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An analytic proof of the planar quantitative isoperimetric inequality

Une démonstration analytique de l'inégalité isopérimétrique quantitative dans le plan

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ABSTRACT

We give an analytic proof of the quantitative isoperimetric inequality in the plane and give an estimation of the upper bound of the constant via maximizing the L^∞-norm of the gradient of solutions to the Poisson equation.

RÉSUMÉ

On donne une démonstration analytique de l'inégalité isopérimétrique quantitative dans le plan, et on établit une estimation de la borne supérieure de la constante en maximisant la norme L^∞ du gradient de la solution de l'équation de Poisson.

1. Introduction

For any Borel set E in $\mathbb{R}^n$, $n \geq 2$, let $P(E)$ denote E's perimeter, which is defined via

$$P(E) := \sup\left\{ \int_E \operatorname{div}\phi(x)\,\mathrm{d}x : \phi \in C_c^\infty(\mathbb{R}^n),\ |\phi| \leq 1 \right\}.$$

The classical isoperimetric inequality states that if E is a Borel set in $\mathbb{R}^n$, $n \geq 2$, with finite Lebesgue measure, i.e., $m(E) < \infty$, then it holds that

$$n\omega_n^{1/n} m(E)^{(n-1)/n} \leq P(E), \tag{1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$; see [18]. It is well known that equality holds in (1) if and only if E is a ball.

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The isoperimetric inequality has been proved in a variety of ways. For example, in the original paper by De Giorgi [9], the isoperimetric inequality was proved for the first time in the general framework of sets of finite perimeter. More proofs can be referred to, e.g., [4,12], etc. In this paper we focus on a quantitative version of the isoperimetric inequality. Supposing that E is a Borel set in $\mathbb{R}^n$ with $0 < m(E) < \infty$, $n \geq 2$, we define the *isoperimetric deficit* as

$$D(E) := \frac{P(E)}{n\omega_n^{1/n} m(E)^{(n-1)/n}} - 1 = \frac{P(E) - P(B)}{P(B)},$$

where B is a ball having the same volume as E, we also define the *Fraenkel asymmetry index* as

$$\lambda(E) := \min\left\{ \frac{m(E\Delta(x+B))}{m(E)} \middle| x \in \mathbb{R}^n \right\}.$$

The sharp quantitative isoperimetric inequality states that there exists such a constant $C = C(n) > 0$ that

$$\frac{\lambda(E)^2}{D(E)} \leq C(n). \tag{2}$$

This inequality was conjectured by Hall [15] in 1992. In 2008, Fusco, Maggi and Pratelli [11] came up with the first proof of the sharp quantitative isoperimetric inequality; see also Figalli, Maggi and Pratelli [13] and Cicalese and Leonardi [7] for different proofs. Fusco and Julin [14] recently have proved a stronger form of the quantitative isoperimetric inequality. After that, some efforts have been done to find the best constant in (2), that is

$$C_{\text{best}} := \min\left\{ C > 0 : \frac{\lambda(E)^2}{D(E)} \leq C, \forall \text{E is a Borel set in } \mathbb{R}^n \right\}. \tag{3}$$

In fact, this is a challenging problem and few results are known. Only in dimension $n = 2$, but within the class of convex sets, the minimizers for the above problem have been identified by Campi [3], and later by Alvino, Ferone, Nitsch [1] via a slightly different approach. Notice that it was given in [1] that in these cases $C_{\text{best}} \simeq 2.465574$. For further developments, we refer to Cicalese and Leonardi [6]. However, for general sets, the problem of finding the best constant is still open; see [8].

Recently, Cicalese and Leonardi [7] have solved Hall's conjecture concerning the best constant for the quantitative isoperimetric inequality in $\mathbb{R}^2$ in the small asymmetry regime, by showing that for any Borel set $E \subset \mathbb{R}^2$ with finite measure, it holds that

$$D(E) \geq \frac{\pi}{8(4-\pi)}\lambda(E)^2 + o(\lambda(E)^2).$$

In [8], Cicalese and Leonardi further determined the best constants for the asymptotic estimate of the quantitative isoperimetric inequality in $\mathbb{R}^2$, and established existence and regularity of minimizers for the problem (3).

In this paper, following the approach from [17], we consider the problem (2) in the plane, and give an estimation of the upper bound of the constant C via maximizing the L^∞-norm of the gradient of solutions to the Poisson equation.

