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Partial differential equations

Courant-sharp eigenvalues of a two-dimensional torus

Valeurs propres Courant-strictes d'un tore bidimensionnel

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we determine, in the case of the Laplacian on the flat two-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$, all the eigenvalues having an eigenfunction that satisfies Courant's theorem with equality (Courant-sharp situation). Following the strategy of Å. Pleijel (1956) [18], the proof is a combination of a lower bound (à la Weyl) of the counting function, with an explicit remainder term, and of a Faber–Krahn inequality for domains on the torus (deduced as in the work of P. Bérard and D. Meyer from an isoperimetric inequality), with an explicit upper bound on the area.

R É S U M É

Cette note vise à déterminer quelles sont les valeurs propres du laplacien sur le tore plat $(\mathbb{R}/\mathbb{Z})^2$ qui ont une fonction propre réalisant le cas d'égalité dans le théorème de Courant (situation Courant-stricte). Nous suivons la stratégie de Å. Pleijel (1956) [18], qui associe une borne inférieure de type loi de Weyl pour la fonction de comptage et une inégalité de type Faber–Krahn. Comme dans les travaux de P. Bérard et D. Meyer, cette dernière est déduite d'une inégalité isopérimétrique, avec une condition de petitesse, ici explicitée, sur l'aire du domaine.

1. Introduction

Let us first recall two classical results on the eigenvalues and eigenfunctions of the Dirichlet–Laplacian on a bounded domain Ω in the plane. According to a well-known result by R. Courant (in [8]), an eigenfunction (real valued) associated with the k-th eigenvalue $\lambda_k(\Omega)$ of this operator has at most k nodal domains. In [18], Å. Pleijel sharpened this result by showing that, for a given domain, an eigenfunction associated with $\lambda_k(\Omega)$ has less than k nodal domains, except for a finite number of indices k. This was generalized in [5] by P. Bérard and D. Meyer to the case of a compact Riemannian manifold, with or without boundary, in any dimension. It has been shown by I. Polterovich in [19], using estimates from [20], that the analogous result also holds for the Neumann–Laplacian on a planar domain with a piecewise-analytic boundary.

These results leave open the question of determining, for a specific domain or manifold, the cases of equality. It is stated in [18] that when Ω is a square, equality can only occur for eigenfunctions having one, two or four nodal domains,

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associated with the first, the second (which has multiplicity two), and the fourth eigenvalue respectively. The proof in [18] is however incomplete and was corrected by P. Bérard and B. Helffer in [3]. The case of the sphere is treated in [11], see also [16,17] for a related study. The cases of an equilateral torus and an equilateral triangle are investigated in [4], and the case of the Neumann-Laplacian in a square is treated in [12]. In this note, we will show that for the flat torus $(\mathbb{R}/\mathbb{Z})^2$, equality holds only for eigenfunctions having one or two nodal domains, respectively associated with the first and the second eigenvalue (this last eigenvalue has multiplicity four). This complements the result [9, Theorem 7.1], which determines the cases of equality for a flat torus $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/b\mathbb{Z})$ with b^2 irrational.

Let us give a more precise statement of the above result, and fix some notation that will be used in the following. In the rest of this paper, $\mathbb{T}^2$ stands for the two-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$ equipped with the standard flat metric, and $-\Delta_{\mathbb{T}^2}$ stands for the Laplace–Beltrami operator on $\mathbb{T}^2$. If Ω is an open set in $\mathbb{T}^2$ with a sufficiently regular boundary, we write $(\lambda_k(\Omega))_{k\geq 1}$ for the eigenvalues of $-\Delta_{\mathbb{T}^2}$ in Ω with a Dirichlet boundary condition on $\partial\Omega$, arranged in non-decreasing order and counted with multiplicity. In particular, $\lambda_k(\mathbb{T}^2)$ is the k-th eigenvalue of $-\Delta_{\mathbb{T}^2}$ (in that case the boundary is empty). If u is an eigenfunction of $-\Delta_{\mathbb{T}^2}$, we call *nodal domains of* u the connected components of $\mathbb{T}^2 \setminus u^{-1}(\{0\})$, and we denote by $\mu(u)$ the cardinal of the set of nodal domains. With any eigenvalue λ of $-\Delta_{\mathbb{T}^2}$, we associate the integer

$$\nu(\lambda) = \min\{k \in \mathbb{N}^* : \lambda_k(\mathbb{T}^2) = \lambda\}.$$

We say that an eigenvalue λ of $-\Delta_{\mathbb{T}^2}$ is *Courant-sharp* if there exists an associated eigenfunction u such that $\mu(u) = \nu(\lambda)$. Following [10], we also use the adjective *Courant-sharp* for such an eigenfunction u. We will prove the following result.

