C. R. Acad. Sci. Paris, Ser. I 353 (2015) 505–510

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex analysis

Bound for the fifth coefficient of certain starlike functions

Borne pour le cinquième coefficient des fonctions étoilées

V. Ravichandran, Shelly Verma

Department of Mathematics, University of Delhi, Delhi 110 007, India

ARTICLE INFO

Article history:
Received 12 January 2015
Accepted 4 March 2015
Available online 1 April 2015

Presented by the Editorial Board

ABSTRACT

For two different choices of φ, the sharp bound for the fifth coefficient of a normalized analytic function f satisfying $zf'(z)/f(z) \prec \varphi(z)$ is obtained by using a bound for a polynomial in the coefficients of functions with positive real part. Our proof uses a characterization of functions with positive real part in terms of certain positive semi-definite Hermitian form and certain well-known coefficient inequalities for functions with positive real part are shown to follow easily from this characterization.

RÉSUMÉ

Nous appuyant sur une majoration de la valeur absolue d'un polynôme en les coefficients de fonctions de partie réelle positive, nous obtenons une majoration précise de la valeur absolue du cinquième coefficient d'une fonction analytique f normalisée, satisfaisant $zf'(z)/f(z) \prec \varphi(z)$, pour deux choix différents de φ. Notre preuve utilise une caractérisation des fonctions de partie réelle positive en termes de certaines formes hermitiennes semi-définies positives. Des inégalités bien connues pour ces fonctions de partie réelle positive résultent aussi sans difficulté de cette caractérisation.

1. Introduction and preliminaries

Let $\mathcal{S}$ be the class of all normalized univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let φ be an analytic univalent function with positive real part mapping $\mathbb{D}$ onto domains symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. For such φ, Ma and Minda [6] introduced the classes:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \quad \text{and} \quad \mathcal{K}(\varphi) = \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

E-mail addresses: vravi68@gmail.com (V. Ravichandran), jmdsv.maths@gmail.com (S. Verma).

http://dx.doi.org/10.1016/j.crma.2015.03.003

The sharp bounds for the first two coefficients of functions in these subclasses have been determined by using the Feketo–Szegö coefficient functional $|a_3 - \mu a_2^2|$ for $\mathcal{K}(\varphi)$ in [6] and the bound for a_4 in [2, Theorem 1, p. 38]. In this paper, our aim is to determine the sharp bound for the fifth coefficient a_5 of functions belonging to the subclasses

$$\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z}) \quad \text{and} \quad \mathcal{S}_{RL}^* := \mathcal{S}^*\left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right).$$

Sokół [9] conjectured that $|a_n| \le 1/(2(n-1))$ for $f \in \mathcal{S}_L^*$. Mendiratta et al. [8] conjectured that $|a_n| \le (5-3\sqrt{2})/(2(n-1))$ for $f \in \mathcal{S}_{RL}^*$. They have respectively verified them for $n = 2, 3, 4$. In this paper, we verify these two conjectures for $n = 5$. The proof uses an estimate for a nonlinear coefficient functional for functions with positive real part, which we prove by making use of a characterization of functions with positive real part in terms of certain positive semi-definite Hermitian form. As a consequence of this characterization, certain well-known coefficient inequalities for functions with positive real part are shown to follow easily.

The class $\mathcal{P}$ of functions with positive real part consists of all analytic functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$. For this class, the following lemma related to the Hermitian form is well known (see Ali [1]).

Lemma 1.1. *(See [3].) A function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$ if and only if*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j + \sum_{k=1}^{\infty} c_k z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty} c_{k+1} z_{k+j}\right|^2\right\} \ge 0$$

for every sequence $\{z_k\}$ of complex numbers that satisfy $\limsup_{k\to\infty} |z_k|^{1/k} < 1$.

2. Coefficient inequalities for functions with positive real part

By using Lemma 1.1, we first prove the following result.

