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Partial differential equations

Global-in-time existence of weak solutions to Kolmogorov's two-equation model of turbulence

Sur l'existence globale en temps des solutions faibles pour le modèle de turbulence à deux équations de Kolmogorov

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A R T I C L E I N F O

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A B S T R A C T

We consider Kolmogorov's model for the turbulent motion of an incompressible fluid in $\mathbb{R}^3$. This model consists in a Navier–Stokes-type system for the mean flow $\boldsymbol{u}$ and two further partial differential equations: an equation for the frequency ω and for the kinetic energy k each. We investigate this system of partial differential equations in a cylinder $\Omega\times\,]0,T[$ ($\Omega\subset\mathbb{R}^3$ cube, $0<T<+\infty$) under spatial periodic boundary conditions on $\partial\Omega\times\,]0,T[$ and initial conditions in $\Omega\times\{0\}$. We present an existence result for a weak solution $\{\boldsymbol{u},\omega,k\}$ to the problem under consideration, with ω, k obeying the inequalities $c_1+t\le\frac{1}{\omega}\le t+c_2$ and $\frac{k^{1/2}}{\omega}\ge c_3t^{1/2}$ ($c_1,c_2,c_3=\text{const}>0$).

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R É S U M É

On considére le modèle de Kolmogorov pour l'écoulement turbulent d'un liquide incompressible dans $\mathbb{R}^3$. Ce modèle consiste d'un système de type Navier–Stokes pour la vitesse moyenne $\boldsymbol{u}$ d'écoulement et de deux équations aux derivées partielles additionnelles : une équation pour la fréquence ω et une pour l'énergie cinétique k de turbulence. Nous considérons ce système d'équations aux derivées partielles dans un cylindre $\Omega\times\,]0,T[$ ($\Omega\subset\mathbb{R}^3$ cube, $0<T<+\infty$) avec des conditions aux limites périodiques spatiales sur $\partial\Omega\times\,]0,T[$ et des conditions initiales dans $\Omega\times\{0\}$. Nous présentons un résultat sur l'existence d'une solution faible $\{\boldsymbol{u},\omega,k\}$ du problème envisagé où ω, k vérifient les inégalités $c_1+t\le\frac{1}{\omega}\le t+c_2$ et $\frac{k^{1/2}}{\omega}\ge c_3t^{1/2}$ ($c_1,c_2,c_3=\text{const}>0$).

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En [7], Kolmogorov postula le système d'équations aux derivées partielles suivant pour la description de l'écoulement turbulent d'un liquide incompressible dans $\mathbb{R}^3$:

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$$\operatorname{div}\boldsymbol{u}=0,\quad \frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\operatorname{div}\Big(\frac{k}{\omega}\,\boldsymbol{D}(\boldsymbol{u})\Big)-\nabla p+\boldsymbol{f}, \tag{1}$$

$$\frac{\partial \omega}{\partial t}+\boldsymbol{u}\cdot\nabla\omega=\operatorname{div}\Big(\frac{k}{\omega}\,\nabla\omega\Big)-\omega^2,\quad \frac{\partial k}{\partial t}+\boldsymbol{u}\cdot\nabla k=\operatorname{div}\Big(\frac{k}{\omega}\,\nabla k\Big)+\frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2-k\omega. \tag{2}$$

Ici $\boldsymbol{u}=(u_1,u_2,u_3)$ denote le champ des vitesses moyennes, p la pression moyenne, $\omega>0$ une fréquence, et $k\ge 0$ l'énergie cinétique moyenne de turbulence ; $\boldsymbol{D}(\boldsymbol{u})=\frac{1}{2}(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top)$ denote le tenseur des vitesses moyennes de déformation.

