C. R. Acad. Sci. Paris, Ser. I 353 (2015) 617–622

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex analysis

The Fekete–Szegö functional associated with k-th root transformation using quasi-subordination

La fonctionnelle de Fekete–Szegö associée aux transformations racine k^e utilisant la subordination

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A R T I C L E   I N F O

Article history:
Received 14 January 2015
Accepted after revision 30 March 2015
Available online 8 May 2015

Presented by the Editorial Board

A B S T R A C T

Quasi-subordination is an underlying concept in the area of complex function theory. It is an interesting topic that unifies the concept of both subordination and majorization. There has been no work in this area for the past three decades except possibly a recent article (Haji Mohd and Darus, Fekete–Szegö problems for quasi-subordination classes, Abstr. Appl. Anal. 2012 (2012) 192956, 14 p.) [8]. Exploiting this article, we provide an estimate with k-th root transform for certain classes of analytic univalent functions using quasi-subordination. The authors sincerely hope that this article will revive this concept and encourage other researchers to work in this quasi-subordination in the near-future in the area of complex function theory.

R É S U M É

La quasi-subordination est un concept sous-jacent de la théorie des fonctions complexes. C'est un sujet intéressant, qui unifie les concepts de subordination et de majoration. Il n'y a pas eu de travaux dans ce domaine au cours des trois dernières décennies, à part, peut-être, un article récent (Haji Mohd and Darus, Fekete–Szegö problems for quasi-subordination classes, Abstr. Appl. Anal. 2012 (2012) 192956, 14 p.) [8]. Exploitant cet article, nous donnons une estimation avec les transformations racine k^e pour certaines classes de fonctions analytiques univalentes, utilisant la quasi-subordination. Les auteurs forment le vœu que cet article ravive l'intérêt pour ce concept et encourage, dans un proche avenir, d'autres chercheurs à le considérer en théorie des fonctions complexes.

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http://dx.doi.org/10.1016/j.crma.2015.03.016
1631-073X/

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open disk $\Delta = \{z : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in Δ. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto the region starlike with respect to 1, which is symmetric with respect to x-axis. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient $|a_2|$ of normalized univalent functions readily yields the growth and distortion bounds for univalent functions. The Fekete–Szegö coefficient functional $|a_3 - \mu a_2^2|$, also arises naturally in the investigation of the univalency of analytic functions. Several authors have investigated the Fekete–Szegö functional for functions in various subclasses of univalent and multivalent functions, see related works in Refs. [1], [2], [4], [5], [6] and [11]. A function $f(z)$ is subordinate to a function $g(z)$, written as $f(z) \prec g(z)$, provided that there is a function $w(z)$, analytic Δ, with $w(0) = 0$ such that $|w(z)| < 1$ and $f(z) = g[w(z)]$ for $z \in \Delta$. In particular, if the function $g(z)$ is univalent in Δ, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$. A function $f \in \mathcal{A}$ is starlike iff $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ or equivalently if $\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$. Ma and Minda [12] have given a unified treatment of various subclasses consisting of starlike and convex functions for which either one of the quantities $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. The unified class $\mathcal{S}^*(\varphi)$ introduced by Ma and Minda [12] consists of functions $f \in \mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi(z)$, $z \in \Delta$ and the corresponding class $C(\varphi)$ of convex functions $f \in \mathcal{A}$ satisfying $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$, $z \in \Delta$. Ma and Minda [12] obtained estimates for the first few coefficients and determined bounds for the associated Fekete–Szegö functional. Many other authors have also investigated the bounds for the Fekete–Szegö coefficient functional for various classes of $\mathcal{S}$ (for example, see the work of [1,2,7]). In [14] Robertson introduced the concept of quasi-subordination. An analytic function $f(z)$ is quasi-subordinate to an analytic function $g(z)$, in the open unit disk if there exist analytic functions φ and w, with $w(0) = 0$ such that $|\varphi(z)| \le 1$, $|w(z)| < 1$ and $f(z) = \varphi(z) g[w(z)]$. Then we write $f(z) \prec_q g(z)$. If $\varphi(z) = 1$, then the quasi-subordination reduces to the subordination. Also, if $w(z) = z$, then $f(z) = \varphi(z) g(z)$, and in this case we say that $f(z)$ is majorized by $g(z)$ and it is written as $f(z) \prec\prec g(z)$ in Δ. Hence it is obvious that quasi-subordination is the generalization of subordination as well as majorization. It is unfortunate that the concept of quasi-subordination is so for an underlying concept in the area of complex function theory, although it deserves much attention as it unifies the concept of both subordination and majorization. There has been no work in this area for the past three decades, except possibly a recent article published in [8]. Exploiting this article, we provide an estimate with k-th root transform for certain classes of analytic univalent functions using quasi-subordination. The authors sincerely hope that this article will revive and encourage other researchers to work on this quasi-subordination in the near-future in the area of complex function theory. Further, we refer to [3,10,13] for works related to quasi-subordination. Haji Mohd and Darus [8] obtained the bounds of $|a_2|$ and $|a_3|$ for the following three classes defined by quasi-subordination as follows.

