal{J}^{\frac{s}{2}}\varphi\|_{L^p}\int_0^1 \tau^{\frac{s}{2}-1}d\tau + \|\varphi\|_{L^1}\int_1^{+\infty}\tau^{-\frac{D}{2p'}-1}d\tau\right)$$
$$\le \frac{C}{\Gamma(s)}\|g\|_{L^{p'}}\left(\|\mathcal{J}^{\frac{s}{2}}\varphi\|_{L^p} + \|\varphi\|_{L^1}\right),$$

and we obtain that the function $u(t,x)$ is in $L^p(\mathbb{G})$ for all $t>0$. □

**Lemma 4.2.** *$u(t,x)$ belongs to the domain of the operator $\mathcal{J}$.*

**Proof.** We study the following quantity

$$\left\langle\frac{H_\rho u(t,\cdot)-u(t,\cdot)}{\rho}; g\right\rangle_{L^p\times L^{p'}} = \frac{1}{\Gamma(s)}\int_{\mathbb{G}}\frac{1}{\rho}\left(H_\rho\left(\int_0^{+\infty}H_\tau\mathcal{J}^s\varphi\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) - \int_0^{+\infty}H_\tau\mathcal{J}^s\varphi\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right) g\, dx$$
$$= \frac{1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}\left(\frac{H_{\rho+\tau}\mathcal{J}^s\varphi - H_\tau\mathcal{J}^s\varphi}{\rho}\right) e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\, g\, dx.$$

Taking $\rho\to 0^+$ we have

$$\lim_{\rho\to 0^+}\left\langle\frac{H_\rho u(t,\cdot)-u(t,\cdot)}{\rho}; g\right\rangle_{L^p\times L^{p'}} = \frac{1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}\mathcal{J}H_\tau\mathcal{J}^s\varphi\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\, g\, dx$$
$$\le \frac{\|g\|_{L^{p'}}}{\Gamma(s)}\left(\int_0^1\|\mathcal{J}^{s+1}H_\tau\varphi\|_{L^p}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}} + \int_1^{+\infty}\|\mathcal{J}^{s+1}H_\tau\varphi\|_{L^p}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right)$$
$$\le \frac{\|g\|_{L^{p'}}}{\Gamma(s)}\left(\int_0^1\|\mathcal{J}^{\frac{s}{2}+1}\varphi\|_{L^p}\|\mathcal{J}^{\frac{s}{2}}h_\tau\|_{L^1}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right.$$
$$\left.+\int_1^{+\infty}\|\varphi\|_{L^p}\|\mathcal{J}^{s+1}h_\tau\|_{L^1}\, e^{-\frac{t^2}{4\tau}}\frac{d\tau}{\tau^{1-s}}\right).$$

Now, using inequalities stated in Lemma 3.2 for the heat kernel we obtain:

$$\lim_{\rho\to 0^+}\left\langle \frac{H_\rho u(t,\cdot)-u(t,\cdot)}{\rho}; g\right\rangle_{L^p\times L^{p'}} \leq \frac{C\|g\|_{L^{p'}}}{\Gamma(s)}\left(\|\mathcal{J}^{\frac{s}{2}+1}\varphi\|_{L^p}+\|\varphi\|_{L^p}\right) < +\infty. \qquad \square$$

**Proposition 4.3.** *The function $u(t,x)$ defined by (7) and associated with an initial data $\varphi\in\mathcal{S}(\mathbb{G})$ satisfies, in the $L^p$-sense, the equation $\partial_t^2 u(t,x)+\frac{1-2s}{t}\partial_t u(t,x)-\mathcal{J}u(t,x)=0$.*