Au lieu de $\mathbb{R}^3$, nous considérons le système (1), (2) dans un cube $\Omega=(\,]0,a[\,)^3$ $(0<a<+\infty$ fixé). En conséquence, nous étudions (1), (2) dans le cylindre $Q_T=\Omega\times\,]0,T[$ $(0<T<+\infty$ quelconque) et posons des conditions aux limite périodiques spatiales sur $\partial\Omega\times\,]0,T[$ et des conditions initiales dans $\Omega\times\{0\}$ pour les fonctions $\{\boldsymbol{u},\omega,k\}$. Nous présentons un théorème d'existence d'une solution faible pour ce problème, la formulation faible de l'équation pour k renfermant une mesure défaut. Comme dans [12,13] nous commencons par démontrer l'existence faible $\{\boldsymbol{u}_\varepsilon,\omega_\varepsilon,k_\varepsilon\}$ $(\omega_\varepsilon\ge 0,\ k_\varepsilon\ge 0$ p. p. dans Q_T ; $\varepsilon>0)$ pour un problème approximatif. Les estimations suivantes sont fondamentales pour continuer les raisonnements :

1. *L'estimation* $\frac{k_\varepsilon}{\varepsilon+\omega_\varepsilon}\ge\frac{k_*}{\varepsilon(1+T\omega^*)+\omega^*}>0$ *p. p. dans* Q_T (voir (20) ci-dessous). Pour établir cette estimation, nous utilisons un variante du principe du maximum dans Q_T pour les fonctions ω_ε et k_ε.

2. *L'estimation* $\int_{Q_T}\big(k_\varepsilon^{4p/3}+|\nabla k_\varepsilon|^p\big)\le \mathrm{Const}\ \forall\,1\le p<2$, $\int_{Q_T}(k_\varepsilon|\nabla k_\varepsilon|)^q\le\mathrm{Const}\ \forall\,1\le q<\frac{8}{7}$. Pour cela, nous développons la technique de test de l'équation pour k_ε avec la fonction $1-\frac{1}{(1+k_\varepsilon)^\delta}$ $(0<\delta<1)$ (voir [12,13]).

Nous obtenons l'existence d'une solution faible du problème envisagé en effectuant le passage à la limite pour $\{\boldsymbol{u}_\varepsilon,\omega_\varepsilon,k_\varepsilon\}$ lorsque $\varepsilon\to 0$.

1. Introduction

In [7], Kolmogorov postulated the following system of partial differential equations as a model for the turbulent motion of an incompressible fluid in $\mathbb{R}^3$:

$$\operatorname{div}\boldsymbol{u}=0,\quad \frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\operatorname{div}\Big(\frac{k}{\omega}\,\boldsymbol{D}(\boldsymbol{u})\Big)-\nabla p+\boldsymbol{f}, \tag{1}$$

$$\frac{\partial \omega}{\partial t}+\boldsymbol{u}\cdot\nabla\omega=\operatorname{div}\Big(\frac{k}{\omega}\,\nabla\omega\Big)-\omega^2,\quad \frac{\partial k}{\partial t}+\boldsymbol{u}\cdot\nabla k=\operatorname{div}\Big(\frac{k}{\omega}\,\nabla k\Big)+\frac{k}{\omega}\,\big|\boldsymbol{D}(\boldsymbol{u})\big|^2-k\omega. \tag{2}$$

Here, $\boldsymbol{u}=(u_1,u_2,u_3)$ denotes the mean velocity, p the mean pressure, $k=\frac{1}{3}\,\overline{|\widetilde{\boldsymbol{u}}|^2}$ the mean turbulent kinetic energy ($\widetilde{\boldsymbol{u}}=$ fluctuation velocity) and $\omega>0$ denotes a frequency associated with the dissipation of turbulent kinetic energy ($\boldsymbol{D}(\boldsymbol{u})=\frac{1}{2}(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^\top)$ mean strain-rate). $\boldsymbol{f}$ represents a given external force. The paper [7] originated from Kolmogorov's theory of turbulence published in 1941. A detailed presentation of this theory is given, e.g., in [5] (see also the article by Yaglom [15, pp. 488–503]). A discussion of (1), (2) and other two-equation models of turbulence can be found in [14], [16, Chap. 4.3].