Theorem 1. *For any Borel set $E \subset \mathbb{R}^2$ with finite Lebesgue measure, it holds that $\lambda(E)^2 \leq 16D(E)$.*

The key of the proof is to establish explicit bound of $||\nabla u_{\chi_E}||_{L^\infty(\mathbb{R}^2)}$, where u_{χ_E} is the solution to the Poisson equation $\Delta u_{\chi_E} = -\chi_E$ on $\mathbb{R}^2$ for those E having $\lambda(E) > 0$. Here and in what follows, for any measurable function $f \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, we let u_f be the solution to the Poisson equation $\Delta u_f = -f$ on $\mathbb{R}^2$, which satisfies $\limsup_{|x|\to\infty} \frac{u_f(x)}{|x|} < +\infty$. Notice that the sharp upper bounds of $||\nabla u_E||_{L^\infty(\mathbb{R}^n)}$ for general E have been found by Cianchi [5].

Lemma 2. *For any Borel set $E \subset \mathbb{R}^2$ with $0 < m(E) < \infty$, let u_{χ_E} be the solution to the Poisson equation $\Delta u_{\chi_E} = -\chi_E$ in $\mathbb{R}^2$. Then it holds that*

$$\|\nabla u_{\chi_E}(x)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{m(E)^{1/2}}{2\sqrt{\pi}}\left(\sqrt{1-\frac{\lambda(E)}{2}} + \sqrt{1+\frac{\lambda(E)}{2}} - 1\right). \tag{4}$$

By using Lemma 2 and a duality argument from [17], we shall see that for any Borel set E with finite Lebesgue measure, it holds

$$m(E)^{1/2} \leq \frac{P(E)}{2\sqrt{\pi}}\left(\sqrt{1-\frac{\lambda(E)}{2}} + \sqrt{1+\frac{\lambda(E)}{2}} - 1\right). \tag{5}$$

When dealing with the capacity increasing as a function of $\lambda(E)$, Hall, Hayman and Weitsman [16] proved that

$$P(E)^2 \geq 4\pi(1 + k\lambda(E)^2/4)\, m(E),$$

where $k = 1/4$ if E is connected and $k = 1/6$ otherwise. Notice that, the inequality (5) improves the above inequality since

$$4\pi\, m(E) \leq P(E)^2 \left(\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1\right)^2 \leq \frac{P(E)^2}{1 + \lambda(E)^2/8}.$$

This improvement can also be used to improve the constant k_2 in [16, Theorem 2.1]; we leave the details for interested readers.

2. Proofs

Proof of Lemma 2. We first assume that E is an open set. Notice that each component of ∇u_{χ_E} is harmonic in E and $\mathbb{R}^2 \setminus \bar{E}$, $|\nabla u|^2$ is continuous and sub-harmonic in E and $\mathbb{R}^2 \setminus \bar{E}$. Therefore, we have that

$$\|\nabla u_{\chi_E}\|_{L^\infty(\mathbb{R}^2)} = \max_{x \in \partial E} |\nabla u_{\chi_E}(x)|.$$

Up to a translation and rotation, we can restrict ourselves to maximizing $-(\partial/\partial x_1) u_{\chi_E}(0)$; see Cianchi [5]. That is

$$-\frac{\partial}{\partial x_1} u_{\chi_E}(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_E(y) y_1 |y|^{-2}\, \mathrm{d}y. \tag{6}$$

First of all, for any open set E, we do a little adjustment to E to get a new open set $\tilde{E}$, which satisfies

$$\tilde{E} := \begin{cases} E, & \text{if } E \subset \mathbb{R}^2_+ \\ (E \cap \mathbb{R}^2_+) \cup F, & \text{otherwise.} \end{cases}$$

Here, $\mathbb{R}^2_+ := \{x \in \mathbb{R}^2 :\ x_1 > 0\}$ and $F \subset \mathbb{R}^2_+$ is an open set, $m(F) = m(E \setminus \mathbb{R}^2_+)$ and $F \cap E \cap \mathbb{R}^2_+ = \varnothing$. It is obvious that we can increase the value of $-(\partial/\partial x_1) u_{\chi_E}(0)$ through this kind of adjustment, which means

$$-\frac{\partial}{\partial x_1} u_{\chi_E}(0) \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{\tilde{E}}(y) y_1 |y|^{-2} \mathrm{d}y.$$

