C. R. Acad. Sci. Paris, Ser. I 353 (2015) 579–582

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Lie algebras/Functional analysis

A note on the Kirillov model for representations of $\mathrm{GL}_n(\mathbb{C})$

Une note sur le modèle de Kirillov des représentations de $\mathrm{GL}_n(\mathbb{C})$

Alexander Kemarsky

Technion, Mathematics, Department of Mathematics, Haifa, Israel

A R T I C L E   I N F O

Article history:
Received 24 December 2014
Accepted 3 April 2015
Available online 29 April 2015

Presented by Michèle Vergne

A B S T R A C T

Let $G = \mathrm{GL}_n(\mathbb{C})$ and $1 \neq \psi : \mathbb{C} \to \mathbb{C}^\times$ be an additive character. Let U be the subgroup of upper triangular unipotent matrices in G. Denote by θ the character $\theta : U \to \mathbb{C}$ given by

$$\theta(u) := \psi(u_{1,2} + u_{2,3} + \ldots + u_{n-1,n}).$$

Let P be the mirabolic subgroup of G consisting of all matrices in G with the last row equal to $(0, 0, \ldots, 0, 1)$. We prove that if π is an irreducible generic representation of $\mathrm{GL}_n(\mathbb{C})$ and $\mathcal{W}(\pi, \psi)$ is its Whittaker model, then the space $\{f|_P : P \to \mathbb{C} : f \in \mathcal{W}(\pi, \psi)\}$ contains the space of infinitely differentiable functions $f : P \to \mathbb{C}$ that satisfy $f(up) = \psi(u) f(p)$ for all $u \in U$ and $p \in P$ and that have a compact support modulo U. A similar result was proven for $\mathrm{GL}_n(F)$, where F is a p-adic field by Gelfand and Kazhdan (1975) [1] and for $\mathrm{GL}_n(\mathbb{R})$ by Jacquet (2010) [2].

R É S U M É

Soit $G = \mathrm{GL}_n(\mathbb{C})$ et $1 \neq \psi : \mathbb{C} \to \mathbb{C}^\times$ un caractère additif non trivial. Soit U le sous-groupe des matrices triangulaires supérieures unipotentes de G. Notons $\theta : U \to \mathbb{C}$ le caractère donné par

$$\theta(u) := \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}).$$

Soit P le sous-groupe mirabolique constitué des matrices de G dont la dernière ligne est $(0, 0, \ldots, 0, 1)$. Nous montrons que, si π est une représentation irréductible générique de G et si $\mathcal{W}(\pi, \psi)$ est son modèle de Whittaker, alors l'espace $\{f|_P : P \to \mathbb{C} : f \in \mathcal{W}(\pi, \psi)\}$ contient l'espace des fonctions $f : P \to \mathbb{C}$ infiniment différentiables, qui satisfont $f(up) = \psi(u) f(p)$ pour tout $u \in U$ et $p \in P$ et qui ont un support compact modulo U. Un résultat similaire a été établi pour $\mathrm{GL}_n(F)$, où F est un corps p-adique, par Gelfand et Kazhdan (1975) [1] et pour $GL_n(\mathbb{R})$ par Jacquet (2010) [2].

E-mail address: alexkem@tx.technion.ac.il.

http://dx.doi.org/10.1016/j.crma.2015.04.002
1631-073X/© 2015 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Let F be the field $\mathbb{R}$ or $\mathbb{C}$. Let $G_n(F)$ be the group $\mathrm{GL}_n(F)$, let $P(F)$ be the mirabolic subgroup in $G_n(F)$ consisting of matrices in $G_n(F)$ with the last row equal to $(0, 0, \ldots, 0, 1)$. Let $U_n(F)$ ($\overline{U_n(F)}$ respectively) be the unipotent subgroups consisting of upper triangular unipotent matrices (lower triangular unipotent matrices respectively) in $G_n(F)$. We fill fix a field F and will abbreviate G for the group $G(F)$. Let $\psi : F \to \mathbb{C}^\times$ be a non-trivial additive character of F and denote by θ_n a character on U_n given by

$$\theta_n(x) := \psi(x_{12} + x_{23} + \ldots + x_{n-1,n})$$

for $x \in U$. Let π be a unitary irreducible representation of G_n on a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$. We let $\mathcal{V}$ be the space of the G_n smooth vectors, equipped with the topology defined by the semi-norms

$$v \to \|\mathrm{d}\pi(X)v\|$$

with $X \in \mathcal{U}(G_n)$. We let $\mathcal{V}'$ be the topological conjugate dual. We have inclusions