Definition 1.1. Let the class $\mathcal{S}_q^*(\phi)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$\frac{zf'(z)}{f(z)} - 1 \prec_q \phi(z) - 1.$$

Definition 1.2. Let the class $\mathcal{S}_q^*(\alpha, \phi)$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} - 1 \prec_q \phi(z) - 1.$$

Definition 1.3. Let the class $M_q(\alpha, \phi)$, $\alpha \ge 0$ consist of function $f \in \mathcal{A}$ satisfying the quasi-subordination

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \phi(z) - 1.$$

Throughout this paper, we assume $\phi(z)$ is an analytic function with $\phi(0) = 1$.

Definition 1.4. Let k be a positive integer. A domain D is said to be *k-fold symmetric* if a rotation of D about the origin through an angle $\frac{2\pi}{k}$ carries D to itself. A function $f(z)$ is said to be *k-fold symmetric* in Δ if for every z in Δ

$$f\left(e^{\frac{2\pi i}{k}}z\right) = e^{\frac{2\pi i}{k}} f(z).$$

In 1916, Gronwall shows that if $f(z)$ is regular and *k-fold symmetric* in Δ, then it has a power series expansion of the form

$$f(z) = b_1 z + b_{k+1} z^{k+1} + b_{2k+1} z^{2k+1} + \cdots = \sum_{n=0}^{\infty} b_{nk+1} z^{nk+1}. \tag{1.2}$$

Conversely, if $f(z)$ is given by the power series (1.2), then $f(z)$ is *k-fold symmetric* inside the circle of convergence of the series. For a univalent function $f(z)$ of the form in (1.1), the k-th root transform is defined by

$$F(z) = \left[f\left(z^k\right)\right]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.3}$$

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. Many authors have also investigated the bounds for the Fekete–Szegö coefficient functional for various classes [1,2,7]. Let $\mathcal{A}^*$ denote the class of all analytic functions f of the form (1.1) such that $f^{-1}(\{0\}) \cap \Delta = \{0\}$, that is the origin is the unique zero of f in Δ.

In this paper, we obtain Fekete–Szegö estimates for the function in the classes $\mathcal{S}_q^*(\phi)$, $\mathcal{S}_q^*(\alpha, \phi)$ and $M_q(\alpha, \phi)$, if $f \in \mathcal{A}^*$. We need the following lemma to prove our results.

Lemma 1.1. *(See [9].) If $w \in \Omega$ and $w(z) = w_1 z + w_2 z^2 + \ldots$ $(z \in \Delta)$, then*

$$|w_2 - t w_1^2| \le \max\{1, |t|\}$$

for any complex number t. The result is sharp for the function $w(z) = z^2$ or $w(z) = z$.

In this paper $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ and $\varphi(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \ldots$, $B_1 \in \mathbb{R}$, $B_1 > 0$, and $|C_n| \le 1$.

2. Main results

Theorem 2.1. *If $f \in \mathcal{A}^*$ given by (1.1) belongs to $\mathcal{S}_q^*(\phi)$ and F is the k-th root transformation of f given by (1.3), then $|b_{k+1}| \le \frac{B_1}{k}$, $|b_{2k+1}| \le \frac{1}{2k}\left[B_1 + \max\left\{B_1, \frac{B_1^2}{k} + |B_2|\right\}\right]$ and for any complex number μ*

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{1}{2k}\left[B_1 + \max\left\{B_1, \frac{|1-2\mu|}{k} B_1^2 + |B_2|\right\}\right].$$

Proof. Let $f \in \mathcal{A}^*$ belongs to class $\mathcal{S}_q^*(\phi)$. Then there exist analytic functions φ and w with $|\varphi(z)| \le 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} - 1 = \varphi(z)[\phi(w(z)) - 1]. \tag{2.1}$$

We have

$$\frac{zf'(z)}{f(z)} - 1 = a_2 z + (-a_2^2 + 2a_3) z^2 + \cdots$$

and

$$\varphi(z)[\phi(w(z)) - 1] = B_1 C_0 w_1 z + [B_1 C_1 w_1 + C_0(B_1 w_2 + B_2 w_1^2)] z^2 + \cdots. \tag{2.2}$$