Theorem 1. *The only Courant-sharp eigenvalues of* $-\Delta_{\mathbb{T}^2}$ *are* $\lambda_k(\mathbb{T}^2)$ *with* $k \in \{1, 2, 3, 4, 5\}$.

The proof follows the approach used by Å. Pleijel in [18] and in the case of a compact manifold by P. Bérard and D. Meyer in [5] (see also [1,2]). We first establish a Faber–Krahn-type inequality for domains in $\mathbb{T}^2$ whose area is sufficiently small. We deduce it from an isoperimetric inequality proved in [13, 7]. We then combine this information with an explicit lower bound of the counting function (similar to Weyl's law) to show that large eigenvalues cannot be Courant-sharp.

Let us point out that interest in Courant-sharp eigenvalues has grown recently thanks to their connection to minimal partition problems. This appears clearly in the paper [10], where the authors consider the following problem: given a two-dimensional domain Ω and an integer k, find a *k-partition* of Ω, that is to say a family $(D_i)_{1\leq i\leq k}$ of k open, connected, and disjoint subsets of Ω, that minimizes the energy

$$\max_{1\leq i\leq k} \lambda_1(D_i).$$

Such a k-partition is said to be *minimal*. Existence and regularity for minimal partitions are proved in [10]. Following [10], let us say that a k-partition is *nodal* if it is the family of the nodal domains for some eigenfunction of the Dirichlet Laplacian on Ω. It is shown in [10] that a nodal partition is minimal if, and only if, the corresponding eigenfunction is Courant-sharp. Finding nodal minimal partitions is therefore equivalent to finding Courant-sharp eigenfunctions. In particular, Theorem 1 has the following consequence.

Corollary 2. *If* $k \geq 3$, *minimal k-partitions of* $\mathbb{T}^2$ *are not nodal.*

The problem of finding minimal k-partitions of $\mathbb{T}^2$, with $k \geq 3$, is studied in [14]. In this paper, a numerical method, based on [6], is used to produce candidates to be minimal partitions for $k \in \{3, 4, 5\}$. They seem to be tilings of $\mathbb{T}^2$ by hexagons or squares.

2. Proof of the theorem

2.1. Faber–Krahn inequality

Let us first recall an isoperimetric inequality, which is a special case of [13, 7].

Proposition 3. *If* Ω *is an open set in* $\mathbb{T}^2$ *such that* $|\Omega| \leq \frac{1}{\pi}$, *we have the inequality*

$$\mathcal{H}^1(\partial\Omega)^2 \geq 4\pi|\Omega|. \quad (1)$$

In this proposition, $|\Omega|$ stands for the usual two-dimensional area measure of Ω and $\mathcal{H}^1(\Omega)$ for the one-dimensional Hausdorff measure of $\partial\Omega$. This inequality is also proved in [15], by a more elementary method than in [13].

Proposition 4. *If* Ω *is an open set in* $\mathbb{T}^2$ *such that* $|\Omega| \leq \frac{1}{\pi}$, *then*

$$\lambda_1(\Omega)|\Omega| \geq \pi j_{0,1}^2. \quad (2)$$

The constant $j_{0,1}$ in Eq. (2) is the first positive zero of the Bessel function of the first kind J_0. Let us note that $\pi j_{0,1}^2$ is the value of the product $\lambda_1(D)|D|$, when D is a disk in the Euclidean plane $\mathbb{R}^2$. As in the planar case, the proof uses the Schwarz symmetrization of the level sets $\Omega_t = \{x : u(x) > t\}$, where u is a positive eigenfunction associated with $\lambda_1(\Omega)$ and $t > 0$. We go through the same steps as in [5, I.9], or [7, III.3]. Note that since $|\Omega_t| \le |\Omega| \le \frac{1}{\pi}$ for all $t > 0$, we can use inequality (1).

2.2. Weyl's law with explicit bounds

For $\lambda \ge 0$, we define the *counting function* by $N(\lambda) = \sharp\{k : \lambda_k(\mathbb{T}^2) \le \lambda\}$.