Lemma 2.1. *Let α, β, γ and a satisfy the inequalities $0 < \alpha < 1$, $0 < a < 1$ and*

$$8a(1-a)((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2) + \alpha(1-\alpha)(\beta - 2a\alpha)^2 \le 4\alpha^2(1-\alpha)^2 a(1-a). \tag{2.1}$$

If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then

$$|\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - (3/2)\beta c_1^2 c_2 - c_4| \le 2.$$

Proof. Let the sequence $\{z_k\}$ of complex numbers be given by $z_0 = \alpha c_3 - \beta c_1 c_2 + \gamma c_1^3$, $z_1 = a c_2 - \delta c_1^2$, $z_2 = b c_1$, $z_3 = -1$, $z_k = 0$ for all $k \ge 4$. For this choice of $\{z_k\}$, Lemma 1.1 gives

$$\begin{aligned}
&|\gamma c_1^4 + a c_2^2 + (\alpha + b) c_1 c_3 - (\beta + \delta) c_1^2 c_2 - c_4|^2 \\
&\quad \le |(2\alpha - 1)c_3 + (a + b - 2\beta)c_1 c_2 + (2\gamma - \delta)c_1^3|^2 + |(2a-1)c_2 + (b - 2\delta)c_1^2|^2 \\
&\qquad - |(a+b)c_1 c_2 - \delta c_1^3 - c_3|^2 + (2b-1)^2|c_1|^2 - |bc_1^2 - c_2|^2 + 4 - |c_1|^2 \\
&\quad = 4\alpha(\alpha-1)\left|c_3 + \left(\frac{\alpha(a+b) + \beta(1-2\alpha)}{2\alpha(\alpha-1)}\right)c_1 c_2 + \left(\frac{\alpha(\gamma-\delta) + \gamma(\alpha-1)}{2\alpha(\alpha-1)}\right)c_1^3\right|^2 \\
&\qquad + \frac{1}{\alpha(\alpha-1)}\left|(\alpha\delta - \gamma)c_1^3 + (\alpha(a+b) - \beta)c_1 c_2\right|^2 + \frac{2}{\alpha(1-\alpha)}(\alpha\delta - \gamma)^2|c_1|^6 \\
&\qquad + \frac{2}{\alpha(1-\alpha)}(\alpha(a+b) - \beta)^2|c_1|^2|c_2|^2 + 4a(a-1)\left|c_2 - \frac{v}{2}c_1^2\right|^2 + \frac{(\delta - ab)^2}{a(1-a)}|c_1|^4 + 4b(b-1)|c_1|^2 + 4 \\
&\quad \le \frac{2}{\alpha(1-\alpha)}(\alpha\delta - \gamma)^2|c_1|^6 + \frac{8}{\alpha(1-\alpha)}(\alpha(a+b) - \beta)^2|c_1|^2 + \frac{(\delta - ab)^2}{a(1-a)}|c_1|^4 + 4b(b-1)|c_1|^2 + 4,
\end{aligned}$$

where

$$v = \frac{\delta(1-a) + a(b-\delta)}{a(1-a)}.$$

For $b = \alpha$, $\delta = \beta/2$, the above inequality becomes

$$\begin{aligned}
&|\gamma c_1^4 + ac_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4|^2 \\
&\le \frac{2}{\alpha(1-\alpha)}\left(\frac{\alpha\beta}{2} - \gamma\right)^2 |c_1|^6 + \frac{1}{a(1-a)}\left(\frac{\beta}{2} - a\alpha\right)^2 |c_1|^4 \\
&\quad + \left(\frac{8}{\alpha(1-\alpha)}(\alpha(a+\alpha) - \beta)^2 + 4\alpha(\alpha-1)\right)|c_1|^2 + 4 \\
&= px^3 + qx^2 + rx + 4,
\end{aligned}$$

where $x = |c_1|^2 \in [0, 4]$, and the numbers p, q, r are given by

$$\begin{aligned}
&p = 2(\alpha\beta/2 - \gamma)^2/(\alpha(1-\alpha)), \\
&q = (\beta/2 - a\alpha)^2/(a(1-a)), \\
&r = 8(\alpha(a+\alpha) - \beta)^2/(\alpha(1-\alpha)) + 4\alpha(\alpha-1).
\end{aligned}$$