Therefore,

$$a_2 = B_1 C_0 w_1 \tag{2.3}$$

and

$$a_3 = \frac{1}{2}[B_1 C_1 w_1 + B_1 w_2 C_0 + C_0(B_2 + B_1^2 C_0) w_1^2]. \tag{2.4}$$

For a function f given by (1.1), a computation shows that

$$[f(z^k)]^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left[\frac{1}{k}a_3 - \frac{1}{2}\left(\frac{k-1}{k^2}\right){a_2}^2\right] z^{2k+1} + \cdots. \tag{2.5}$$

Upon equating the coefficients of z^{k+1} and z^{2k+1} in view of (1.2) and (2.5), we get

$$b_{k+1} = \frac{1}{k}a_2, \text{ and } b_{2k+1} = \frac{a_3}{k} - \frac{1}{2}\left(\frac{k-1}{k^2}\right){a_2}^2. \tag{2.6}$$

Further, from (2.3), (2.4) and (2.6), one can easily get $b_{k+1} = \frac{1}{k}B_1 w_1 C_0$, and $b_{2k+1} = \frac{1}{2k}\left[B_1 C_1 w_1 + B_1 w_2 C_0 + B_2 C_0 w_1^2 + \frac{B_1^2 w_1^2 C_0^2}{k}\right]$. Also, for any complex number μ,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k}\left\{w_1 C_1 + w_2 C_0 - \left[-\frac{B_1 C_0}{k}(1-2\mu) - \frac{B_2}{B_1}\right] C_0 w_1^2\right\}.$$

Using the inequalities $|C_n| \le 1$, $|w_n(z)| \le 1$, we get $|b_{k+1}| \le \frac{B_1}{k}$, and

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{B_1}{2k}\left\{1 + \left|w_2 - \left[-\frac{B_1 C_0}{k}(1-2\mu) - \frac{B_2}{B_1}\right]{w_1}^2\right|\right\}.$$

An application of Lemma 1.1 to $\left|w_2 - \left[-\frac{B_1 C_0}{k}(1-2\mu) - \frac{B_2}{B_1}\right]{w_1}^2\right|$, yields

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{B_1}{2k}\left[1 + \max\left\{1, \left|-\frac{B_1 C_0}{k}(1-2\mu) - \frac{B_2}{B_1}\right|\right\}\right].$$

Since

$$\left|-\frac{B_1 C_0}{k}(1-2\mu) - \frac{B_2}{B_1}\right| \le \frac{B_1|1-2\mu|}{k} + \left|\frac{B_2}{B_1}\right|,$$

we conclude that

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{1}{2k}\left[B_1 + \max\left\{B_1, \frac{|1-2\mu|}{k}B_1^2 + |B_2|\right\}\right].$$

For $\mu = 0$, we get $|b_{2k+1}| \le \frac{1}{2k}\left[B_1 + \max\left\{B_1, \frac{B_1^2}{k} + |B_2|\right\}\right]$. This essentially completes the proof of Theorem 2.1. □

Theorem 2.2. *If $f \in \mathcal{A}^*$ is given by (1.1) belongs to the class $M_q(\alpha, \phi)$ and F is the k-th root transformation of f given by (1.3), then*

$$|b_{k+1}| \le \frac{B_1}{k(1+\alpha)}, |b_{2k+1}| \le \frac{1}{2k(1+2\alpha)}\left[B_1 + \max\{\left\{B_1, \left|\frac{k\alpha + (1+2\alpha)}{k(1+\alpha)^2}\right| B_1^2 + |B_2|\right\}\right]$$

and for any complex number μ

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{1}{2k(1+2\alpha)}\left[B_1 + \max\left\{B_1, \left|\frac{k\alpha + (1+2\alpha)(1-2\mu)}{k(1+\alpha)^2}\right| B_1^2 + |B_2|\right\}\right].$$

Proof. If $f \in \mathcal{A}^*$ belongs to the class $M_q(\alpha, \phi)$, $\alpha \ge 0$, then there exist analytic functions φ and w with $|\varphi(z)| \le 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = \varphi(z)[\phi(w(z)) - 1]. \tag{2.7}$$

Then we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = (1+\alpha)a_2 z + [-(1+3\alpha)a_2^2 + 2(1+2\alpha)a_3]z^2 + \cdots. \tag{2.8}$$

From (2.2), (2.7) and (2.8), we have

$$a_2 = \frac{B_1 C_0 w_1}{1+\alpha} \tag{2.9}$$

and

$$a_3 = \frac{1}{2(1+2\alpha)} \left[B_1 C_1 w_1 + B_1 w_2 C_0 + \left(B_2 C_0 + \frac{(1+3\alpha) B_1^2 C_0^2}{(1+\alpha)^2} \right) w_1^2 \right]. \tag{2.10}$$