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'.$$

The positive-definite scalar product on $\mathcal{V} \times \mathcal{V}$ extends to $\mathcal{V} \times \mathcal{V}'$. Recall π is said to be generic if there is an element $\lambda \neq 0$ of $\mathcal{V}'$ such that

$$(\pi(u)v, \lambda) = \theta_n(u)\,(v, \lambda)$$

for all $v \in \mathcal{V}$ and $u \in U_n$. Up to a scalar factor, the vector λ is unique [4]. For every $v \in \mathcal{V}$ we define a function W_v on G_n by

$$W_v(g) = (\pi(g)v, \lambda).$$

We let $\mathcal{W}(\pi, \psi)$ be the Whittaker model of π, that is, the space spanned by the functions

$$g \mapsto (\pi(g)v, \lambda)\,,$$

where v is in $\mathcal{V}$. We identify $\mathcal{V}$ and $\mathcal{W}(\pi, \psi)$. Then the scalar product induced by $\mathcal{H}$ on $\mathcal{V}$ is a scalar multiple of the scalar product defined by the convergent integral

$$(v_1, v_2) := \int\limits_{U_{n-1}\backslash G_{n-1}} W_{v_1}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \overline{W_{v_2}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}} \mathrm{d}g,$$

see [3]. We may assume the scalar product on $\mathcal{V}$ is equal to this convergent integral.

We denote by $C_c^\infty(\theta_{n-1}, G_{n-1})$ the space of smooth and compactly supported modulo U_{n-1} functions $f : G_{n-1} \to \mathbb{C}$ such that $f(ug) = \theta_n(u) f(g)$ for all $u \in U_{n-1}$, $g \in G_{n-1}$.

In this note, we prove the following theorem for $F = \mathbb{C}$.

Theorem 1. *Let π be a generic unitary irreducible representation of $G_n (:= G_n(\mathbb{C}))$. Given $\phi \in C_c^\infty(\theta_{n-1}, G_{n-1})$ there is a unique $W_\phi \in \mathcal{V}$ such that, for all $g \in G_{n-1}$,*

$$W_\phi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \phi(g).$$

Furthermore, the map $\phi \to W_\phi$ is continuous.

An analogous result was proven for $G_n(F)$, F a p-adic field by Gelfand and Kazhdan [1]. Recently it was proven by Jacquet for $G_n(\mathbb{R})$ [2]. Our proof for $G_n(\mathbb{C})$ is similar to Jacquet's proof. We sketch here the main steps of the proof. There is only one technical lemma that should be re-proven for $G_n(\mathbb{C})$.

For a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. For a Lie group G, we denote by $\mathrm{Lie}(G)$ its Lie algebra (over $\mathbb{C}$), by $\mathcal{U}(G)$ its universal enveloping algebra, by $Z(\mathcal{U}(G))$ the center of $\mathcal{U}(G)$. Denote the image of $X \in \mathrm{Lie}(G)$ under the natural embedding in $\mathcal{U}(G)$ by D_X. We will identify $\mathrm{Lie}(G)$ with its image in $\mathcal{U}(G)$. Note that D_X can be realized as the linear differential operator on the space $C_c^\infty(G)$ corresponding to the derivative in the direction X.

Let us denote the elementary matrix with 1 in the (i, j) place and 0 in all other places by E_{ij}. Let us denote $D_{E_{ij}} := D_{(i,j)}$. Denote the elementary matrix with $\sqrt{-1}$ in the place (i, j) and 0 in all other places by $D_{\sqrt{-1}(i,j)}$. The set

$$\left\{ D_{ij}, D_{\sqrt{-1}(i,j)} : 1 \le i, j \le n \right\}$$

is a basis of $\mathrm{Lie}(G_n(\mathbb{C}))$. Observe that there are two natural embeddings of Lie algebras

$$i_1, i_2 : \mathrm{Lie}(G_n(\mathbb{R})) \to \mathrm{Lie}(G_n(\mathbb{C}))$$

given by

$$i_1(D_X) = \frac{1}{2}\left(D_X + \sqrt{-1}D_{\sqrt{-1}X}\right),\ i_2(D_X) = \frac{1}{2}\left(D_X - \sqrt{-1}D_{\sqrt{-1}X}\right).$$