Since $p \ge 0$, $q \ge 0$, we have, by using (2.1),

$$\begin{aligned}
px^2 + qx + r &\le 16p + 4q + r \\
&= \frac{1}{\alpha(1-\alpha)a(1-a)}(8a(1-a)((\alpha\beta - 2\gamma)^2 + (\alpha(a+\alpha) - \beta)^2) \\
&\quad + \alpha(1-\alpha)(\beta - 2a\alpha)^2 - 4\alpha^2(1-\alpha)^2 a(1-a)) \le 0
\end{aligned}$$

and hence $px^3 + qx^2 + rx + 4 \le 4$. □

The inequalities in (a), (b), (c) in the following lemma were proved in [4] and the inequality (d) is well known, but here we give a different proof of these inequalities by using Lemma 1.1.

Lemma 2.2. *(See [4].) If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then*

(a) $|c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4| \le 2$;
(b) $|c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5| \le 2$;
(c) $|c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 - c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 6c_1c_2c_3 - c_6| \le 2$;
(d) $|c_n| \le 2$ $(n \ge 1)$.

Proof. All these inequalities follow from Lemma 1.1 for different choices of $\{z_k\}$. The inequality in (a) follows if $\{z_k\}$ is chosen by $z_0 = c_3 - 2c_1c_2 + c_1^3$, $z_1 = -c_1^2 + c_2$, $z_2 = c_1$, $z_3 = -1$, $z_k = 0$ for all $k \ge 4$. The inequality in (b) follows if $z_0 = c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4$, $z_1 = -c_3 + 2c_1c_2 - c_1^3$, $z_2 = -c_2 + c_1^2$, $z_3 = -c_1$, $z_4 = 1$, $z_k = 0$ for all $k \ge 5$. The inequality in (c) follows if $z_0 = c_5 + c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3$, $z_1 = -c_1^4 - c_2^2 - 2c_1c_3 + 3c_1^2c_2 + c_4$, $z_2 = c_3 - 2c_1c_2 + c_1^3$, $z_3 = -c_1^2 + c_2$, $z_4 = c_1$, $z_5 = -1$, $z_k = 0$ for all $k \ge 6$. The inequality in (d) follows if $z_{n-1} = 1$, $z_k = 0$ for all $k \ne n-1$. □

Livingston [5] had proved the following result for the case $\mu = 1$.

Lemma 2.3. *If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then for all $n, m \in \mathbb{N}$,*

$$|\mu c_n c_m - c_{n+m}| \le \begin{cases} 2, & 0 \le \mu \le 1; \\ 2|2\mu - 1|, & \text{elsewhere}. \end{cases}$$

If $0 < \mu < 1$, the inequality is sharp for the function $p(z) = (1 + z^{n+m})/(1 - z^{n+m})$. In the other cases, the inequality is sharp for the function $p(z) = (1+z)/(1-z)$.

Proof. It is enough to prove for the case $n \le m$. For fixed $n, m \in \mathbb{N}$, choose the sequence $\{z_k\}$ of complex numbers by $z_{n-1} = \mu c_m$, $z_{n+m-1} = -1$, $z_k = 0$ for all $k \ne n-1, n+m-1$. Lemma 1.1 gives

$$\begin{aligned}
|\mu c_n c_m - c_{n+m}|^2 &\le |(2\mu - 1)c_m|^2 - |c_m|^2 + 4 = 4\mu(\mu - 1)|c_m|^2 + 4 \\
&\le \begin{cases} 4, & 0 \le \mu \le 1, \\ 4(2\mu - 1)^2, & \text{elsewhere}. \end{cases}
\end{aligned}$$

For $0 < \mu < 1$, the inequality is sharp for the function $p(z) = (1 + z^{n+m})/(1 - z^{n+m})$.

For $\mu \le 0$ or $\mu \ge 1$, the inequality is sharp for the function $p(z) = (1+z)/(1-z)$. □

Corollary 2.4. *If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then, for $\mu \le 1$,*

$$|\mu c_n c_{2n} - c_n^3| \le 4(2-\mu), \quad \text{and} \quad |\mu c_n^2 c_{2n} - c_n^4| \le 8(2-\mu).$$

The inequalities are sharp for the function $p(z) = (1+z)/(1-z)$.

