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Dynamical systems

The C^0 general density theorem for geodesic flows

Le théorème de densité de Pugh C^0 pour les flots géodésiques

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A R T I C L E   I N F O

A B S T R A C T

Given a closed Riemannian manifold, we prove the C^0-general density theorem for continuous geodesic flows. More precisely, we prove that there exists a residual (in the C^0-sense) subset of the continuous geodesic flows such that, in that residual subset, the geodesic flow exhibits dense closed orbits.

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R É S U M É

Étant donnée une variété riemannienne compacte sans bord, nous démontrons un théorème de densité C^0-générique pour les flots géodésiques et, plus précisément, nous prouvons qu'il existe une partie C^0-résiduelle de l'ensemble des flots géodésiques continus, telle que tout flot dans cette partie admet un ensemble dense d'orbites périodiques.

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1. Introduction: basic definitions and statement of the results

1.1. The geodesic flow and mechanical Hamiltonians

A Riemannian manifold (M, g) is a C^∞-manifold with an Euclidean inner product g_x in each $T_x M$ that varies smoothly with respect to $x \in M$. Therefore a Riemannian metric is a smooth section $g \colon M \to \mathrm{Symm}_2^+(TM)$, where $\mathrm{Symm}_2^+(TM)$ is the set of positive bilinear and symmetric forms in TM. Given a Riemannian metric g on TM, we denote by $d_{TM}(\cdot, \cdot)$ the geodesic distance associated with g on TM. Note that since all Riemannian metrics are Lipschitz equivalent on compact subsets, the choice of the metric on TM is not important.

The geodesic flow of the metric g of class C^2 is the flow on TM defined by

$$
\phi_g^t \colon \begin{array}{ccc} TM & \longrightarrow & TM \\ (x, v) & \longmapsto & (\gamma_{x,v}^g(t), \dot\gamma_{x,v}^g(t)), \end{array}
$$

where $\gamma_{x,v}^g \colon \mathbb{R} \longrightarrow M$ denotes the geodesic starting at x with initial velocity v, $x \in M$, $v \in T_x M$. Since the speed of the geodesic is constant, we can consider the flow restricted to $UT(M) := \{(x, v) \in TM : g_x(v, v) = 1\}$.

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Clearly, the orbit of a point $(x, v) \in UT(M)$ consists of the tangent vectors to the geodesic defined by (x, v) and periodic orbits for the geodesic flow correspond to closed geodesics on M. We write $Per(\phi_g^t) \subset UT(M)$ for the set of periodic orbits. Recall that $(x, v) \in UT(M)$ belongs to the *nonwandering set* of ϕ_g^t, denoted by $\Omega(\phi_g^t)$, if for every neighborhood V of (x, v) there exists $t_n \to \infty$ such that $\phi_g^{t_n}(V) \cap V \neq \emptyset$. We say that $(x, v) \in UT(M)$ is a ϕ_g^t-*recurrent point*, and we denote this set by $R(\phi_g^t)$, if given any neighborhood V of (x, v), there exists t_n such that $\phi_g^{t_n}((x, v)) \in V$. We have $R(\phi_g^t) \subset \Omega(\phi_g^t)$. From now on we assume that M is closed and with dimension ≥ 2. We observe that the Liouville volume is invariant under the geodesic flow. Thus, Poincaré's recurrence theorem (see, e.g., [10]) asserts that almost every point (in the Lebesgue measure) is recurrent. Therefore, we conclude that Lebesgue almost every point is a nonwandering point and so $UT(M) = \Omega(\phi_g^t)$.

The geodesic flow is a subclass of the Hamiltonian flows. Actually, for the metric g on M, we have the associated Hamiltonian defined in the cotangent bundle T^*M, endowed with the canonical symplectic form, by:

$$H \colon \quad T^*M \quad \longrightarrow \quad \mathbb{R}$$
$$(x, p) \quad \longmapsto \quad \tfrac{1}{2}(\|p\|_x^*)^2,$$

where $\| \cdot \|^*$ stands for the dual norm on the cotangent bundle. Usually, we call H the *mechanical Hamiltonian* because it is given only by the kinetic energy and free of potential.