From (1.2) and (2.5), equating the coefficients of z^{k+1} and z^{2k+1}, we get

$$b_{k+1} = \frac{1}{k} a_2 \tag{2.11}$$

and

$$b_{2k+1} = \frac{a_3}{k} - \left(\frac{k-1}{2k^2} \right) a_2^2. \tag{2.12}$$

From (2.9), (2.10), (2.11) and (2.12), we have $b_{k+1} = \dfrac{B_1 w_1 C_0}{k(1+\alpha)}$ and

$$b_{2k+1} = \frac{1}{2k(1+2\alpha)} \left[B_1 w_1 C_1 + B_1 w_2 C_0 + B_2 C_0 w_1^2 + \frac{(1+3\alpha) B_1^2 C_0^2 w_1^2}{(1+\alpha)^2} - \frac{(1+2\alpha) B_1^2 C_0^2 w_1^2}{(1+\alpha)^2} + \frac{(1+2\alpha) B_1^2 C_0^2 w_1^2}{k(1+\alpha)^2} \right].$$

Therefore, for any complex number μ,

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k(1+2\alpha)} \left[w_1 C_1 + C_0 \left\{ w_2 + \left(\frac{B_2}{B_1} + \frac{\alpha B_1 C_0}{(1+\alpha)^2} + \frac{(1+2\alpha)(1-2\mu) B_1 C_0}{k(1+\alpha)^2} \right) w_1^2 \right\} \right].$$

Using the inequalities $|C_n| \le 1$ and $|w_n(z)| \le 1$, we get $|b_{k+1}| \le \frac{B_1}{k(1+\alpha)}$, and

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{B_1}{2k\,(1+2\alpha)} \left\{ 1 + \left| w_2 - \left[-\frac{B_2}{B_1} - \left(\frac{k\alpha + (1+2\alpha)\,(1-2\mu)}{k(1+\alpha)^2} \right) B_1 \right] {w_1}^2 \right| \right\}.$$

Applying Lemma 1.1 to

$$\left| w_2 - \left[-\frac{B_2}{B_1} - \left(\frac{k\alpha + (1+2\alpha)\,(1-2\mu)}{k(1+\alpha)^2} \right) B_1 \right] {w_1}^2 \right|,$$

yields

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{1}{2k(1+2\alpha)} \left[B_1 + \max \left\{ B_1, \left| \frac{k\alpha + (1+2\alpha)(1-2\mu)}{k(1+\alpha)^2} \right| B_1^2 + |B_2| \right\} \right].$$

For $\mu = 0$, we have

$$|b_{2k+1}| \le \frac{1}{2k(1+2\alpha)} \left[B_1 + \max \left\{ B_1, \left| \frac{k\alpha + (1+2\alpha)}{k(1+\alpha)^2} \right| B_1^2 + |B_2| \right\} \right],$$

which completes the proof of Theorem 2.2. □

Theorem 2.3. *If $f \in \mathcal{A}^*$ given by (1.1) belongs to $\mathcal{S}_q^*(\alpha, \phi)$ and F is the k-th root transformation of f given (1.3) then*

$$|b_{k+1}| \le \frac{B_1}{k(1+2\alpha)}, \tag{2.13}$$

$$|b_{2k+1}| \le \frac{1}{2k(1+3\alpha)} \left[B_1 + \max \left\{ B_1, \left| \frac{k\alpha - (1+3\alpha)}{k(1+2\alpha)^2} \right| B_1^2 + |B_2| \right\} \right], \tag{2.14}$$

and for any complex number μ

$$|b_{2k+1} - \mu b_{k+1}^2| \le \frac{1}{2k(1+3\alpha)} \left[B_1 + \max \left\{ B_1, \left| \frac{k\alpha - (1-2\mu)(1+3\alpha)}{k(1+2\alpha)^2} \right| B_1^2 + |B_2| \right\} \right]. \tag{2.15}$$

Proof. If $f \in \mathcal{A}^*$ belongs to class $\mathcal{S}_q^*(\alpha, \phi)$, $\alpha \geq 0$, then there exist analytic functions φ and w with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} - 1 = \varphi(z)[\phi(w) - 1].$$

Further, we have

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} - 1 = a_2(1+2\alpha)z + [2(1+3\alpha)a_3 - (1+2\alpha)a_2^2]z^2 + \cdots.$$

By applying a similar technique as in Theorem 2.1 and Theorem 2.2, one can obtain the inequality (2.13), (2.14) and (2.15) which essentially proves Theorem 2.3. □

Authors contributions All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements

The authors sincerely thank the referee for a careful reading of the original manuscript. The work of the third author is supported by a grant from Department of Science and Technology, Government of India vide ref: SR/FTP/MS-022/2012 under fast-track scheme.

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