We have

$$\mathrm{Lie}\,(G_n(\mathbb{C})) = i_1\,(\mathrm{Lie}\,(G_n(\mathbb{R}))) \oplus i_2\,(\mathrm{Lie}\,(G_n(\mathbb{R}))).$$

That is, the complex vector space $\mathrm{Lie}\,(G_n(\mathbb{C}))$ is a direct sum of the vector spaces

$$i_1\,(\mathrm{Lie}\,(G_n(\mathbb{R}))),\ i_2\,(\mathrm{Lie}\,(G_n(\mathbb{R}))),$$

and for all $X, Y \in \mathrm{Lie}\,(G_n\,(\mathbb{R}))$ we have $[i_1(X), i_2(Y)] = 0$.

The following lemma is the key lemma in the proof of Theorem 1 for $G_n(\mathbb{R})$.

Lemma 1. *Consider a* $\mathrm{Lie}(G_n(F))$ *module* V *and a vector* $0 \neq \lambda \in V$. *Suppose* $D_{ij}\lambda = 0$ *for every pair of indices* $1 \le i, j \le n$ *such that* $j > i + 1$. *Suppose also that for every* $1 \le i \le n-1$ *there is a constant* $c_i \neq 0$ *such that* $D_{i,(i+1)}\lambda = c_i\lambda$. *If* $F = \mathbb{C}$ *suppose also that* $D_{\sqrt{-1}i,j}\lambda = 0$ *for every pair of indices* $1 \le i, j \le n$ *such that* $j > i+1$ *and for every* $1 \le i \le n-1$ *there is a constant* $c_{\sqrt{-1}i} \neq 0$ *such that* $D_{\sqrt{-1}(i,i+1)}\lambda = c_{\sqrt{-1}i}\lambda$. *Then*

$$\mathrm{Lie}\left(\overline{U_n(F)}\right)\lambda \subset \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F))\lambda,$$

$$\mathcal{U}\left(\overline{U_n(F)}\right)\lambda \subset \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F))\lambda,$$

$$\mathcal{U}\left(\overline{G_n(F)}\right)\lambda \subset \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F))\lambda.$$

Proof. We will use the corresponding result for the case $\mathrm{Lie}(G_n(\mathbb{R}))$ and deduce from it the result for $\mathrm{Lie}(G_n(\mathbb{C}))$. Note that it is enough to prove the first inclusion, as the second inclusion and the third inclusion follows from the first by an application of theorem of Poincaré–Birkhoff–Witt. See [2], the first lines of the proof of Lemma 3.

To prove the first inclusion, we have to prove that

$$D_X\lambda \in \mathcal{U}(G_{n-1}(F))Z(\mathcal{U}(G_n(F))\lambda$$

for

$$X \in \{E_{n1}, E_{n2}, \ldots, E_{nn}\} \cup \left\{E_{\sqrt{-1}(n,1)}, E_{\sqrt{-1}(n,2)}, \ldots, E_{\sqrt{-1}(n,n)}\right\}.$$

By the corresponding result for $\mathrm{Lie}(G_n(\mathbb{R}))$, we know that for the Lie algebras $i_1\,(\mathrm{Lie}\,(G_{n-1}(\mathbb{R})))$ and $i_2\,(\mathrm{Lie}\,(G_{n-1}(\mathbb{R})))$ and $X \in \{E_{n1}, E_{n2}, \ldots, E_{nn}\}$, we have

$$D_{i_{1,2}(X)}\lambda \in \mathcal{U}\left(i_{1,2}\,(\mathrm{Lie}\,(G_{n-1}(\mathbb{R})))\right) Z\left(\mathcal{U}\left(i_{1,2}\,(\mathrm{Lie}\,(G_{n-1}(\mathbb{R})))\right)\right)\lambda \subset \mathcal{U}\,(G_{n-1}(\mathbb{C}))\,Z\,(\mathcal{U}\,(G_{n-1}(\mathbb{C})))\,\lambda.$$