In this way, we just need to consider the open set $E \subset \mathbb{R}^2_+$. Considering the special form of the integral on the right-hand side of (6), it is reasonable to define the set $S(E)$,

$$S(E) = \left\{x \in \mathbb{R}^2 :\ x_1 \geq 0, 2x_1 |x|^{-2} > \sqrt{\pi/m(E)}\right\}.$$

Clearly, $S(E)$ is a disk in $\mathbb{R}^2$ satisfying $m(S(E)) = m(E)$. Then

$$\max_{F : m(F) = m(E)} \left(-\frac{\partial}{\partial x_1} u_{\chi_F}(0)\right) \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{S(E)}(y) y_1 |y|^{-2}\, \mathrm{d}y.$$

Letting $c = m(E)$ in the following calculation, we find that:

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_{S(E)}(y) y_1 |y|^{-2}\, \mathrm{d}y &= \frac{1}{2\pi} \int_0^{\sqrt{c/\pi}} \int_0^{2\pi} \frac{r(\sqrt{c/\pi} + r\cos\theta)}{r^2 + 2r\sqrt{c/\pi}\cos\theta + c/\pi}\, \mathrm{d}\theta\, \mathrm{d}r \\
&= \frac{1}{\pi} \int_0^{\sqrt{c/\pi}} r\, \mathrm{d}r \int_0^{+\infty} \frac{\sqrt{c/\pi} + r(1-t^2)/(1+t^2)}{r^2 + 2r\sqrt{c/\pi}(1-t^2)/(1+t^2) + c/\pi} \times \frac{2}{1+t^2}\, \mathrm{d}t \\
&= \frac{1}{\pi} \int_0^{\sqrt{c/\pi}} r\pi \sqrt{\pi/c}\, \mathrm{d}r = \frac{\sqrt{m(E)}}{2\sqrt{\pi}}.
\end{aligned}$$

Consequently,

$$\|\nabla u_{\chi_E}\|_{L^\infty(\mathbb{R}^2)} \leq \left|\nabla u_{\chi_{S(E)}}(0)\right| = -\frac{\partial}{\partial x_1} u_{\chi_{S(E)}}(0) = \frac{1}{2\sqrt{\pi}} m(E)^{1/2}. \tag{7}$$

Let $a = \lambda(E)$ and suppose that $0 < a < 2$. Let us estimate $\|\nabla u_{\chi_E}\|_{L^\infty(\mathbb{R}^2)}$. By the above arguments, we see that, up to rotation and translation, it suffices to estimate $-\partial u_{\chi_E}/\partial x_1(0)$, and we may assume that $E \subset \mathbb{R}^2_+$. Let $B_0 := \left\{x \in \mathbb{R}^2 : |x|^2 < 2\sqrt{m(E)/\pi}x_1\right\}$ and write

$$-\frac{\partial}{\partial x_1}u_{\chi_E}(0) = \frac{1}{2\pi}\int_E \frac{y_1}{|y|^2}\mathrm{d}y = \frac{1}{2\pi}\left(\int_{B_0\cap E} \frac{y_1}{|y|^2}\mathrm{d}y + \int_{B_0^c\cap E} \frac{y_1}{|y|^2}\mathrm{d}y\right).$$

By the inequality (7), we see that

$$\int_{B_0\cap E} \frac{y_1}{|y|^2}\mathrm{d}y \le \sqrt{\pi}m(B_0\cap E)^{1/2} = \sqrt{\pi}\sqrt{m(B_0) - m(E^c\cap B_0)} = \sqrt{\pi}\sqrt{m(B_0) - m(E\cap B_0^c)},$$

since $m(E) = m(B_0)$, and

$$\int_{B_0^c\cap E} \frac{y_1}{|y|^2}\mathrm{d}y = \int_{B_0\cup(B_0^c\cap E)} \frac{y_1}{|y|^2}\mathrm{d}y - \int_{B_0} \frac{y_1}{|y|^2}\mathrm{d}y \le \sqrt{\pi}m(B_0\cup(B_0^c\cap E))^{1/2} - \sqrt{\pi}m(B_0)^{1/2}.$$

Combining the above estimates, we find that

$$-\frac{\partial}{\partial x_1}u_{\chi_E}(0) \le \frac{1}{2\sqrt{\pi}}\left(\sqrt{m(B_0) - m(E\cap B_0^c)} + \sqrt{m(B_0) + m(B_0^c\cap E)} - m(B_0)^{1/2}\right).$$