Instead of studying (1), (2) in the whole $\mathbb{R}^3$, we consider this system in a cube $\Omega=(]0,a[)^3$ $(0<a<+\infty$ fixed) and complete it by spatial periodic boundary conditions with respect to Ω. Let $\partial\Omega$ denote the boundary of Ω. We define $\Gamma_i=\partial\Omega\cap\{x_i=0\}$, $\Gamma_{i+3}=\partial\Omega\cap\{x_i=a\}$ $(i=1,2,3)$.

Let $0<T<+\infty$. We study systems (1), (2) in the cylinder $Q_T=\Omega\times\,]0,T[$ with the following conditions:

$$\left.\begin{aligned}&\boldsymbol{u}\Big|_{\Gamma_i\times]0,T[}=\boldsymbol{u}\Big|_{\Gamma_{i+3}\times]0,T[},\ \text{analogously for } p,\omega,k,\\ &\boldsymbol{D}(\boldsymbol{u})\Big|_{\Gamma_i\times]0,T[}=\boldsymbol{D}(\boldsymbol{u})\Big|_{\Gamma_{i+3}\times]0,T[},\ \text{analogously for } \nabla\omega,\ \nabla k,\end{aligned}\right\} \tag{3}$$

$$\boldsymbol{u}=\boldsymbol{u}_0,\quad \omega=\omega_0,\quad k=k_0 \text{ in } \Omega\times\{0\}. \tag{4}$$

The aim of this Note is to present an existence result for a weak solution $\{\boldsymbol{u},\omega,k\}$ to (1)–(4).

2. Statement of the main result

Let X denote a real normed space with norm $|\cdot|_X$, let X^* be its dual and let $\langle x^*,x\rangle_X$ denote the dual pairing of $x^*\in X^*$ and $x\in X$. The symbol $C_w([0,T];X)$ stands for the vector space of all mappings $u:[0,T]\to X$ such that, for every $x^*\in X^*$, the function $t\mapsto\big\langle x^*,u(t)\big\rangle_X$ is continuous on $[0,T]$. Next, by $L^p(0,T;X)$ $(1\le p\le+\infty)$, we denote the vector space of all equivalence classes of Bochner measurable mappings $u:[0,T]\to X$ such that the function $t\mapsto\big|u(t)\big|_X$ is in $L^p(0,T)$ (cf. [2, Chap. III, §3, Chap. IV, §3], [3, App.], [4]).

For bounded domains $\Omega\subset\mathbb{R}^n$ $(n\ge 2)$ with Lipschitz boundary we denote by $W^{1,p}(\Omega)$ $(1\le p<+\infty)$ the usual Sobolev space.

In what follows, let $\Omega=(\,]0,a[\,)^3$ be the cube introduced above. We define

$$W^{1,p}_{\mathrm{per}}(\Omega)=\Big\{u\in W^{1,p}(\Omega);\ u\Big|_{\Gamma_i}=u\Big|_{\Gamma_{i+3}}\ (i=1,2,3)\Big\},\qquad \boldsymbol{W}^{1,p}_{\mathrm{per,diver}}(\Omega)=\big\{\boldsymbol{u}\in\boldsymbol{W}^{1,p}_{\mathrm{per}}(\Omega);\ \operatorname{div}\boldsymbol{u}=0\text{ a.e. in }\Omega\big\}$$

(bold-faced letters refer to vector valued mappings as well as to Banach spaces of such mappings). The conditions on the data are:

$$\boldsymbol{f} \in \boldsymbol{L}^2(Q_T); \quad \boldsymbol{u}_0 \in \overline{\boldsymbol{C}^\infty_{\mathrm{per,div}}(\overline{\Omega})}^{\|\cdot\|_{\boldsymbol{L}^2(\Omega)}}, \tag{5}$$

$$\left.\begin{aligned} &\omega_0 \textit{ measurable in } \Omega,\ \omega_* \le \omega_0(x) \le \omega^* \textit{ for a.e. } x \in \Omega\ (\omega_*, \omega^* = \text{const} > 0),\\ &k_0 \in L^1(\Omega),\ k_0(x) \ge k_* = \text{const} > 0 \textit{ for a.e. } x \in \Omega. \end{aligned}\right\} \tag{6}$$

The following theorem is the main result of our paper.