1.2. Statement of the result and some history

Let $\mathcal{R}^k(M)$ denote the set of C^k Riemannian metrics in M. For $\ell \in \{1, \dots, k\}$ we endow $\mathcal{R}^k(M)$ with the C^ℓ-topology. A property in $\mathcal{R}^k(M)$, endowed with the C^ℓ topology is said to be C^ℓ-*generic* if it holds in a C^ℓ-residual subset of $\mathcal{R}^k(M)$. In particular, since when $\ell = k$ the set $\mathcal{R}^k(M)$ is a Baire space, by Baire's Category theorem (see [12]) a C^k-residual subset of $\mathcal{R}^k(M)$ is C^k-dense in $\mathcal{R}^k(M)$. Consequently, a C^k-residual subset of $\mathcal{R}^k(M)$ is C^ℓ-dense in $\mathcal{R}^k(M)$, for any $\ell \in \{1, \dots, k\}$.

Let $C^0([a, b] \times UT(M), UT(M))$ be the set of continuous maps from $[a, b] \times UT(M)$ into $UT(M)$. We say that ϕ is a C^0 flow on the unit tangent bundle $UT(M)$ if $\phi \colon \mathbb{R} \times UT(M) \to UT(M)$ is such that $\phi(t, \cdot)$ is a homeomorphism and $\phi(\cdot, (x, v))$ is a C^1 curve on $UT(M)$. As usual, we denote a flow by ϕ^t. We are interested in volume-preserving flows, i.e., the Lebesgue measure is ϕ^t-invariant. We let $\mathcal{F}^0(UT(M))$ be the set of volume-preserving C^0 flows on $UT(M)$. For any $a < b$, we define the map:

$$\rho_{a,b} \colon \quad \mathcal{F}^0(UT(M)) \quad \longrightarrow \quad C^0([a, b] \times UT(M), UT(M))$$
$$\phi^t \quad \longmapsto \quad \phi^t|_{[a,b] \times UT(M)}$$

The compact-open topology, that we shall denote by τ, is the smallest one making $\rho_{a,b}$ continuous (see [19,8]). We endow $\mathcal{F}^0(UT(M))$ with τ. Let $\mathcal{G}^1(UT(M))$ be the set of geodesic flows associated with metrics in $\mathcal{R}^2(M)$. We define the set of *continuous geodesic flows* by the τ-closure of $\mathcal{G}^1(UT(M))$ and denote this set by $\mathcal{G}^0(UT(M)) \subset \mathcal{F}^0(UT(M))$. Observe that $(\mathcal{G}^0(UT(M)), \tau)$ is Baire [8, §2.4]. Moreover, if $\phi_{g_1}^t, \phi_{g_2}^t \in \mathcal{G}^1(UT(M))$ and g_1 and g_2 are C^1-close, then, as a consequence of Gronwall's inequality (see, e.g., [16]), we get that $\phi_{g_1}^t$ and $\phi_{g_2}^t$ are τ-close. A property in $\mathcal{F}^0(UT(M))$ is said to be τ-*generic* if it holds in a τ-residual subset of $\mathcal{F}^0(UT(M))$. Once again, by Baire's Category theorem, a τ-residual subset of $\mathcal{F}^0(UT(M))$ is τ-dense in $\mathcal{F}^0(UT(M))$.

Given any continuous geodesic flow $\phi \in \mathcal{G}^0(UT(M))$, the definitions of the nonwandering and recurrent sets of ϕ^t can be given analogously to the ones given for a geodesic flow $\phi_g^t \in \mathcal{G}^1(UT(M))$.

Considering C^0-closures of Hamiltonian flows is a subject of growing interest and it is quite related to Gromov–Eliasberg symplectic rigidity (see, e.g., [3,13,15,22]). In particular, we mention a recent result on the rigidity of C^0-geodesic flows [14, §6] which, in rough terms, says that, if the sequence of metrics $g_n \in \mathcal{R}^2(M)$ converges in the C^0-sense to g and its geodesic flows are uniformly Cauchy, then the geodesic flows associated with g_n, $\phi_{g_n}^t$, converge to the geodesic flow of g.

