C. R. Acad. Sci. Paris, Ser. I 353 (2015) 415–419

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Lie algebras/Topology

Dirac families for loop groups as matrix factorizations

Familles d'opérateurs de Dirac pour les groupes de lacets et factorisations en matrices

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A R T I C L E I N F O

Article history:
Received 18 January 2015
Accepted 24 February 2015
Available online 18 March 2015

Presented by Michèle Vergne

A B S T R A C T

We identify the category of integrable lowest-weight representations of the loop group LG of a compact Lie group G with the category of twisted, conjugation-equivariant *curved Fredholm complexes* on the group G: namely, the twisted, equivariant *matrix factorizations* of a super-potential built from the loop rotation action on LG. This lifts the isomorphism of K-groups of [3–5] to an equivalence of categories. The construction uses families of Dirac operators.

R É S U M É

On identifie la catégorie des représentations intégrables de plus bas poids du groupe de lacets LG d'un groupe de Lie compact G avec la catégorie des complexes de Fredholm tordus, courbés et équivariants pour conjugaison sur le groupe G : plus précisément, les *factorisations en matrices* d'un potentiel provenant de la rotation des lacets dans LG. Cette construction relève l'isomorphisme de K-groupes de [3–5] en une équivalence de catégories. La construction fait appel aux familles d'opérateurs de Dirac.

1. Introduction and background

The group LG of smooth loops in a compact Lie group G has a remarkable class of linear representations whose structure parallels the theory for compact Lie groups [10]. The defining stipulation is the existence of a circle action on the representation, with finite-dimensional eigenspaces and spectrum bounded below, intertwining with the loop rotation action on LG. We denote the rotation circle by $\mathbb{T}_r$; its infinitesimal generator L_0 represents the *energy* in a conformal field theory.

Noteworthy is the *projective nature* of these representations, described (when G is semi-simple) by a *level* $h \in H^3_G(G;\mathbb{Z})$ in the equivariant cohomology for the adjoint action of G on itself. The representation category $\mathfrak{Rep}^h(LG)$ at a given level h is semi-simple, with finitely many simple isomorphism classes. Irreducibles are classified by their *lowest weight* (plus some supplementary data when G is not simply connected [5, Ch. IV]).

http://dx.doi.org/10.1016/j.crma.2015.02.011

In a series of papers [3–5], the authors, jointly with Michael Hopkins, construct $K^0\mathfrak{Rep}^h(LG)$ in terms of a twisted, conjugation-equivariant topological K-theory group. To wit, when G is connected, as we shall assume throughout this paper,[1] we have

$$K^0\,\mathfrak{Rep}^h(LG) \cong K_G^{\tau+\dim G}(G), \tag{1.1}$$

with a twisting $\tau \in H^3_G(G;\mathbb{Z})$ related to h, as explained below.

Remark 1.1. One loop group novelty is a *braided tensor* structure[2] on $\mathfrak{Rep}^h(LG)$. The structure arises from the *fusion product* of representations, relevant to 2-dimensional conformal field theory. The K-group in (1.1) carries a Pontryagin product, and the multiplications match in (1.1).

The map from representations to topological K-classes is implemented by the following *Dirac family*. Calling $\mathcal{A}$ the space of connections on the trivial G-bundle over S^1, the quotient stack $[G:G]$ under conjugation is equivalent to $[\mathcal{A}:LG]$ under the gauge action, via the holonomy map $\mathcal{A} \to G$. Denote by $\mathbf{S}^\pm$ the (lowest-weight) modules of spinors for the loop space $L\mathfrak{g}$ of the Lie algebra and by $\psi(A): \mathbf{S}^\pm \to \mathbf{S}^\mp$ the action of a Clifford generator A, for $d + Adt \in \mathcal{A}$. A representation $\mathbf{H}$ of LG leads to a family of Fredholm operators over $\mathcal{A}$,

$$\not{D}_A : \mathbf{H}\otimes\mathbf{S}^+ \to \mathbf{H}\otimes\mathbf{S}^-, \quad \not{D}_A := \not{D}_0 + \mathrm{i}\psi(A) \tag{1.2}$$