Theorem. *Assume (5) and (6). Then there exists a triple of measurable functions $\{\boldsymbol{u}, \omega, k\}$ in Q_T such that*

$$\frac{\omega_*}{1+t\omega_*} \le \omega(x,t) \le \frac{\omega^*}{1+t\omega^*}, \quad k(x,t) \ge \frac{k_*}{1+t\omega_*} \ \textit{for a.e.}\ (x,t) \in Q_T, \tag{7}$$

$$\frac{1}{\omega^*} + t \le \frac{1}{\omega(x,t)} \le t + \frac{1}{\omega_*}, \quad \frac{k^{1/2}(x,t)}{\omega(x,t)} \ge \Big(\frac{k_*}{\omega^*}t\Big)^{1/2} \ \textit{for a.e.}\ (x,t) \in Q_T, \tag{8}$$

$$\left.\begin{aligned} &\boldsymbol{u} \in C_w\big([0,T]; \boldsymbol{L}^2(\Omega)\big) \cap L^2\big(0,T; \boldsymbol{W}^{1,2}_{\mathrm{per,div}}(\Omega)\big),\\ &\omega \in C_w\big([0,T]; L^2(\Omega)\big) \cap L^2\big(0,T; W^{1,2}_{\mathrm{per}}(\Omega)\big), \quad k \in L^\infty\big(0,T; L^1(\Omega)\big), \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} &\int_{Q_T} \big(k^{4p/3} + |\nabla k|^p\big) < +\infty \quad \forall\, 1 \le p < 2, \ \int_{Q_T} \big(k|\nabla k|\big)^q < +\infty \quad \forall\, 1 \le q < \frac{8}{7},\\ &\int_{Q_T} \frac{|\nabla k|^2}{(1+k)^{1+\delta}} < +\infty \quad \forall\, 0 < \delta < 1; \quad \int_{Q_T} \frac{k}{\omega}\big(|\boldsymbol{D}(\boldsymbol{u})|^2 + |\nabla\omega|^2\big) < +\infty, \end{aligned}\right\} \tag{10}$$

$$\boldsymbol{u}' \in \bigcup_{1\le p<2} L^{8p/(4p+3)}\Big(0,T; \big(\boldsymbol{W}^{1,8p/(4p-3)}_{\mathrm{per,div}}(\Omega)\big)^*\Big), \quad \omega' \in \bigcup_{1\le p<2} L^{8p/(4p+3)}\Big(0,T; \big(W^{1,8p/(4p-3)}_{\mathrm{per}}(\Omega)\big)^*\Big), \tag{11}$$

$$\left.\begin{aligned} &\int_0^T \big\langle \boldsymbol{u}'(t), \boldsymbol{v}(t)\big\rangle_{\boldsymbol{W}^{1,r}_{\mathrm{per,div}}} \mathrm{d}t - \int_{Q_T} (\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla\boldsymbol{v} + \int_{Q_T} \frac{k}{\omega}\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\boldsymbol{v})\\ &\quad = \int_{Q_T} \boldsymbol{f}\cdot\boldsymbol{v} \ \forall\, \boldsymbol{v} \in \bigcup_{r>16/5} L^r\big(0,T; W^{1,r}_{\mathrm{per}}(\Omega)\big), \quad \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 \ \textit{a.e. in}\ \Omega, \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} &\int_0^T \big\langle \omega'(t), \varphi(t)\big\rangle_{W^{1,r}_{\mathrm{per}}} \mathrm{d}t - \int_{Q_T} \omega\boldsymbol{u}\cdot\nabla\varphi + \int_{Q_T} \frac{k}{\omega}\nabla\omega\cdot\nabla\varphi\\ &\quad = -\int_{Q_T} \omega^2\varphi \ \forall\, \varphi \in \bigcup_{r>16/5} L^r\big(0,T; W^{1,r}_{\mathrm{per}}(\Omega)\big), \quad \omega(\cdot,0) = \omega_0 \ \textit{a.e. in}\ Q_T, \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} &\exists \textit{ bounded Radon measure } \mu \textit{ on the Borel } \sigma\textit{-algebra of } \overline{Q}_T \textit{ such that}\\ &-\int_{Q_T} k\frac{\partial z}{\partial t} - \int_{Q_T} k\boldsymbol{u}\cdot\nabla z + \int_{Q_T} \frac{k}{\omega}\nabla k\cdot\nabla z = \int_\Omega k_0(x)z(x,0)\mathrm{d}x + \int_{Q_T} \Big(\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - k\omega\Big)z + \int_{\overline{Q}_T} z\mathrm{d}\mu\\ &\forall\, z \in C^1(\overline{Q}_T),\ z\Big|_{\Gamma_i\times\,]0,T[} = z\Big|_{\Gamma_{i+3}\times\,]0,T[} \quad (i=1,2,3), \quad z(\cdot,T) = 0. \end{aligned}\right\} \tag{14}$$