Thus,

$$\frac{1}{2}\left(D_{n,r}\lambda + \sqrt{-1}D_{\sqrt{-1}n,r}\lambda\right),\ \frac{1}{2}\left(D_{n,r}\lambda - \sqrt{-1}D_{\sqrt{-1}n,r}\lambda\right) \in \mathcal{U}\,(G_{n-1}(\mathbb{C}))\,Z\,(\mathcal{U}\,(G_{n-1}(\mathbb{C})))\,\lambda.$$

By adding and subtracting the two terms on the left-hand side of the last formula, we obtain

$$D_{n,r}\lambda,\ D_{\sqrt{-1}n,r}\lambda \in \mathcal{U}\,(G_{n-1}(\mathbb{C}))\,Z\,(\mathcal{U}\,(G_{n-1}(\mathbb{C})))\,\lambda.$$

The lemma is proved. □

For the convenience of the reader, we rewrite here only formulations of the lemmas that are needed to prove Theorem 1. All the proofs are identical to that written in [2], so we see no need to repeat the proofs here.

From now on $F = \mathbb{C}$. Consider the space $\mathcal{V}_{n-1}$ of G_{n-1} smooth vectors in $\mathcal{H}$.

Lemma 2. *We have continuous inclusions*

$$\mathcal{V} \subset \mathcal{V}_{n-1} \subset \mathcal{H} \subset \mathcal{V}'_{n-1} \subset \mathcal{V}'.$$

Lemma 3. *The vector* λ *which is appriori in* $\mathcal{V}'$ *belongs to* $\mathcal{V}'_{n-1}$.

Given $\phi \in C_c^\infty(G_{n-1})$, we set

$$u_\phi := \int_{g \in G_{n-1}} \phi(g)\pi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \lambda \, dg.$$

Lemma 4. *The vector u_ϕ belongs to $\mathcal{V}_{n-1}$, in particular to $\mathcal{H}$. The map*

$$\phi \to u_\phi, \; C_c^\infty(G_{n-1}) \to \mathcal{H}$$

is continuous.

Lemma 5. *For every $X \in \mathcal{U}(G_n)$ and every $\phi \in C_c^\infty(G_{n-1})$, the vector u_ϕ, which, a priori, is in $\mathcal{V}'$, is in fact in $\mathcal{H}$. Moreover, there is a continuous semi-norm μ on $C_c^\infty(G_{n-1})$ such that, for every $\phi \in C_c^\infty(G_{n-1})$,*

$$\|d\pi(X)u_\phi\| \le \mu(\phi).$$

We note that Lemma 1 is used to prove the last lemma.

Lemma 6. *Suppose $v_0 \in \mathcal{H}$ is a vector such that for every $X \in \mathcal{U}(G_n)$ the vector $d\pi(X)v_0$, which a priori is in $\mathcal{V}'$, is in fact in $\mathcal{H}$. Then v_0 is in $\mathcal{V}$.*

Proposition 1. *For every $\phi \in C_c^\infty(G_{n-1})$ the vector*

$$u_\phi := \int_{g \in G_{n-1}} \phi(g)\pi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \lambda \, dg$$

is in $\mathcal{V}$. Furthermore, the map

$$\phi \mapsto u_\phi, \; C_c^\infty(G_{n-1}) \to \mathcal{V}$$

is continuous.

Lemma 7. *If $v = u_\phi$ then*

$$W_v \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \phi_0(g),$$

where

$$\phi_0(g) = \int_{U_{n-1}} \phi((gu)^{-1}) \overline{\theta_{n-1}}(u) \, du.$$

I would like to thank Omer Offen and Erez Lapid for posing me this question, and Dmitry Gourevitch and Hervé Jacquet for their interest in this project.

Acknowledgement

The research was supported by ISF grant No. 1394/12.

References

[1] I.M. Gelfand, D. Kazhdan, Representations of the group $GL(n, K)$ where K is a local field, in: Lie Groups and Their Representations, Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971, Halsted, New York, 1975, pp. 95–118.
[2] H. Jacquet, Distinction by the quasi-split unitary group, Isr. J. Math. 178 (1) (2010) 269–324.
[3] H. Jacquet, J.A. Shalika, On Euler products and the classification of automorphic representations I, Amer. J. Math. 103 (3) (1981) 499–558.
[4] J.A. Shalika, The multiplicity one theorem for $GL(n)$, Ann. Math. 100 (2) (1974) 171–193.