where $\not{D}_0$ is built from a certain Dirac operator [7] on the loop group.[3] The family is projectively LG-equivariant; dividing out by the subgroup $\Omega G \subset LG$ of based loops leads to a projective, G-equivariant Fredholm complex on G, whose K-theory class $\left[(\not{D}_\bullet, \mathbf{H}\otimes\mathbf{S}^\pm)\right] \in K_G^{\tau+*}(G)$ is the image of $\mathbf{H}$ in the isomorphism (1.1). When $\dim G$ is odd, $\mathbf{S}^+ = \mathbf{S}^-$ and skew-adjointness of $\not{D}_A$ leads instead to a class in K^1. The twisting τ is the level of $\mathbf{H}\otimes\mathbf{S}$ as an LG-representation, with a (G-dependent) shift from the level h of $\mathbf{H}$.

The shifts are best explained in the world of super-categories, with $\mathbb{Z}/2$ gradings on morphisms and objects; odd simple objects have as endomorphisms the rank one Clifford algebra Cliff(1), and in the semi-simple case, they contribute a free generator to K^1 instead of K^0. Consider the τ-projective representations of LG with compatible action of Cliff($L\mathfrak{g}$), thinking of them as modules for the (not so well-defined) crossed product $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$. They form a semi-simple super-category $\mathfrak{SRep}^\tau$, and the isomorphism (1.1) becomes

$$K^*\,\mathfrak{SRep}^\tau\,(LG \ltimes \mathrm{Cliff}(L\mathfrak{g})) \cong K_G^{\tau+*}(G) \tag{1.3}$$

with the advantage of having no shift in degree or twisting. (For simply connected G, both sides live in degree $* = \dim\mathfrak{g}$, but both parities can be present for general G.) This isomorphism is induced by the Dirac families of (1.2): a super-representation $\mathbf{SH}^\pm$ of $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$ can be coupled to the Dirac operators $\not{D}_A$ without a choice of factorization as $\mathbf{H}\otimes\mathbf{S}^\pm$.

2. The main result

There is a curious mismatch in (1.3): the isomorphism is induced by a functor of underlying Abelian categories, from $\mathbb{Z}/2$-graded representations to twisted Fredholm bundles over G, but this functor is far from an equivalence. The category $\mathfrak{SRep}^\tau$ is semi-simple (in the graded sense discussed), but that of twisted Fredholm complexes is not so; we can even produce continua of non-isomorphic objects in any given K-class, by compact perturbation of a Fredholm family.

Here, we redress this problem by incorporating a *super-potential*, a celebrity in the algebraic geometry of 2-dimensional physics (the "B-model"). As explained by Orlov[4] [8], this deforms the category of complexes of vector bundles into that of *matrix factorizations*: the 2-*periodic, curved complexes* with curvature equal to the super-potential W. Our W has Morse critical points, leading to a semi-simple super-category with one generator for each critical point; the generators are precisely the Dirac families of (1.2) on irreducible LG-representations. The artifice of introducing W is redeemed by its natural topological origin in the *loop rotation* $\mathbb{T}_r$-action on the stack $[G:G]$. The $\mathbb{T}_r$-action is evident in the presentation $[\mathcal{A}:LG]$, but it rigidifies to a $B\mathbb{Z}$-action on the stack. Furthermore, for twistings τ transgressed from BG, the $B\mathbb{Z}$-action lifts to the G-equivariant *gerbe* G^τ over G which underlies the K-theory twisting. The logarithm of this lift is $2\pi\mathrm{i}W$.

Remark 2.1. The conceptual description of a super-potential as logarithm of a $B\mathbb{Z}$-action on a category of sheaves is worked out in [9]; the matrix factorization category is the *Tate fixed-point category* for the $B\mathbb{Z}$-action. For varieties, W is a function and $\exp(2\pi\mathrm{i}W)$ generates a $B\mathbb{Z}$-action on sheaves; on a stack, a geometric underlying action can also be present, as in this case. With respect to [9], our W_τ below should be re-scaled to take integer values at all critical points; we will omit this detail in order to better connect with the formulas in [4,5].

