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Functional analysis

KK-theory for some graph C^*-algebras

KK-théorie de certaines C^*-algèbres de graphes

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A R T I C L E I N F O

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A B S T R A C T

In [2], the first author had presented a method to obtain a K-equivalence between full and reduced free products of nuclear unital C^*-algebras. The object of this note is to show how it can be extended to free products with amalgamation over a finite dimensional algebra. As a consequence the K-equivalence holds for graph-C^*-algebras whose edge stabilizers are all finite dimensional.

R É S U M É

Dans [2], le premier auteur avait présenté une méthode pour obtenir l'équivalence en K-théorie entre les produits libres pleins ou réduits de C^*-algèbres nucléaires unifères. Nous montrons ici comment étendre ce résultat aux produits libres amalgamés au-dessus d'une algèbre de dimension finie. Ceci permet alors de démontrer le même type de résultat pour les graphes d'algèbres quand les stabilisateurs des arêtes sont de dimension finie.

1. Introduction

Given two unital C^*-algebras A_1 and A_2 with a common unital sub-algebra B and conditional expectation onto it, we can construct two types of amalgamated free products over B: the full one, latter denoted as A_m, which satisfies the universal property of amalgamated free products in the category of unital C^*-algebras, and the reduced one, defined by D. Voiculescu in [6], later denoted as A_r. It is already known that in the case of C*-algebras of K-amenable groups, the canonical morphism from A_m to A_r has an inverse in $KK(A_r, A_m)$, which implies, in particular, that they have the same K-groups. It has been generalized by Fima and Freslon to amenable quantum groups in [1]. In 1994, the first author had also proved a result for any nuclear C^*-algebras and amalgamation over $\mathbb{C}$. This is this latter result that we expand here to amalgamation over a finite-dimensional algebra. The proof follows the line of [2]. First we must state a precise variation of the (relative) Kasparov–Voiculescu theorem to obtain a specific version of the K-nuclearity property of Skandalis [4] for the algebras A_i. Then we verify that the construction of the inverse given in [2] adapts easily to the new situation.

For the whole article, B denotes a finite dimensional C^*-algebra.

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2. Relative K-nuclearity property

Definition 2.1. Let A be a C^*-algebra. An injective representation π of A into a B-Hilbert module E_B is strongly injective if for any minimal central projection p_i in B, the restriction of the representation π to $E_B p_i$ is again injective.

Theorem 2.1. *Let A be a nuclear C^*-algebra with a unital copy of B. Suppose we have a strongly injective representation π of A in some B-Hilbert module E_B. Then for any completely positive map φ from A to $L(H_D)$ where H_D is a Hilbert module over a unital C^*-algebra D that contains also a unital copy of B, there exists a sequence of $V_i \in L(H_D, \ell^2(\mathbb{N}) \otimes E_B \otimes_B D)$ such that $\lim_i \|\varphi(a) - V_i^* 1 \otimes \pi(a) \otimes_B 1_D V_i\| = 0$ and $\varphi(a) - V_i^* 1 \otimes \pi(a) \otimes_B 1_D V_i \in K(H_D), \forall a \in A$.*

It is proved by an easy modification in the proof of Theorem 4 in [3] of the formula for the vector ξ_i (p. 146).

In particular, this holds for $D = A$ and the identity map; therefore, we have an analogue of the K-nuclearity property. Precisely, the following theorem holds.

Theorem 2.2. *With the above hypothesis, there exists a unitary U of $L(\ell^2(\mathbb{N}) \otimes E_B \otimes_B A \otimes C([0,1]) \oplus A \otimes C([0,1]), \ell^2(\mathbb{N}) \otimes E_B \otimes_B A \otimes C([0,1]))$ such that the cycle with A-bimodule $\ell^2(\mathbb{N}) \otimes E_B \otimes_B A \otimes C([0,1])^2$ with left action of A as $U(1 \otimes \pi \otimes_B 1_A \otimes 1 \oplus L_A \otimes 1)U^* \oplus 1 \otimes \pi \otimes_B 1_A \otimes 1$ (where L_A is the left action of A onto itself) and the flip operator is an element of $KK(A, A)$ that is degenerated at $t = 1$.*

It is possible, furthermore, because B is finite dimensional, to obtain the following additional property.

Proposition 2.1. *In the precedent theorem, we can modify the unitary U by a compact perturbation such that for all $b \in B$, $U(1 \otimes \pi(b) \otimes_B 1_A \otimes 1 \oplus L_A(b) \otimes 1)U^* = 1 \otimes \pi(b) \otimes_B 1_A \otimes 1$.*

The proof is first reduced to the case where $B = \mathbb{C} \oplus \mathbb{C}$. It is then about two continuous paths of projections whose difference is compact and even 0 at the starting point. In that situation, there exists a continuous path of unitaries of the form $Id + compact$ that can conjugate one path to the other.

3. Free product of K-cycles and applications

Let A_i be collection of nuclear unital C^*-algebras with a common unital sub-algebra B of finite dimension. Assume furthermore that there exists a conditional expectation of A_i over B such that the GNS representation is strongly injective. Following [2], we can show:

Theorem 3.1. *The canonical map from the full amalgamated (over B) free product A_m of the A_i's onto the reduced amalgamated free product (of Voiculescu) A_r with respect to the conditional expectations has an inverse in $KK(A_r, A_m)$.*

Because of Ueda's remark [5], Theorem 3.1 has an immediate application to HNN extensions as follows.

Theorem 3.2. *Let A be a nuclear unital C^*-algebra with a unital finite dimensional sub-algebra B and θ an injective unital morphism from B to A. Suppose that there exists a conditional expectation E of A onto B such that the GNS representation of E is strongly injective, then the full and reduced HNN extensions associated with these data are equivalent in KK-theory.*

In [1], Fima introduced the notion of graph C^*-algebras. It can be reconstructed by induction via amalgamated free products and HNN extensions. Hence, we get the following theorem.

Theorem 3.3. *Suppose we have a finite graph of C^*-algebras such that the stabilizers of vertices A_v are nuclear and the stabilizers of edges B_e are finite dimensional. Assume also that the conditional expectations from A_v onto B_e if v is an edge of e all give GNS representations that are strongly injective. Then the full and reduced C^*-algebras associated with these data are equivalent in KK-theory.*

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