Given a Riemannian metric g that generates a geodesic flow ϕ_g^t, a central question in dynamical systems is to know whether the periodic orbits of ϕ_g^t are dense in $\Omega(\phi_g^t)$. The aim of the present paper is to prove the celebrated Pugh's general density theorem [18] for continuous geodesic flows, i.e. to show that:

Theorem 1. *There exists a τ-residual subset $\mathcal{G}$ of $\mathcal{G}^0(UT(M))$ such that $UT(M) = \overline{Per(\phi^t)}$, for any $\phi^t \in \mathcal{G}$.*

We recall that Klingenberg and Takens theorem (see [11]) assure that, for a dense subset of metrics on a compact manifold M, we have infinitely many closed geodesics (see also [20] on more general generic assumptions). It is worth noting that on manifolds with negative curvature, the geodesic flow is Anosov and so, by Anosov closing lemma (see, e.g., [10]), the closed geodesics are dense in the manifold without any need of generic considerations.

The first attempt to obtain the dissipative version of Theorem 1 for homeomorphisms was in [17] by Palis, Pugh, Shub, and Sullivan. However, the proof in [17] was not complete as it was observed in [7]. The complete proof is due to Hurley [9] using dissipative arguments. Hurley's proof uses the fact that we can create a C^0-stable periodic *sink* by a small C^0-perturbation and Brouwer's fixed point theorem guarantees a fixed point for every C^0-close homeomorphism. Clearly, this strategy is meaningless for the volume-preserving setting because sinks simply do not exist. With respect to the volume-preserving context, Daalderop and Fokkink proved (see [5, Proposition 4]) that the general density theorem holds. In the

present paper, we mainly follow the arguments in [7], adapting them to the geodesic flow framework and combine it with the recently proved closing lemma [21]. It is interesting to remark that the general density theorem was first established in the C^1-class by Pugh [18] in the late 1960s and much later by Pugh and Robinson for volume-preserving and symplectic diffeomorphisms as also for Hamiltonians (see [19]). Actually, the general density theorem is a direct consequence of the combination of the closing lemma and the stability and persistence of nondegenerated closed orbits given, for example, from the hyperbolic (and also elliptic in the conservative case) structure. This stability and persistence holds in the smooth case, but sadly the notion of hyperbolicity (and ellipticity) is no longer valid in the topological context, as it is perceptively observed by Rifford in [21]: *"The Pugh C^1-Closing Lemma has strong consequences on the structure of the flow of generic vector fields... It is worth noticing that our result is not striking enough to infer relevant properties for generic geodesic flows (for instance, the existence of a hyperbolic periodic orbit is not stable under C^0 perturbations on the dynamics)."* Fortunately, we have at hand a range of topological techniques, which will allow us to reach a generic result of undeniable interest.

2. Perturbation results

Let ϕ_g^t be the geodesic flow of a Riemannian metric $g \in \mathcal{R}^2(M)$ acting on $UT(M)$, the unit tangent bundle of M. Let $\pi : UT(M) \to M$ be the canonical projection. Non-trivial closed geodesics on M are in one-to-one correspondence to the periodic orbits of ϕ_g^t. From now on we denote by θ_γ the representative of the closed geodesic γ by choosing a single point in the orbit, say $\theta_\gamma = (x, v)$. Given a closed orbit $\gamma = \{\phi_g^t(\theta) : t \in [0, a]\}$ of period a, we can define the *Poincaré map* $\mathcal{P}_g(\Sigma, \gamma)$ as follows: one can choose a local $(2 \dim(M) - 2)$-hypersurface Σ in $UT(M)$ containing θ and transversal to γ such that there are open neighborhoods Σ_0 and Σ_a of θ in Σ and a differentiable arrival function $\delta : \Sigma_0 \to \mathbb{R}$ with $\delta(\theta) = a$ such that the map $\mathcal{P}_g(\Sigma, \gamma) : \Sigma_0 \to \Sigma_a$ given by $v \mapsto \phi_g^{\delta(v)}(v)$ is a diffeomorphism.

Given a closed geodesic $c : \mathbb{R}/\mathbb{Z} \to M$, all iterates $c^m : \mathbb{R}/\mathbb{Z} \to M$; $c^m(t) = c(mt)$ for a positive integer m are closed geodesics too.