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1 Twisted loop groups show up when G is disconnected [5].

2 When G is not simply connected, there is a constraint on h.

3 The normalized operator $(-2)^{-1/2}\not{D}_0$ is the square root G_0 of L_0 in the super-Virasoro algebra.

4 Orlov discusses complex algebraic vector bundles; we found no exposition for equivariant Fredholm complexes in topology, and a discussion is planned for our follow-up paper.

To spell this out, recall that a stack is an instance of a category, and a $B\mathbb{Z}$-action thereon is described by its generator, an automorphism of the identity functor. This is a section over the space of objects, valued in automorphisms, which is central for the groupoid multiplication. For $[G:G]$, the relevant section is the identity map $G \to G$, from objects to morphisms. Intrinsically, $[G:G]$ is the mapping stack from $B\mathbb{Z}$ to BG, and the $B\mathbb{Z}$-action in question is the self-translation action of $B\mathbb{Z}$. This rigidifies the geometric $\mathbb{T}_r$-action on the homotopy equivalent spaces $LBG \sim BLG \sim \mathcal{A}/LG$.

A class $\hat{\tau} \in H^4(BG;\mathbb{Z})$ transgresses to a $\tau \in H^3_G(G;\mathbb{Z})$, with the latter having a natural $\mathbb{T}_r$-equivariant refinement. This can also be rigidified, as follows. The exponential sequence lifts $\hat{\tau}$ uniquely to $H^3(BG;\mathbb{T})$, the group cohomology with smooth circle coefficients. That defines a Lie 2-group $G^{\hat{\tau}}$, a multiplicative $\mathbb{T}$-gerbe over G. (Multiplicativity encodes the original $\hat{\tau}$.) The mapping stack from $B\mathbb{Z}$ to $BG^{\hat{\tau}}$ is the quotient $[G^{\hat{\tau}}:G^{\hat{\tau}}]$ under conjugation, and carries the $B\mathbb{Z}$-action from the self-translations of the latter. Because $B\mathbb{T} \hookrightarrow G^{\hat{\tau}}$ is strictly central, the self-conjugation action of $G^{\hat{\tau}}$ factors through G, and the quotient stack $[G^{\hat{\tau}}:G]$ is our $B\mathbb{Z}$-equivariant gerbe over $[G:G]$ with band $\mathbb{T}$. We denote this central circle by $\mathbb{T}_c$, to distinguish it from $\mathbb{T}_r$.

The $B\mathbb{Z}$-action gives an automorphism $\exp(2\pi i W_\tau)$ of the identity of $[G^{\hat{\tau}}:G]$, lifting the geometric one on $[G:G]$. Concretely, $[G^{\hat{\tau}}:G]$ defines a $\mathbb{T}_c$-central extension of the stabilizer of $[G:G]$, and $\exp(2\pi i W_\tau)$ is a trivialization of its fiber over the automorphism g at the point $g \in G$ (see Section 3 below). The logarithm W_τ is multi-valued and only locally well-defined; nevertheless, the category $\mathrm{MF}^\tau_G(G;W_\tau)$ of twisted matrix factorizations is well-defined, and its objects are represented by τ-twisted G-equivariant Fredholm complexes over G curved by $W_\tau + \mathbb{Z}\cdot\mathrm{Id}$.

Theorem 2.2. *The following defines an equivalence of categories from $\mathfrak{SRep}^\tau$ to $\mathrm{MF}^\tau_G(G;-2W_\tau)$: a graded representation $\mathbf{SH}^\pm$ goes to the twisted and curved Fredholm family $(\not{D}_\bullet, \mathbf{SH}^\pm)$ whose value at the connection $d + A\,dt \in \mathcal{A}$ is the τ-projective LG-equivariant curved Fredholm complex*

$$\not{D}_A = \not{D}_0 + i\psi(A) : \mathbf{SH}^+ \rightleftarrows \mathbf{SH}^-.$$

Remark 2.3.