In addition, the following inequalities hold for a.e. $t \in\,]0,T[$:

$$\frac{1}{2}\int_\Omega \big|\boldsymbol{u}(x,t)\big|^2\mathrm{d}x + \int_0^t\int_\Omega \frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 \le \frac{1}{2}\int_\Omega \big|\boldsymbol{u}_0\big|^2 + \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}, \tag{15}$$

$$\int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}(x,t)\big|^2 + k(x,t)\Big)\mathrm{d}x + \int_0^t\int_\Omega k\omega \le \int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}_0\big|^2 + k_0\Big) + \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}, \tag{16}$$

Remarks. 1. Obviously, (8) follows from (7). Except for the additive constants $\frac{1}{\omega^*}$ and $\frac{1}{\omega_*}$ in (8), the estimates for $\frac{1}{\omega}$ are in coincidence with Kolmogorov's theory of turbulence (cf. [5, pp. 100–103], [8, Chap. 33]).

The function $L := \frac{k^{1/2}}{\omega}$ characterizes the "external length scale" of the turbulent motion (see [5, Chap. 7], [8, Chap. 33], [16, Chap. 8.1]). Instead of the growth of L in (8), Kolmogorov [7] claims the weaker growth $L \ge c_0 t^{2/7}$ ($c_0 = \text{const} > 0$).

2. The integral relations in (11) and (12) represent a weak formulation of the $\boldsymbol{u}$-equation and the ω-equation, respectively, with spatial periodic boundary condition (cf. (1), (2), (3)). The derivatives $\boldsymbol{u}'$ and ω' in (11) (and (12), (13)) have to be understood in the sense of distributions from $]0,T[$ into the spaces $(\boldsymbol{W}^{1,8p/(4p-3)}_{\text{per,div}}(\Omega))^*$ and $(W^{1,8p/(4p-3)}_{\text{per}}(\Omega))^*$, respectively. An analogous remark refers to $\{\boldsymbol{u}'_\varepsilon, \omega'_\varepsilon, k'_\varepsilon\}$ below.

3. The defect measure μ in (14) arises from our approximation method for the proof the existence of a weak solution to (1)–(4). The measure μ vanishes, provided the weak solution under consideration satisfies appropriate regularity properties. More precisely:

Let the triple $\{\boldsymbol{u}, \omega, k\}$ satisfy $\omega > 0$, $k > 0$ a.e. in Q_T and let (9)–(16) be fulfilled. If equality holds in both (15) and (16), then (i) $\mu = 0$ and (ii) $\exists\, k' \in \bigcap_{1<s<8/7} L^1\big(0,T;(W^{1,s}_{\text{per}}(\Omega))^*\big)$.