A closed orbit γ (or the corresponding closed geodesic c) is called *nondegenerate* (cf. [4]) if 1 is not an eigenvalue of the linearized Poincaré map $P_c := D_{\gamma(0)} \mathcal{P}_g(\Sigma, \gamma)$. In this case, γ is an isolated closed orbit and $\pi \circ \gamma$ is an isolated closed geodesic.

A Riemannian metric g is called *bumpy* if all the closed orbits of the geodesic flow are nondegenerate. Since $P_{c^m} = P_c^m$, this is equivalent to saying that if $\exp(2\pi i\lambda)$ is an eigenvalue of P_c, then λ is irrational. We state the bumpy metric theorem [1,2]:

Theorem 2.1 (Bumpy metric theorem). *For $2 \leq k \leq \infty$, the set of bumpy metrics of class C^k is a residual subset of $\mathcal{R}^k(M)$.*

Recently, in [21], Rifford was able to overcome a problem that was open for a long time and showed how to close an orbit of the geodesic flow by a small perturbation of the metric in the C^1 topology:

Theorem 2.2 (C^1-closing lemma). *Let g be a Riemannian metric on M of class C^k with $k \geq 3$ (resp. $k = \infty$), $(x, v) \in UT(M)$ and $\epsilon > 0$ be fixed. Then there exists a metric $\tilde{g}$ of class C^{k-1} (resp. C^∞) with $\|\tilde{g} - g\|_{C^1} < \epsilon$ such that the geodesic $\gamma_{(x,v)}^{\tilde{g}}$ is periodic.*

3. Proof of Theorem 1

Let $\phi^t \in \mathcal{G}^0(UT(M))$. We define the set of *weak-periodic points of ϕ^t* by

$$W\,Per(\phi^t) := \{(x, v) = \lim_{n \to \infty}(x_n, v_n) : (x_n, v_n) \in Per(\phi_n^t),\ \phi_n^t \in \mathcal{G}^0(UT(M)) \text{ and } \phi_n^t \xrightarrow{\tau} \phi^t\}$$

We observe that, by the Poincaré recurrence theorem, the geodesic flow ϕ^t is nonwandering on $UT(M)$. Let $\mathcal{G}^2(UT(M))$ be the set of geodesic flows associated with metrics in $\mathcal{R}^3(M)$. Clearly, $\mathcal{G}^2(UT(M))$ is τ-dense in $\mathcal{G}^0(UT(M))$. Therefore, it follows from the C^1-closing lemma (Theorem 2.2), and using Gronwall's inequality, that:

Lemma 3.1. *If $\phi^t \in \mathcal{G}^0(UT(M))$, then $UT(M) = W\,Per(\phi^t)$.*

We say that a closed orbit γ of ϕ^t is *permanent* if any $\tilde{\phi}^t \in \mathcal{G}^0(UT(M))$ and τ-arbitrarily close to ϕ^t has a $\tilde{\phi}^t$-periodic orbit $\tilde{\gamma}$ near γ. Let $\mathcal{P}(\phi^t)$ denote the set of all permanent closed orbits of ϕ^t.

Lemma 3.2. *There exists a τ-residual subset $\mathcal{G}$ of $\mathcal{G}^0(UT(M))$ such that $Per(\phi^t) = \mathcal{P}(\phi^t)$, for any $\phi^t \in \mathcal{G}$.*

Proof. Along this proof, we borrow some arguments developed in [7] together with some elementary fixed point index theory. By Theorem 2.1, there exists a C^2-residual subset $\mathcal{R}_0$ of $\mathcal{R}^2(M)$, hence $\mathcal{R}_0$ is C^1-dense in $\mathcal{R}^2(M)$, such that every metric g in $\mathcal{R}_0$ has all the closed orbits of the geodesic flow nondegenerate. As a consequence of Gronwall's inequality, there exists a τ-dense subset $\mathcal{G}_0$ of $\mathcal{G}^1(UT(M))$ such that every ϕ_g^t in $\mathcal{G}_0$ has all the closed orbits nondegenerate. Since $\mathcal{G}^1(UT(M))$ is τ-dense in $\mathcal{G}^0(UT(M))$, we get that $\mathcal{G}_0$ is τ-dense is $\mathcal{G}^0(UT(M))$.