(i) The factor (-2), stemming from our conventions [5], can be absorbed by scaling the operators.
(ii) Matrix factorizations obtained from irreducible representations are supported on single conjugacy classes, the so-called *Verlinde conjugacy classes* in G, for the twisting τ. These are the supports of the co-kernels of the Dirac families (1.2), [5, §12].
(iii) There is a braided tensor structure on $\mathfrak{SRep}^\tau$ $(LG \ltimes \mathrm{Cliff}(L\mathfrak{g}))$ (without $\mathbb{T}_r$-action). A corresponding structure on $\mathrm{MF}^\tau_G(G, W_\tau)$ should come from the Pontryagin product. We do not know how to spell out this structure, partly because the $\mathbb{T}_r$-action is already built into the construction of MF^τ, and the Pontryagin product is *not* equivariant thereunder.
(iv) The values of the automorphism $\exp(2\pi i W_\tau)$ at the Verlinde conjugacy classes determine the *ribbon element* in $\mathfrak{Rep}^h(LG)$; see [2] for the discussion when G is a torus.

Theorem 2.2 has a $\hat{\tau} \to \infty$ scaling limit, which is needed in the proof. In this limit, the representation category of LG becomes that of G. On the topological side, noting that each $\hat{\tau}$ defines an inner product on $\mathfrak{g}$, we magnify a neighborhood of $1 \in G$ to fix the scale. The τ-central extensions of stabilizers near 1 have natural splittings, and W_τ converges to a super-potential W, a central element of the crossed product algebra $G \ltimes \mathrm{Sym}(\mathfrak{g}^*)$. In a basis ξ_a of $\mathfrak{g}$ with dual basis ξ^a of $\mathfrak{g}^*$, we will find in Section 3 that

$$W = -i \cdot \xi_a(\delta_1) \otimes \xi^a + \frac{1}{2}\sum\nolimits_a \|\xi^a\|^2 \tag{2.1}$$

with $\xi_a(\delta_1)$ denoting the ξ_a-derivative of the delta-function at $1 \in G$. This leads to a G-equivariant matrix factorization category $\mathrm{MF}_G(\mathfrak{g}, W)$ on the Lie algebra.

To describe this limiting case, recall from [5, §4] the G-analogue of the Dirac family (1.2). Kostant's *cubic Dirac operator* [6] on G is left-invariant, and the Peter–Weyl decomposition gives an operator $\not{D}_0 : \mathbf{V} \otimes \mathbf{S}^\pm \to \mathbf{V} \otimes \mathbf{S}^\mp$ for any irreducible representation $\mathbf{V}$ of G, coupled to the spinors $\mathbf{S}^\pm$ on $\mathfrak{g}$. As before, let us work with graded modules $\mathbf{SV}$ for the super-algebra $G \ltimes \mathrm{Cliff}(\mathfrak{g})$.

Theorem 2.4. *Sending $\mathbf{SV}^\pm$ to $(\not{D}_\bullet, \mathbf{SV}^\pm)$, the curved complex over $\mathfrak{g}$ given by*

$$\mathfrak{g} \ni \mu \mapsto \not{D}_\mu = \not{D}_0 + i\psi(\mu) : \mathbf{SV}^+ \leftrightarrows \mathbf{SV}^-$$

provides an equivalence of super-categories from graded $G \ltimes \mathrm{Cliff}(\mathfrak{g})$-modules $\mathbf{SV}^\pm$ to G-equivariant, $(-2W)$-matrix factorizations over $\mathfrak{g}$.

With λ denoting the lowest weight of V and $T(\mu)$ the μ-action on $\mathbf{SV}$, we have [5, Cor. 4.8]

$$\not{D}_\mu^2 = -\|\lambda_V + \rho\|^2 + 2i \cdot T(\mu) - \|\mu\|^2 \in (-2W) + \mathbb{Z}.$$