Notice that the function

$$f(x) = \sqrt{m(E) - x} + \sqrt{m(E) + x} - m(E)^{1/2}$$

is decreasing on $[0, m(E)]$. Since $m(E\cap B_0^c) \ge m(E)\lambda(E)/2$, we finally see that

$$\|\nabla u_{\chi_E}\|_{L^\infty(\mathbb{R}^2)} \le \frac{m(E)^{1/2}}{2\sqrt{\pi}}\left(\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1\right).$$

For general cases, E being measurable, we choose a sequence of open sets $\{E_j\}_{j\in\mathbb{N}}$ such that $E \subset E_j$ and $\lim_{j\to\infty} m(E_j \setminus E) = 0$. Then we have

$$\begin{aligned}\|\nabla u_{\chi_E}\|_{L^\infty(\mathbb{R}^2)} &\le \|\nabla u_{\chi_{E_j}}\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u_{\chi_{E_j\setminus E}}\|_{L^\infty(\mathbb{R}^2)} \\ &\le \frac{m(E_j)^{1/2}}{2\sqrt{\pi}}\left(\sqrt{1 - \frac{\lambda(E_j)}{2}} + \sqrt{1 + \frac{\lambda(E_j)}{2}} - 1\right) + \left\|\int_{E_j\setminus E} \frac{1}{2\pi|\cdot - y|}\mathrm{d}y\right\|_{L^\infty(\mathbb{R}^2)}.\end{aligned}$$

Letting $j \to \infty$, we can conclude that

$$\|\nabla u_{\chi_E}\|_{L^\infty(\mathbb{R}^2)} \le \frac{m(E)^{1/2}}{2\sqrt{\pi}}\left(\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1\right),$$

which completes the proof. □

We are now in position to prove the main result.

Proof of Theorem 1. We begin by recalling that the perimeter of the measurable set E satisfies

$$P(E) = \inf_{\varphi_k}\left\{\liminf_{k\to\infty}\int_{R^n} |\nabla\varphi_k|\mathrm{d}x : \varphi_k \in C^1(\mathbb{R}^2), \varphi_k \to \chi_E \text{ in } L^1(\mathbb{R}^2) \text{ and } |\varphi_k| \le 1\right\},$$

see [2,10]. Therefore we can choose a subsequence, denoting still by $\{\varphi_k\}_{k\in\mathbb{N}}$ for simplicity, such that $\|\nabla\varphi_k\|_{L^1(\mathbb{R}^2)} \to P(E)$ and $\varphi_k \to \chi_E$ in $L^1(\mathbb{R}^2)$.

Let u_{χ_E} be the solution to the Poisson equation $\Delta u_{\chi_E} = -\chi_E$. Then for each φ_k, it holds that

$$\int_E \varphi_k\,\mathrm{d}x = -\int_{\mathbb{R}^2} \Delta u_{\chi_E}\cdot\varphi_k\mathrm{d}x = \int_{\mathbb{R}^2} \nabla u_{\chi_E}\cdot\nabla\varphi_k\mathrm{d}x.$$

Letting $k \to \infty$ we find that

$$m(E) = \lim_{k\to\infty} \int_E \varphi_k \, dx \le \lim_{k\to\infty} ||\nabla u_{\chi_E}||_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |D\varphi_k| \, dx = ||\nabla u_{\chi_E}||_{L^\infty(\mathbb{R}^2)} P(E),$$

This and Lemma 2 further imply that

$$m(E)^{1/2} \le \frac{P(E)}{2\sqrt{\pi}} \left(\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1 \right).$$

Let B be a ball with the same measure as E. Then we have

$$\frac{P(E) - P(B)}{P(B)} \ge \frac{2\sqrt{\pi} m(E)^{1/2}}{\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1} \cdot \frac{1}{P(B)} - 1 = \frac{1}{\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1} - 1,$$

which implies that

$$\frac{D(E)}{\lambda(E)^2} \ge \frac{1}{\lambda(E)^2} \left(\frac{1}{\sqrt{1 - \frac{\lambda(E)}{2}} + \sqrt{1 + \frac{\lambda(E)}{2}} - 1} - 1 \right).$$

A direct calculation shows that the function

$$\frac{1}{a^2} \left(\frac{1}{\sqrt{1 - \frac{a}{2}} + \sqrt{1 + \frac{a}{2}} - 1} - 1 \right),$$

defined on $(0, 2)$, attains a minimum of $\frac{1}{16}$ at the origin, i.e. $a = 0$. Therefore, we conclude that $\frac{D(E)}{\lambda(E)^2} \ge \frac{1}{16}$, as desired. □

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