We claim that there exists a τ-residual $\mathscr{G}$ such that any closed orbit of a flow in $\mathscr{G}$ is permanent. We begin by taking a countable base for the topology $\{\mathcal{B}_i\}_{i \in \mathbb{N}}$ of the unit tangent bundle $UT(M)$ consisting of open balls whose boundaries are embedded spheres. The fixed point index will play a crucial role along the proof since, in rough terms, the existence of non-zero index on a set assures a fixed point in that set and, moreover, displaying non-zero index persists under τ-perturbations. Now we define, for every $i \in \mathbb{N}$, the following τ-open subsets of $\mathscr{G}^0(UT(M))$, which a priori do not cover the whole set $\mathscr{G}^0(UT(M))$, in the following way:

(1) $\phi^t \in \mathscr{F}_i$ if $\phi^t(x, v) \neq (x, v)$ for all $(x, v) \in \overline{\mathcal{B}_i}$ and all $t \in]0, 1]$;
(2) $\phi^t \in \mathscr{I}_i$ if there exists $\mathcal{B}_j$ with $\overline{\mathcal{B}_j} \subset \mathcal{B}_i$ such that $\phi^t(x, v) \neq (x, v)$ for all (x, v) in the boundary of $\mathcal{B}_j$ and all $t \in]0, 1]$, and the index of the time-t map ϕ^t (for some $t \in]0, 1]$) in $\mathcal{B}_j$ is non-zero.

Note that every geodesic flow with nondegenerate closed orbits with period $a \in]0, 1]$ belongs to every $\mathscr{F}_i \cup \mathscr{I}_i$. Therefore, $\mathscr{G}_0$ is contained in every

$$\mathscr{D}_{n,i} = \{\phi^t \in \mathscr{G}^0(UT(M)): \phi^{nt} \in \mathscr{F}_i \cup \mathscr{I}_i\},$$

where ϕ^{nt} stands for the flow ϕ^t with speed reparameterization defined by $\phi^{nt}(x, v) = \overbrace{(\phi^t)(\phi^t)...(\phi^t)}^{n-\text{times}}(x, v)$. Since $\mathscr{G}_0$ is τ-dense is $\mathscr{G}^0(UT(M))$, we obtain that every $\mathscr{D}_{n,i}$ is τ-dense in $\mathscr{G}^0(UT(M))$. As $\mathscr{F}_i$ and $\mathscr{I}_i$ are τ-open and $\phi^t \mapsto \phi^{nt}$ is continuous we obtain that every $\mathscr{D}_{n,i}$ is τ-open and dense in $\mathscr{G}^0(UT(M))$. In conclusion, we define the residual subset in the lemma by $\mathscr{G} := \cap \mathscr{D}_{n,i}$.

Let us show that $Per(\phi^t) = \mathscr{P}(\phi^t)$; take $\phi^t \in \mathscr{G}$ and $\gamma \in Per(\phi^t)$ of period a, then for any $\mathcal{B}_i$ which intersects γ, there exist $n \in \mathbb{N}$ and $\mathcal{B}_j$ with $\overline{\mathcal{B}_j} \subset \mathcal{B}_i$ such that $\phi^{nt}(x, v) \neq (x, v)$ for all (x, v) in the boundary of $\mathcal{B}_j$ and all $t \in]0, 1]$, and the index of the time-nt map ϕ^{nt} (for some $t \in]0, 1]$) in $\mathcal{B}_j$ is non-zero. Since this property is persistent for small τ-perturbations of the original flow, we get that γ is permanent. $\square$

Let $UT(M)^\star$ be the set of compact subsets of $UT(M)$ endowed with the Hausdorff topology.