Proposition 5. *We have, for all* $\lambda \ge 0$*,*

$$\frac{\lambda}{4\pi} - \frac{2\sqrt{\lambda}}{\pi} - 3 \le N(\lambda). \tag{3}$$

Proof. The proof consists in counting lattice points contained in a planar region. Indeed, the eigenvalues of $-\Delta_{\mathbb{T}^2}$ are of the form

$$\lambda_{m,n} = 4\pi^2(m^2 + n^2)\,,$$

with $(m,n) \in \mathbb{N}^2$. With each pair of integers (m,n) we associate a finite dimensional space $E_{m,n}$ of eigenfunctions such that

$$L^2(\mathbb{T}^2) = \overline{\bigoplus_{(m,n)} E_{m,n}}.$$

The vector space $E_{m,n}$ is generated by products of trigonometric functions, see for instance the proof of [9, Theorem 2.2] for details. The dimension of $E_{m,n}$ is 1 if $(m,n) = (0,0)$, 2 if either m or n, but not both, is 0, and 4 if $m > 0$ and $n > 0$. Let us denote by $n(\lambda)$ the number of points with positive and integer coordinates contained in the disk of center 0 and radius $\frac{\sqrt{\lambda}}{2\pi}$. Taking the dimension of the spaces $E_{m,n}$ into account, we have the following exact formula for the counting function:

$$N(\lambda) = 4n(\lambda) - 4\left\lfloor \frac{\sqrt{\lambda}}{2\pi} \right\rfloor - 3, \tag{4}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x. By covering the upper right-hand quarter of the disk of center 0 and radius $\frac{\sqrt{\lambda}}{2\pi}$ with squares of side 1, we see that $n(\lambda) \ge \frac{\lambda}{16\pi}$, and we obtain the desired lower bound. □

2.3. Courant-sharp eigenvalues of the torus

We now turn to the proof of Theorem 1. We will use the following lemmas.

Lemma 6. *If λ is an eigenvalue of $-\Delta_{\mathbb{T}^2}$ that has an associated eigenfunction u with k nodal domains, and if $k \ge 4$,*

$$\pi j_{0,1}^2 k \le \lambda.$$

Proof. Since $|\mathbb{T}^2| = 1$, one of the nodal domains of u has an area no larger than $\frac{1}{k}$. Let us denote this nodal domain by D. Since $k \ge 4$, $|D| \le \frac{1}{k} < \frac{1}{\pi}$. According to Proposition 4,

$$\lambda = \lambda_1(D) \ge \frac{\pi j_{0,1}^2}{|D|} \ge \pi j_{0,1}^2 k. \qquad \square$$

Corollary 7. *If λ is a Courant-sharp eigenvalue of $-\Delta_{\mathbb{T}^2}$ with $\nu(\lambda) \ge 4$,*

$$\pi j_{0,1}^2 \nu(\lambda) \le \lambda. \tag{5}$$

Lemma 8. *For all $k \in \mathbb{N}$,*

$$\lambda_k(\mathbb{T}^2) \le \left(4 + 2\sqrt{4 + \pi(k+3)}\right)^2. \tag{6}$$

Table 1
The first 57 eigenvalues.

Tableau 1
Les 57 premières valeurs propres.

$\frac{\lambda}{4\pi^2}$	Indices	Multiplicity	$\nu(\lambda)$
0	$(0,0)$	1	1
1	$(1,0)$, $(0,1)$	4	2
2	$(1,1)$	4	6
4	$(2,0)$, $(0,2)$	4	10
5	$(2,1)$, $(1,2)$	8	14
8	$(2,2)$	4	22
9	$(3,0)$, $(0,3)$	4	26
10	$(3,1)$, $(1,3)$	8	30
13	$(3,2)$, $(2,3)$	8	38
16	$(4,0)$, $(0,4)$	4	46
17	$(4,1)$, $(1,4)$	8	50

Table 2
Table of ratios.

Tableau 2
Tableau des rapports.

k	6	10	14	22	26	30	38	46
$\frac{\lambda_k(\mathbb{T}^2)}{4k\pi^2}$	0.3333	0.4000	0.3571	0.3636	0.3462	0.3333	0.3421	0.3478

Proof. The proof is immediate from the following remark: if λ is a non-negative number such that $N(\lambda) \geq k$, then $\lambda_k(\mathbb{T}^2) \leq \lambda$. The lower bound for $N(\lambda)$ given in inequality (3) then implies the desired upper bound. □

Comparing the lower bound (5) with the upper bound (6), we can easily show that if λ is an eigenvalue of $-\Delta_{\mathbb{T}^2}$ with $\nu(\lambda) \geq 50$, it is not Courant-sharp. Table 1 gives the first fifty-seven eigenvalues of $-\Delta_{\mathbb{T}^2}$. It shows that we have to test inequality (5) for $\lambda = \lambda_k(\mathbb{T}^2)$ with $k \in \{6, 10, 14, 22, 26, 30, 38, 46\}$. Table 2 displays the ratio

$$\frac{\lambda_k(\mathbb{T}^2)}{4k\pi^2},$$

which should be greater than

$$\frac{j_{0,1}^2}{4\pi} \simeq 0.4602$$

in case $\lambda_k(\mathbb{T}^2)$ is Courant-sharp. This does not happen in the cases considered, and therefore Theorem 1 is proved.

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