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Superposition with subunitary powers in Sobolev spaces

Superposition avec des puissances sous-unitaires dans les espaces de Sobolev

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ABSTRACT

Let $0 < a < 1$ and set $\Phi(t) = |t|^a$, $\forall t \in \mathbb{R}$. Let $1 < p < \infty$ and $n \geq 1$. We prove that the superposition operator $u \mapsto \Phi(u)$ maps the Sobolev space $W^{1,p}(\mathbb{R}^n)$ into the fractional Sobolev space $W^{a,p/a}(\mathbb{R}^n)$. We also investigate the case of more general nonlinearities.

RÉSUMÉ

Pour $0 < a < 1$, soit $\Phi(t) = |t|^a$, $\forall t \in \mathbb{R}$. Soient $1 < p < \infty$ et $n \geq 1$. Nous montrons que l'opérateur de superposition $u \mapsto \Phi(u)$ envoie l'espace de Sobolev $W^{1,p}(\mathbb{R}^n)$ dans l'espace de Sobolev fractionnaire $W^{a,p/a}(\mathbb{R}^n)$. Nous examinons aussi la superposition avec des non-linéarités plus générales.

1. Introduction

Let $0 < a < 1$ and set

$$\Phi(t) = |t|^a, \ \forall t \in \mathbb{R}. \tag{1}$$

We consider the superposition operator

$$u \mapsto \Phi(u), \ \forall u \in W^{1,p}(\mathbb{R}^n),$$

and investigate its mapping properties. This question, natural in the context of the analysis of singularities of supercritical viscous Hamilton–Jacobi equations [3], was asked to me by L. Véron.

Our main result is the following

Theorem 1.1. *Let $1 < p < \infty$ and $n \geq 1$. The superposition operator $u \mapsto \Phi(u)$ maps $W^{1,p}(\mathbb{R}^n)$ into $W^{a,p/a}(\mathbb{R}^n)$.*

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As we will see, the above theorem does not hold when $p = 1$.

An immediate consequence of Theorem 1.1 is the answer to the original question in [3].

Corollary 1.2. *Let $0 < a < 1$, $1 < p < \infty$ and $n \geq 1$. Let $\widetilde{\Phi}(t) = |t|^a \operatorname{sgn} t$, $\forall t \in \mathbb{R}$. The superposition operator $u \mapsto \widetilde{\Phi}(u)$ maps $W^{1,p}(\mathbb{R}^n)$ into $W^{a,p/a}(\mathbb{R}^n)$.*

The heart of the proof of Theorem 1.1 consists in establishing the estimate

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\Phi(u(y)) - \Phi(u(x))|^{p/a}}{|y-x|^{n+p}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\mathbb{R}^n} |\nabla u(x)|^p\,\mathrm{d}x,\ \forall u \in W^{1,p}(\mathbb{R}^n). \tag{2}$$

Estimate (2) is a special case of a family of inequalities involving more general even nonlinearities Φ which are non-decreasing on $[0, \infty)$. Assume that, on $[0, \infty)$, we may write:

$$\Phi = \zeta + g, \text{ with } \zeta \text{ continuous concave non decreasing}$$
$$\text{such that } \zeta(0) = 0 \text{ and with } g \text{ non increasing.} \tag{3}$$

This assumption is satisfied, e.g., when, on $[0, \infty)$, the function Φ is non decreasing and either (a) continuous and concave or (b) Lipschitz continuous. If (3) holds, then it is always possible to choose in (3) a function ζ such that in addition

$$\zeta : [0, \infty) \to [0, \infty) \text{ is a concave homeomorphism} \tag{4}$$

and

$$\text{the restriction to } (0, \infty) \text{ of the inverse map } \Psi = \zeta^{-1} \text{ is } C^2 \text{ and } \Psi''(t) > 0,\ \forall t > 0. \tag{5}$$

[Indeed, every concave function ζ as in (3) is dominated by a concave continuous function $\widetilde{\zeta} : [0, \infty) \to [0, \infty)$ such that $\widetilde{\zeta}(0) = 0$, $\widetilde{\zeta} \in C^\infty((0, \infty))$, $\widetilde{\zeta}'(t) > 0$, $\widetilde{\zeta}''(t) < 0$, $\forall t > 0$ and $\lim_{t\to\infty} \widetilde{\zeta}(t) = \infty$. Any such $\widetilde{\zeta}$ satisfies (4) and (5). In addition, if we define $\widetilde{g} = g + \zeta - \widetilde{\zeta}$, then $u = \widetilde{\zeta} + \widetilde{g}$ and $\widetilde{g}$ is non increasing.]