3. Outline of the proof

3.1. Executive summary

The category $\mathrm{MF}^{\tau}_G(G; W_\tau)$ sheafifies over the conjugacy classes of G. Near a $g \in G$ with centralizer Z, the stack $[G:G]$ is modeled on a neighborhood of 0 in the adjoint quotient $[\mathfrak{z}:Z]$ of the Lie algebra $\mathfrak{z}$, via $\mathfrak{z} \ni \zeta \mapsto g \cdot \exp(2\pi\zeta)$. The equivariant gerbe $[G^{\hat\tau}:G]$ is locally trivialized (possibly on a finite cover of Z) uniquely up to discrete choices, differing by Z-characters. We will compute W_τ locally in those terms in $Z \ltimes C^\infty(\mathfrak{z})$, recovering (2.1), up to a (g-dependent) central translation in $\mathfrak{z}$. We then show that MF^τ vanishes near *singular* elements g. Assuming for brevity that $\pi_1(G)$ is torsion-free, we are then left with the case when Z is the maximal torus $T \subset G$, where the super-potential W_τ turns out to have Morse critical points, located precisely at the Verlinde conjugacy classes. The local category is freely generated by the respective Atiyah–Bott–Schapiro Thom complex; the latter is quasi-isomorphic to our Dirac family for a specific irreducible representation, associated with the Verlinde class [5, §12].

3.2. Crossed module description

We will describe $G^{\hat\tau}$ as a *Whitehead crossed module* [11]. This is an exact sequence of groups

$$\mathbb{T}_c \rightarrowtail K \xrightarrow{\varphi} H \twoheadrightarrow G,$$

equipped with an action $\alpha : H \to \mathrm{Aut}(K)$ which lifts the self-conjugation of H and factors the self-conjugation of K. Call h an H-lift of $g \in G$ and C the pre-image of Z in H. Define the central extension $\widetilde{Z}$ by means of a $\mathbb{T}_c$-central extension $\widetilde{C}$ of C trivialized over $\varphi(K) \cap C$, as follows.[5]

The commutator $c \mapsto hch^{-1}c^{-1}$ gives a crossed homomorphism $\chi : C \to \varphi(K)$ with respect to the conjugation action of C on $\varphi(K)$. The lift α lets χ pull back the central extension $K \twoheadrightarrow \varphi(K)$ to one $\widetilde{C} \twoheadrightarrow C$; further, $\widetilde{C}$ is trivialized over $\varphi(K)$, since $\alpha(h)$ identifies the fibers of K over c and hch^{-1}, when $c \in \varphi(K)$. Finally, noticing that $hhh^{-1}h^{-1} = 1$ trivializes the fiber of $\widetilde{C}$ over $c = h$ and gives our $\exp(2\pi \mathrm{i} W_\tau)$ at $g \in Z$.

3.3. Local computation of W_τ

Following [1], take $K = \Omega^\tau G$, the τ-central extension of the group of smooth maps $[0, 2\pi] \to G$ sending $\{0, 2\pi\}$ to 1, and $H = \mathscr{P}_1 G$, the group of smooth paths starting at $1 \in G$ but free at the end. With the $\hat\tau$-inner product $\langle . | . \rangle$, the crossed module action of $\gamma \in H$ on the Lie algebra $\mathrm{i}\mathbb{R} \oplus \Omega\mathfrak{g}$ of K is

$$\gamma.(x \oplus \omega) = \left(x - \frac{\mathrm{i}}{2\pi} \int_0^{2\pi} \langle \gamma^{-1} \mathrm{d}\gamma | \omega \rangle \right) \oplus \mathrm{Ad}_\gamma(\omega) \tag{3.1}$$

extending the Ad-action of $\Omega^\tau G$ [10, Prop. 4.3.2], and exponentiating to an H-action on K.[6]

Lift g to $h = \exp(t\mu) \in \mathscr{P}_1 G$, $\mu \in \frac{1}{2\pi} \log g$, and assume first that Z centralizes μ. Instead of the entire group C of Section 3.2, consider the subgroup $\mathscr{P}_1 Z$ of paths in Z. This centralizes h, trivializing $\widetilde{C}$ over $\mathscr{P}_1 Z$. In this 'lucky' trivialization, $W_\tau = 0$. However, over $\Omega Z = \varphi(K) \cap \mathscr{P}_1 Z$, the trivialization of Section 3.2 differs from the lucky one by adding the (exponentiated) character

$$\omega \mapsto -\frac{\mathrm{i}}{2\pi} \int_0^{2\pi} \langle \mu | \omega \rangle \mathrm{d}t,$$

as per formula (3.1). We can trivialize $\widetilde{Z}$ locally by extending this to a character of $\mathscr{P}_1 Z$, accomplished by exponentiating the same integral. Now, $2\pi \mathrm{i} W_\tau(g) = \pi \mathrm{i} \|\mu\|^2 \oplus 2\pi\mu \in \mathrm{i}\mathbb{R} \oplus \mathfrak{g}$.