**Proof.** Let $g\in L^{p'}(\mathbb{G})$, we have:

$$\begin{aligned}
&\left\langle \partial_t^2 u(t,\cdot)+\frac{1-2s}{t}\partial_t u(t,\cdot)-\mathcal{J}u(t,\cdot); g\right\rangle_{L^p\times L^{p'}}\\
&=\int_{\mathbb{G}}\frac{1}{\Gamma(s)}\int_0^{+\infty} H_\tau\mathcal{J}^s\varphi\left(\frac{t^2}{4\tau^2}-\frac{1}{2\tau}\right)\mathrm{e}^{-\frac{t^2}{4\tau}}\frac{\mathrm{d}\tau}{\tau^{1-s}}g\,\mathrm{d}x\\
&\quad+\int_{\mathbb{G}}\frac{1-2s}{t}\left(\frac{-1}{\Gamma(s)}\int_0^{+\infty}H_\tau\mathcal{J}^s\varphi\frac{t\,\mathrm{e}^{-\frac{t^2}{4\tau}}}{2\tau}\frac{\mathrm{d}\tau}{\tau^{1-s}}\right)g\,\mathrm{d}x-\int_{\mathbb{G}}\mathcal{J}u(t,\cdot)g\,\mathrm{d}x\\
&=\int_{\mathbb{G}}\left(\frac{1}{\Gamma(s)}\int_0^{+\infty}H_\tau\mathcal{J}^s\varphi\left(\frac{t^2}{4\tau^2}+\frac{s-1}{\tau}\right)\mathrm{e}^{-\frac{t^2}{4\tau}}\frac{\mathrm{d}\tau}{\tau^{1-s}}\right)g\,\mathrm{d}x-\int_{\mathbb{G}}\mathcal{J}u(t,\cdot)g\,\mathrm{d}x.
\end{aligned}$$

At this point, we can perform an integration by parts in the first integral with respect to the variable $\tau$ to obtain:

$$\left\langle \partial_t^2 u(t,\cdot)+\frac{1-2s}{t}\partial_t u(t,\cdot)-\mathcal{J}u(t,\cdot); g\right\rangle_{L^p\times L^{p'}}=\frac{-1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}\partial_\tau\left(H_\tau\mathcal{J}^s\varphi\right)\mathrm{e}^{-\frac{t^2}{4\tau}}\frac{\mathrm{d}\tau}{\tau^{1-s}}g\,\mathrm{d}x-\int_{\mathbb{G}}\mathcal{J}u(t,\cdot)g\,\mathrm{d}x.$$

But, since $\partial_\tau H_\tau f=-\mathcal{J}H_\tau f$, we obtain:

$$\begin{aligned}
&\left\langle \partial_t^2 u(t,\cdot)+\frac{1-2s}{t}\partial_t u(t,\cdot)-\mathcal{J}u(t,\cdot); g\right\rangle_{L^p\times L^{p'}}=\frac{1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}\mathcal{J}\left(H_\tau\mathcal{J}^s\varphi\right)\mathrm{e}^{-\frac{t^2}{4\tau}}\frac{\mathrm{d}\tau}{\tau^{1-s}}g\,\mathrm{d}x-\int_{\mathbb{G}}(\mathcal{J}u)g\,\mathrm{d}x\\
&\left\langle \partial_t^2 u(t,\cdot)+\frac{1-2s}{t}\partial_t u(t,\cdot)-\mathcal{J}u(t,\cdot); g\right\rangle_{L^p\times L^{p'}}=\int_{\mathbb{G}}\mathcal{J}u(t,\cdot)g\,\mathrm{d}x-\int_{\mathbb{G}}\mathcal{J}u(t,\cdot)g\,\mathrm{d}x=0. \qquad \square
\end{aligned}$$

It remains to prove the last part of our main theorem and this is done with the next proposition.