Lemma 3.3. *The map $\mathfrak{P}: \mathscr{G}^0(UT(M)) \to UT(M)^\star$, where $\mathscr{G}^0(UT(M))$ is endowed with the τ-topology and $UT(M)^\star$ is endowed with the Hausdorff topology, and defined by $\mathfrak{P}(\phi^t) = \overline{Per(\phi^t)}$ is lower semicontinuous on the residual $\mathscr{G}$ given by Lemma 3.2.*

Proof. We must prove that for any $\tilde{\phi}^t \in \mathscr{G}$, and any $\epsilon > 0$, there exists a neighborhood V of $\tilde{\phi}^t$ such that $\mathfrak{P}(\tilde{\phi}^t) \subseteq \mathcal{B}_\epsilon(\mathfrak{P}(\phi^t))$ for all $\phi^t \in V$, or, in other words, there are no implosions of the number of closed orbits when we perturb $\tilde{\phi}^t$. But Lemma 3.2 says that $Per(\tilde{\phi}^t) = \mathscr{P}(\tilde{\phi}^t)$ and the proof is completed by recalling the definition of permanent closed orbit. $\square$

Proof of Theorem 1. Noting that $UT(M) = \Omega(\phi^t)$, we will prove that $\Omega(\phi^t) = \overline{Per(\phi^t)}$ for some τ-generic flow in $\mathscr{G}^0(UT(M))$. From Lemma 3.3 the map $\mathfrak{P}: \mathscr{G}^0(UT(M)) \to UT(M)^\star$ defined by $\mathfrak{P}(\phi^t) = \overline{Per(\phi^t)}$ is lower semicontinuous on $\mathscr{G}$. It is well-known (see [6]) that the continuity points of $\mathfrak{P}|_\mathscr{G}$ are a residual subset $\mathscr{G}_1 \subset \mathscr{G}$, hence a residual subset of $\mathscr{G}^0(UT(M))$. Let us see that if $\phi^t \in \mathscr{G}_1$ (i.e. ϕ^t is a continuity point of $\mathfrak{P}|_\mathscr{G}$), then $UT(M) = \overline{Per(\phi^t)}$. The non-obvious inclusion is $UT(M) \subset \overline{Per(\phi^t)}$. Assume, by contradiction, that there exists $(x, v) \in UT(M) \setminus \overline{Per(\phi^t)}$. By Lemma 3.1, let $\{\phi_n^t\}_{n \in \mathbb{N}} \subset \mathscr{G}^0(UT(M))$ and $\{(x_n, v_n)\}_{n \in \mathbb{N}} \subset Per(\phi_n^t)$ be such that $\phi_n^t \to_\tau \phi^t$ and $\lim_{n \to \infty}(x_n, v_n) = (x, v)$. By Lemma 3.2 there exists $\{\tilde{\phi}_n^t\}_{n \in \mathbb{N}}$ such that $(\tilde{x}_n, \tilde{v}_n) \in \mathscr{P}(\tilde{\phi}_n^t)$ and each $\tilde{\phi}_n^t$ becomes τ-arbitrarily close to ϕ_n^t.

Since $\mathscr{G}^0(UT(M))$ endowed with the τ-topology is a Baire space, we get that $\mathscr{G}$ is dense in $\mathscr{G}^0(UT(M))$. Therefore, there exist $\{\overline{\phi}_n^t\}_{n \in \mathbb{N}} \subset \mathscr{G}$ and $\{(\overline{x}_n, \overline{v}_n)\}_{n \in \mathbb{N}} \subset Per(\overline{\phi}_n^t)$ such that

$$d_{TM}((\tilde{x}_n, \tilde{v}_n), (\overline{x}_n, \overline{v}_n)) < \frac{1}{n}$$

and $\overline{\phi}_n^t$ becomes τ-arbitrarily close to $\tilde{\phi}_n^t$.

We conclude that $\overline{\phi}_n^t \to_\tau \phi^t$ and $\lim_{n \to \infty}(\overline{x}_n, \overline{v}_n) = (x, v)$. Then, since ϕ^t is a continuity point of $\mathfrak{P}|_\mathscr{G}$ we have that $\lim_{n \to \infty}\mathfrak{P}(\overline{\phi}_n^t) = \mathfrak{P}(\phi^t)$, i.e., $\lim_{n \to \infty}\overline{Per(\overline{\phi}_n^t)} = \overline{Per(\phi^t)}$. Finally, we observe that $(x, v) \in \overline{Per(\overline{\phi}_n^t)}$, or equivalently, $(x, v) \in \overline{Per(\phi^t)}$ which is a contradiction with the assumption that $(x, v) \in UT(M) \setminus \overline{Per(\phi^t)}$. $\square$

Acknowledgements

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