Even when Z does not centralize μ, W_τ is determined (for $\pi_1(G)$ torsion-free) by restriction to a maximal torus. Continuity also pins it down: the assumption on μ can be satisfied for generic g.

3.4. Vanishing of singular contributions

Take for simplicity $g = 1$, $Z = G$, W on $\mathfrak{g}$ as in (2.1), plus possibly a central linear term μ. Koszul duality equates the localized category $\mathrm{MF}^\tau_G(\mathfrak{g}; W)$ with the super-category of modules over the differential super-algebra

$$\left(G \ltimes \mathrm{Cliff}(\mathfrak{g}), [\not{D}_\mu, \_] \right), \quad \text{with} \quad \not{D}_\mu = \not{D}_0 + \mathrm{i}\psi(\mu)$$

[5] The trivialization will be normalized by C-conjugation, thus descending the central extension to Z.

[6] Acting on other components of ΩG requires more topological information from $\hat\tau$.

of Theorem 2.4. Ignoring $\not{D}_\mu$, the algebra is semi-simple, with simple modules the $\mathbf{V}\otimes\mathbf{S}^\pm$. Now, $\not{D}_\mu^2 = -\|\lambda_V + \mu + \rho\|^2$ cannot vanish for any $\mathbf{V}$ for non-abelian $\mathfrak{z}$, so $[\not{D}_\mu, \not{D}_\mu]$ provides a homotopy between 0 and the central unit $\not{D}_\mu^2$. This makes the super-category of graded modules quasi-equivalent to 0.

3.5. Globalization for the torus

We describe the stack $[T^{\hat{\tau}} : T]$ and potential W_τ in the presentation $T = [\mathfrak{t} : \Pi]$ of the torus as a quotient of its Lie algebra by $\Pi \cong \pi_1(T)$. Lifted to $\mathfrak{t}$, the gerbe of stabilizers $\tilde{T}$ is trivial with band $\mathbb{T}_c \times T$. The descent datum under translation by $p \in \Pi$ is the shearing automorphism of $\mathbb{T}_c \times T$ given by the $\mathbb{T}_c$-valued character $\exp\langle p|\log t\rangle$, $t \in T$. In the same trivialization over $\mathfrak{t}$, the super-potential is

$$2\pi \mathrm{i} W_\tau(\mu) = \pi \mathrm{i}\|\mu\|^2 \oplus 2\pi\mu \in \mathrm{i}\mathbb{R} \oplus \mathfrak{t}.$$

With Λ denoting the character lattice of T, the crossed product algebra of the stack $[T^\tau : T]$ can be identified with the functions on $\left(\coprod_{\lambda\in\Lambda} \mathfrak{t}_\lambda\right)/\Pi$, with the action of Π by simultaneous translation on Λ and $\mathfrak{t}$. On the sheet $\lambda \in \Lambda$, $W_\tau = -\langle\lambda|\mu\rangle + \|\mu\|^2/2$ has a single Morse critical point at $\mu = \lambda$.

It follows that the super-category $\mathrm{MF}_T^\tau(T; W_\tau)$ is semi-simple, with one generator of parity $\dim \mathfrak{t}$ at each point in the kernel of the isogeny $T \to T^*$ derived from the quadratic form $\hat{\tau} \in H^4(BT; \mathbb{Z})$. The kernel comprises precisely the Verlinde points in T [2], concluding the proof of our main result.

Acknowledgements

This research was partially supported by NSF-funded Focussed Research Group grants DMS-1160461 and DMS-1160328.

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