**Proposition 4.4.** *Let $\varphi\in\mathcal{S}(\mathbb{G})$ be an initial data of the extension problem (6), let $u(t,x)$ be the function defined by the formula (7), then in the $L^p$-sense we have the limit $\lim_{t\to 0^+}t^{1-2s}\partial_t u(t,x)=-C(s)\mathcal{J}^s\varphi(x)$.*

**Proof.** For $g\in L^{p'}(\mathbb{G})$ we have:

$$\langle t^{1-2s}\partial_t u(t,\cdot); g\rangle_{L^p\times L^{p'}}=\int_{\mathbb{G}}t^{1-2s}\partial_t u(t,x)g(x)\,\mathrm{d}x=\frac{-1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}H_\tau\mathcal{J}^s\varphi(x)\frac{t^{2-2s}}{2\tau^{2-s}}\mathrm{e}^{-\frac{t^2}{4\tau}}\mathrm{d}\tau\,g(x)\,\mathrm{d}x.$$

Making $u=\frac{\tau}{t^2}$, we obtain $\langle t^{1-2s}\partial_t u(t,\cdot); g\rangle_{L^p\times L^{p'}}=\frac{-1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}H_{t^2u}\mathcal{J}^s\varphi(x)\frac{\mathrm{e}^{-\frac{1}{4u}}}{2u^{2-s}}\mathrm{d}u\,g(x)\,\mathrm{d}x$. Now, taking $t\to 0^+$, we have:

$$\begin{aligned}
\lim_{t\to 0^+}\langle t^{1-2s}\partial_t u(t,\cdot); g\rangle_{L^p\times L^{p'}}&=\lim_{t\to 0^+}\frac{-1}{\Gamma(s)}\int_{\mathbb{G}}\int_0^{+\infty}H_{t^2u}\mathcal{J}^s\varphi(x)\frac{\mathrm{e}^{-\frac{1}{4u}}}{2u^{2-s}}\mathrm{d}u\,g(x)\,\mathrm{d}x\\
&=\frac{-1}{\Gamma(s)}\int_{\mathbb{G}}\mathcal{J}^s\varphi(x)\left(\int_0^{+\infty}\frac{\mathrm{e}^{-\frac{1}{4u}}}{2u^{2-s}}\mathrm{d}u\right)g(x)\,\mathrm{d}x=-C(s)\int_{\mathbb{G}}\mathcal{J}^s\varphi(x)g(x)\,\mathrm{d}x. \qquad \square
\end{aligned}$$

**Remark 1.** Nilpotent Lie groups have some special properties compared to general polynomial volume-growth Lie groups, see details in [8]. However, all the properties used here for the heat kernel and for the spectral decomposition for a Laplacian satisfying the Hörmander condition remains true in the general setting of polynomial volume-growth Lie groups (see again the book [8]). Theorem 2.1 is still valid in this general setting, as the proof follows the same lines as those exposed here.

## Acknowledgement

We would like to thank the referee for helpful comments and suggestions.

## References

[1] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Differ. Equ. 32 (2007) 1245–1260.
[2] F. Ferrari, B. Franchi, Hanarck inequality for fractional sub-Laplacians in Carnot groups, Math. Z. 279 (2015) 435–458.
[3] R. Frank, M.D.M. González, D.D. Monticelli, J. Tan, An extension problem for the CR fractional Laplacian, Adv. Math. 270 (2015) 97–137.
[4] G. Furioli, C. Melzi, A. Veneruso, Littlewood–Paley decomposition and Besov spaces on Lie groups of polynomial growth, Math. Nachr. 279 (9–10) (2006) 1028–1040.
[5] J.E. Galé, P.J. Miana, P.R. Stinga, Extension problems and fractional operators: semi-groups and wave equations, arXiv:1410.6051.
[6] M. Kemmppainen, S. Sjögren, J.L. Torrea, Wave extension problem for the fractional Laplacian, J. Evol. Equ. 2 (2014) 343–368.
[7] P. Stinga, J. Torrea, Extension problem and Harnack's inequality for some fractional operators, Commun. Partial Differ. Equ. 35 (11) (2010) 2092–2122.
[8] N.T. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge Tracts in Mathematics, vol. 100, 1992.