Proof. The case $\mu = 0$ is obvious and so assume that $\mu \neq 0$. An application of Lemma 2.3 shows that

$$|\mu c_n c_{2n} - c_n^3| = |\mu||c_n| \left| c_{2n} - \frac{1}{\mu} c_n^2 \right| \le 4(2-\mu),$$

and

$$|\mu c_n^2 c_{2n} - c_n^4| = |\mu||c_n|^2 \left| c_{2n} - \frac{1}{\mu} c_n^2 \right| \le 8(2-\mu).$$

Both inequalities are sharp for the function $p(z) = (1+z)/(1-z)$. □

Remark 2.5. The first inequality for $\mu \ge 6$ and the second inequality for $\mu \ge 4$ in Corollary 2.4 have been proved by Ma and Minda [7].

3. Bound for fifth coefficient

We now determine the bound for the fifth coefficient of functions in the subclass $\mathcal{S}^*(\varphi)$ for two different choices of φ.

Theorem 3.1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.*

(a) *If $f \in S_L^*$, then $|a_5| \le \frac{1}{8}$.*

(b) *If $f \in S_{RL}^*$, $|a_5| \le \frac{5 - 3\sqrt{2}}{8} \approx 0.0946699$.*

The estimates are sharp.

Proof. First, we express the coefficient a_n $(n = 2, 3, 4, 5)$ of the function $f \in \mathcal{S}^*(\varphi)$ in terms of the coefficient of the function $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ and of a function with the positive real part. Let $p(z) := zf'(z)/f(z) = 1 + b_1 z + b_2 z^2 + \cdots$. This equation readily shows that

$$(n-1)a_n = \sum_{k=1}^{n-1} b_k a_{n-k} \quad \text{for } n > 1. \tag{3.1}$$

Since φ is univalent and $p \prec \varphi$, the function

$$p_1(z) = \frac{1 + \varphi^{-1}(p(z))}{1 - \varphi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$$

belongs to $\mathcal{P}$. Equivalently,

$$p(z) = \varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right).$$

Using the last equation, the coefficients b_i can be expressed in terms of c_i and B_i. Precisely, we have

$$b_1 = \frac{1}{2} B_1 c_1,$$

$$b_2 = \frac{1}{4}\left((B_2 - B_1)c_1^2 + 2B_1 c_2\right),$$

$$b_3 = \frac{1}{8}((B_1 - 2B_2 + B_3)c_1^3 + 4(B_2 - B_1)c_1 c_2 + 4B_1 c_3),$$

and

$$b_4 = \frac{1}{16}((-B_1 + 3B_2 - 3B_3 + B_4)c_1^4 + 6(B_3 - 2B_2 + B_1)c_1^2 c_2 + 4(B_2 - B_1)c_2^2 + 8(B_2 - B_1)c_1 c_3 + 8B_1 c_4).$$

Therefore, by using the expressions for b_k in (3.1), we get

$$a_2 = b_1 = \frac{1}{2}B_1 c_1,$$

$$a_3 = \frac{1}{8}((B_1^2 - B_1 + B_2)c_1^2 + 2B_1 c_2),$$

$$a_4 = \frac{1}{48}((B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_1 - 4B_2 + 2B_3)c_1^3 + 2(3B_1^2 - 4B_1 + 4B_2)c_1c_2 + 8B_1c_3),$$

and

$$a_5 = \frac{1}{384}((B_1^4 - 6B_1^3 + 6B_1^2B_2 + 11B_1^2 - 22B_1B_2 + 3B_2^2 + 8B_1B_3 - 6B_1 + 18B_2 - 18B_3 + 6B_4)c_1^4 + 4(3B_1^3 - 11B_1^2 + 11B_1B_2 + 9B_1 - 18B_2 + 9B_3)c_1^2c_2 + 12(B_1^2 - 2B_1 + 2B_2)c_2^2 + 16(2B_1^2 - 3B_1 + 3B_2)c_1c_3 + 48B_1c_4). \tag{3.2}$$

(a) Let $f \in S_L^*$. Then $zf'(z)/f(z) \prec \varphi_L(z)$ where

$$\varphi_L(z) = \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \frac{5}{128}z^4 + \cdots.$$