Note that when Φ is given by (1), we may take $g = 0$ and ζ to be the restriction of Φ to $[0, \infty)$. The same holds when Φ is even and the restriction $\zeta : [0, \infty) \to [0, \infty)$ of Φ is a concave homeomorphism whose inverse satisfies (5).

Define

$$F(t) = \int_0^t \left(\int_0^s [\Psi'(\tau)]^{1-1/p} [\Psi''(\tau)]^{1/p}\,\mathrm{d}\tau \right)^p \mathrm{d}s,\ \forall t \geq 0. \tag{6}$$

We have the following.

Theorem 1.3. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be even and non decreasing on $[0, \infty)$, and satisfy (3)–(5). Let F be given by (6). We have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{F(|\Phi(u(y)) - \Phi(u(x))|)}{|y-x|^{n+p}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\mathbb{R}^n} |\nabla u(x)|^p\,\mathrm{d}x,\ \forall u \in W^{1,p}(\mathbb{R}^n). \tag{7}$$

The reader may check that, when Φ is as in (1) and ζ is the restriction of Φ to $[0, \infty)$, we have $F(t) = C_{p,a} t^{p/a}$, and thus (7) includes (2) as a special case.

2. Proofs

The key estimate is the following special case of (7):

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F(|\Phi(v(y)) - \Phi(v(x))|)}{|y-x|^{1+p}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\mathbb{R}} |v'(x)|^p\,\mathrm{d}x,\ \forall v \in C_c^\infty(\mathbb{R}),\ \forall p \in (1, \infty). \tag{8}$$

Assuming (8) established for the moment, we proceed to the proof of Theorems 1.1 and 1.3, and of Corollary 1.2.

Proof of Theorems 1.1 and 1.3, and of Corollary 1.2.

1. We start from the following calculation, valid for every measurable function $f : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$.

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}} |r|^{n-1} f(x,x+r\omega)\,\mathrm{d}r\,\mathrm{d}s_\omega\,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\int_{\mathbb{R}} |r|^{n-1} f(x'+s\omega, x'+(r+s)\omega)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x'\,\mathrm{d}s_\omega \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}}\int_{\mathbb{R}} |r-s|^{n-1} f(x'+s\omega, x'+r\omega)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}x'\,\mathrm{d}s_\omega. \end{aligned} \tag{9}$$

Let I denote the left-hand side of (7) and let $u \in C_c^\infty(\mathbb{R}^n)$. If we take, in (9),

$$f(x,y) = \frac{F(|\Phi(u(y)) - \Phi(u(x))|)}{|y-x|^{n+p}}$$

and apply (8) with $v(r) = u(x'+r\omega)$, $\forall r \in \mathbb{R}$, we obtain that

$$\begin{aligned} I &\lesssim \int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}} |\nabla u(x'+r\omega)\cdot\omega|^p\,\mathrm{d}r\,\mathrm{d}x'\,\mathrm{d}s_\omega \le \int_{\mathbb{S}^{n-1}}\int_{\omega^\perp}\int_{\mathbb{R}} |\nabla u(x'+r\omega)|^p\,\mathrm{d}r\,\mathrm{d}x'\,\mathrm{d}s_\omega \\ &= |\mathbb{S}^{n-1}|\int_{\mathbb{R}^n} |\nabla u(x)|^p\,\mathrm{d}x, \end{aligned}$$

i.e., (7) holds for $u \in C_c^\infty(\mathbb{R}^n)$.
2. By a standard argument, (7) holds for $u \in W^{1,p}(\mathbb{R}^n)$. In particular, (2) holds.
3. Next, Theorem 1.1 is a consequence of (2) and of the identity $\|\Phi(u)\|_{L^{p/a}(\mathbb{R}^n)}^{p/a} = \|u\|_{L^p(\mathbb{R}^n)}^p$ (with Φ as in (1)).
4. Corollary 1.2 follows from Theorem 1.1 combined with the identity $\widetilde{\Phi}(u) = \Phi(u^+) - \Phi(u^-)$ and with the fact that $u^\pm \in W^{1,p}(\mathbb{R}^n)$ whenever $u \in W^{1,p}(\mathbb{R}^n)$. $\square$

We next proceed to the proof of (8).

Proof of (8).
1. Starting from the inequality

$$0 \le \int_0^s [\Psi'(\tau)]^{1-1/p}[\Psi''(\tau)]^{1/p}\,\mathrm{d}\tau \le \int_0^s [\Psi'(\tau)+\Psi''(\tau)]\,\mathrm{d}\tau = \Psi(s)+\Psi'(s),\ \forall s>0,$$

we find that $F \in C^1([0,\infty))$ and that

$$F'(t) = \left(\int_0^t [\Psi'(s)]^{1-1/p}[\Psi''(s)]^{1/p}\,\mathrm{d}s\right)^p > 0,\ \forall t>0,\ \text{while } F'(0)=0. \tag{10}$$

The identity

$$[F'(t)]^{1-p}[F''(t)]^p = C_p[\Psi'(t)]^{p-1}\Psi''(t),\ \forall t>0, \tag{11}$$

is a consequence of (10) combined with the formula of $F''(t)$ obtained by differentiating (10).
Integrating (11), we obtain

$$\int_0^t [F'(s)]^{1-p}[F''(s)]^p\,\mathrm{d}s = C'_p[\Psi'(t)]^p,\ \forall t>0. \tag{12}$$

If we let, in (12), $t=\zeta(\tau)$, and take into account the fact that $\zeta = \Psi^{-1}$ on $(0,\infty)$, we find that

$$\int_0^{\zeta(\tau)} [F'(s)]^{1-p}[F''(s)]^p\,\mathrm{d}s\,[\zeta'(\tau)]^p = C'_p,\ \forall \tau>0. \tag{13}$$

2. We next note the following Hardy type inequality. If $U \subset \mathbb{R}$ is a bounded open set and if $w \in C(\overline{U}) \cap C^1(U)$ satisfies $w=0$ on ∂U, then

$$\int_{\mathbb{R}\setminus U}\left(\int_U \frac{|w(x)|^p}{|y-x|^{1+p}}\,\mathrm{d}x\right)\mathrm{d}y \le C_p \int_U |w'(x)|^p\,\mathrm{d}x,\ \forall\, p\in(1,\infty). \tag{14}$$

Indeed, by the subadditivity of (14) with respect to U, it suffices to prove (14) when $U=(a,b)$, and in that case integration in y and Hardy's inequality in x lead to

$$\int_{\mathbb{R}\setminus U}\left(\int_U \frac{|w(x)|^p}{|y-x|^{1+p}}\,\mathrm{d}x\right)\mathrm{d}y \le C'_p \int_U \frac{|w(x)|^p}{[\mathrm{dist}\,(x,\partial U)]^p}\,\mathrm{d}x \le C_p \int_U |w'(x)|^p\,\mathrm{d}x.$$

[The idea of using Hardy-type inequalities in the study of superposition operators originates in [1].]

3. Φ being even, it suffices to establish (8) when v is replaced by $|v|$. By a standard argument, it then suffices to establish (8) when $v\in C_c^\infty(\mathbb{R})$ and $v\ge 0$. Moreover, by (3), the monotonicity of Φ on $[0,\infty)$ and the fact that F is non decreasing, we have

$$F(|\Phi(t)-\Phi(s)|)\le F(|\zeta(t)-\zeta(s)|),\ \forall\, s,t\in\mathbb{R}^+,$$

and therefore it suffices to establish (8) when Φ is replaced by ζ. In conclusion, it suffices to establish the estimate

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{F(|\zeta(v(y))-\zeta(v(x))|)}{|y-x|^{1+p}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\mathbb{R}} |v'(x)|^p\,\mathrm{d}x,\ \forall\, v\in C_c^\infty(\mathbb{R},\mathbb{R}^+),\ \forall\, p\in(1,\infty), \tag{15}$$

with ζ satisfying (4) and (5).

4. With $v\in C_c^\infty(\mathbb{R},\mathbb{R}^+)$ and $x,y\in\mathbb{R}$ such that $v(y)<v(x)$, we have

$$\begin{aligned} F(|\zeta(v(y))-\zeta(v(x))|) &= F(\zeta(v(x))-\zeta(v(y))) \\ &= -F(\zeta(v(x))-\zeta(t))\Big|_{t=v(y)}^{t=v(x)} = \int_{v(y)}^{v(x)} [g_t(x)]^p\,\mathrm{d}t, \end{aligned} \tag{16}$$

where

$$\begin{aligned} g_t(x) &= \left[-\frac{\mathrm{d}}{\mathrm{d}t}[F(\zeta(v(x))-\zeta(t))]\right]^{1/p} \\ &= [\zeta'(t)]^{1/p}[F'(\zeta(v(x))-\zeta(t))]^{1/p},\ \forall\, x\in\mathbb{R},\ \forall\, t\in(0,v(x)]. \end{aligned} \tag{17}$$

5. Integrating (16), we find that the left-hand side J of (15) is given by

$$J = 2\int_0^\infty\left(\int_{v(y)<t}\left(\int_{v(x)>t} \frac{[g_t(x)]^p}{|y-x|^{1+p}}\,\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}t. \tag{18}$$

6. Let (with $t>0$ fixed) $U=\{x;\ v(x)>t\}$. Note that U is bounded. By (17) and the last assertion in (10), we have $g_t=0$ on ∂U, and thus we may apply (14) to $x\mapsto g_t(x)$, $\forall\, x\in U$.