Use of Eq. (3.2) shows that the coefficient a_5 is given by

$$a_5 = -\frac{1}{16}\left(\frac{49}{384}c_1^4 - \frac{17}{24}c_1^2c_2 + \frac{1}{2}c_2^2 + \frac{11}{12}c_1c_3 - c_4\right).$$

Applying Lemma 2.1, we get $|a_5| \le 1/8$. Let the function $f_0 : \mathbb{D} \to \mathbb{C}$ be given by

$$f_0(z) = z\exp\left(\int_0^z \frac{\sqrt{1+t^4} - 1}{t}\mathrm{d}t\right) = z + \frac{1}{8}z^5 - \frac{1}{128}z^9 + \cdots.$$

Then $f_0(0) = f_0'(0) - 1 = 0$, $zf_0'(z)/f_0(z) = \varphi_L(z^4)$ and so the function $f_0 \in \mathcal{S}_L^*$. The result is sharp for this function f_0.

(b) Let $f \in S_{RL}^*$. Then $zf'(z)/f(z) \prec \varphi_R(z)$ where

$$\varphi_R(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1 + 2(\sqrt{2} - 1)z}} = 1 + \frac{5 - 3\sqrt{2}}{2}z + \frac{71 - 51\sqrt{2}}{8}z^2 + \frac{589 - 415\sqrt{2}}{16}z^3 + \frac{20043 - 14179\sqrt{2}}{128}z^4 + \cdots.$$

By using Eq. (3.2), we see that a_5 is given by

$$a_5 = -\frac{5 - 3\sqrt{2}}{16}(\gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - (3/2)\beta c_1^2 c_2 - c_4)$$

where $\gamma = (-17456 + 12631\sqrt{2})/1536$, $a = (9\sqrt{2} - 8)/8$, $\alpha = (-29 + 30\sqrt{2})/24$, $\beta = -(263\sqrt{2} - 408)/48$. These numbers α, β, γ and a satisfy the conditions of Lemma 2.1 and hence $|a_5| \le (5 - 3\sqrt{2})/8$. Let the function $f_1 : \mathbb{D} \to \mathbb{C}$ be given by

$$f_1(z) = z\exp\left((\sqrt{2} - 1)\int_0^z \frac{1 - \sqrt{(1 - t^4)/(1 + 2(\sqrt{2} - 1)t^4)}}{t}\mathrm{d}t\right) = z + \frac{5 - 3\sqrt{2}}{8}z^5 + \frac{185 - 132\sqrt{2}}{128}z^9 + \cdots.$$

Then $f_1(0) = f_1'(0) - 1 = 0$, $zf_1'(z)/f_1(z) = \varphi_R(z^4)$ and so the function $f_1 \in \mathcal{S}_{RL}^*$. The result is sharp for this function f_1. □

Acknowledgements

The authors are supported respectively by a grant from University of Delhi and NBHM, Mumbai.

References

[1] R.M. Ali, Coefficients of the inverse of strongly starlike functions, Bull. Malays. Math. Sci. Soc. (2) 26 (1) (2003) 63–71.

[2] R.M. Ali, V. Ravichandran, N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput. 187 (1) (2007) 35–46.

[3] C.R. Leverenz, Hermitian forms in function theory, Trans. Amer. Math. Soc. 286 (2) (1984) 675–688.

[4] R.J. Libera, E.J. Złotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85 (2) (1982) 225–230.

[5] A.E. Livingston, The coefficients of multivalent close-to-convex functions, Proc. Amer. Math. Soc. 21 (1969) 545–552.

[6] W.C. Ma, D. Minda, A unified treatment of some special classes of univalent functions, in: Proceedings of the Conference on Complex Analysis, Tianjin, 1992, in: Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, USA, 1992, pp. 157–169.

[7] W.C. Ma, D. Minda, Uniformly convex functions. II, Ann. Pol. Math. 58 (3) (1993) 275–285.

[8] R. Mendiratta, S. Nagpal, V. Ravichandran, A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli, Int. J. Math. 25 (9) (2014), 17 pp.

[9] J. Sokół, Coefficient estimates in a class of strongly starlike functions, Kyungpook Math. J. 49 (2) (2009) 349–353.