7. Combining (14) (applied with $U=\{x;\ v(x)>t\}$ and with $w=g_t$) with (18) and using the change of variables $\zeta(v(x))-\zeta(t)=s$, we find that

$$\begin{aligned} J &\lesssim \int_{\mathbb{R}}\left(\int_0^{v(x)} \zeta'(t)[F'(\zeta(v(x))-\zeta(t))]^{1-p}[F''(\zeta(v(x))-\zeta(t))]^p\,\mathrm{d}t\right)[\zeta'(v(x))]^p|v'(x)|^p\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\left(\int_0^{\zeta(v(x))} [F'(s)]^{1-p}[F''(s)]^p\,\mathrm{d}s\right)[\zeta'(v(x))]^p|v'(x)|^p\,\mathrm{d}x. \end{aligned} \tag{19}$$

Next (13) (applied with $\tau=v(x)$) combined with (19) leads to (15). The proof of (8) is complete. $\square$

We conclude this note by proving that Theorem 1.1 does not hold when $p=1$. The key observation is that Φ given by (1) does not satisfy the estimate

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\Phi(v(y))-\Phi(v(x))|^{1/a}}{|y-x|^2}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\mathbb{R}} |v'(x)|\,\mathrm{d}x,\ \forall\, v\in C_c^\infty(\mathbb{R}). \tag{20}$$

Indeed, argue by contradiction and assume that (20) holds. By a standard limiting argument, (20) still holds for $v \in \mathrm{BV}(\mathbb{R})$. However, if we take $v = \mathbb{1}_{(0,1)}$, then $v \in \mathrm{BV}(\mathbb{R})$ and the reader may check that the left-hand side of (20) is infinite. Thus (20) does not hold, as claimed.

Using the fact that (20) does not hold and the invariance of the estimate (20) by scaling and translations, we may find a sequence $(v_j)_{j\geq 1} \subset C_c^\infty(\mathbb{R})$ such that

$$\operatorname{supp} v_j \subset (j, j+1), \ \int_{\mathbb{R}} |v_j'(x)|\,\mathrm{d}x = \frac{1}{j^2}, \tag{21}$$

and

$$\int_j^{j+1}\int_j^{j+1} \frac{|\Phi(v_j(y)) - \Phi(v_j(x))|^{1/a}}{|y-x|^2}\,\mathrm{d}x\,\mathrm{d}y \geq \frac{1}{j}. \tag{22}$$

Note that

$$\int_{\mathbb{R}} |v_j(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}} |v_j'(x)|\,\mathrm{d}x = \frac{1}{j^2}. \tag{23}$$

Set $v := \sum_{j\geq 1} v_j$. Using (21), (22) and (23), we find that $v \in W^{1,1}(\mathbb{R})$ and, in addition,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\Phi(v(y)) - \Phi(v(x))|^{1/a}}{|y-x|^2}\,\mathrm{d}x\,\mathrm{d}y &\geq \sum_{j\geq 1}\int_j^{j+1}\int_j^{j+1} \frac{|\Phi(v(y)) - \Phi(v(x))|^{1/a}}{|y-x|^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{j\geq 1}\int_j^{j+1}\int_j^{j+1} \frac{|\Phi(v_j(y)) - \Phi(v_j(x))|^{1/a}}{|y-x|^2}\,\mathrm{d}x\,\mathrm{d}y = \infty.\end{aligned}$$

Thus $v \notin W^{a,1/a}(\mathbb{R})$, i.e., Theorem 1.1 does not hold when $p = 1$ and $n = 1$.

The case where $p = 1$ and $n \geq 2$ is straightforward. Let v be as above and let $\zeta \in C_c^\infty(\mathbb{R}^{n-1})$, $\zeta \not\equiv 0$. Set $u(x', x_n) = \zeta(x')v(x_n)$, $\forall x' \in \mathbb{R}^{n-1}$, $\forall x_n \in \mathbb{R}$. Then clearly $u \in W^{1,1}(\mathbb{R}^n)$. However, we claim that $u \notin W^{a,1/a}(\mathbb{R}^n)$. Indeed, argue by contradiction and assume that $u \in W^{a,1/a}(\mathbb{R}^n)$. By the Fubini property of the space $W^{a,1/a}(\mathbb{R}^n)$ [2, Section 2.5.13, Theorem p. 115], for a.e. $x' \in \mathbb{R}^{n-1}$ we have $u(x', \cdot) \in W^{a,1/a}(\mathbb{R})$. In particular, we have $v \in W^{a,1/a}(\mathbb{R})$, which is a contradiction. Therefore, when $p = 1$ and $n \geq 1$, Theorem 1.1 does not hold.

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