To prove (i), let α be any Lipschitz function on $[0,T]$, $\alpha(t) = 0$ for all $t \in [t_\alpha, T]$ ($0 < t_\alpha < T$). Then (14) continues to hold for functions $z = z(x,t) = \zeta(x)\alpha(t)$, where $\zeta \in W^{1,s}_{\text{per}}(\Omega)$ ($s > 8$, observe (10)). Given $t \in]0,T[$ and $m > \frac{1}{T-t}$ ($m \in \mathbb{N}$), define

$$\alpha_m(\tau) = \begin{cases} 1 & 0 \le \tau \le t, \\ m\Big(t + \dfrac{1}{m} - \tau\Big) & t < \tau < t + \dfrac{1}{m}, \\ 0 & t + \dfrac{1}{m} \le \tau \le T. \end{cases}$$

We insert $z = 1\cdot\alpha_m$ into (14) and obtain

$$m\int_t^{t+1/m}\int_\Omega k(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \ge \int_\Omega k_0 + \int_0^{t+1/m}\int_\Omega \Big(\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - k\omega\Big)\alpha_m + \mu\big(\overline{\Omega}\times[0,t]\big).$$

Letting $m \to +\infty$ it follows that

$$\int_\Omega k(x,t)\,\mathrm{d}x \ge \int_\Omega k_0 + \int_0^t\int_\Omega \Big(\frac{k}{\omega}\big|\boldsymbol{D}(\boldsymbol{u})\big|^2 - k\omega\Big) + \mu\big(\overline{\Omega}\times[0,t]\big)$$

for all Lebesgue points t of the function $\tau \mapsto \int_\Omega k(x,\tau)\,\mathrm{d}x$. Adding this inequality to (15) [with equality therein], one finds

$$\int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}(x,t)\big|^2 + k(x,t)\Big)\mathrm{d}x + \int_0^t\int_\Omega k\omega - \int_\Omega \Big(\frac{1}{2}\big|\boldsymbol{u}_0\big|^2 + k_0\Big) - \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u} \;\ge\; \mu\big(\overline{\Omega}\times[0,t]\big).$$

From (16) [with equality therein], it follows $\mu(\overline{\Omega}\times[0,t]) = 0$. Thus, $\mu(\overline{Q}_T) = 0$.

The claim (ii) can be easily established by routine arguments (cf. [3, Appendix, Prop. A6], [4]).

With (i) and (ii) in hand, we obtain $k \in C([0,T];(W^{1,s}_{\text{per}}(\Omega))^*)$ and $k(\cdot,0) = k_0$ in the sense of $(W^{1,s}_{\text{per}}(\Omega))^*$. Now, (14) turns into the weak formulation of the k-equation.

4. The defect measure μ in (14) reflects the deep problem to establish an energy equality for weak solutions to the Navier–Stokes equations (see also [12,13]). In [9], the author studies a simplified one-equation model of turbulence, where a defect measure appears on p. 397 and 416. We notice that defect measures also occur for other types of nonlinear partial differential equations (cf., e.g., [1,6,10]).

3. Sketch of proof

Let $\Phi \in C([0,+\infty[)$ be a fixed, non-increasing function fulfilling the conditions $0 \le \Phi \le 1$ in $[0,+\infty[$, $\Phi = 1$ in $[0,1]$ and $\Phi = 0$ in $[2,+\infty[$. For $0 < \varepsilon \le 1$, define $\Phi_\varepsilon(\xi) = \Phi(\varepsilon\xi)$, $0 \le \xi < +\infty$.

1° *Existence of an approximate solution.* Fix any $6 < \rho < +\infty$, $3 < \sigma < \frac{11}{3}$. For every $0 < \varepsilon < +\infty$ there exist measurable functions $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ in Q_T such that $\omega_\varepsilon \ge 0$, $k_\varepsilon \ge 0$ a.e. in Q_T,

$$\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\} \in L^\rho\big(0, T; \boldsymbol{W}^{1,\rho}_{\mathrm{per,div}}(\Omega)\big) \times L^4\big(0, T; W^{1,4}_{\mathrm{per}}(\Omega)\big) \times L^\sigma\big(0, T; W^{1,\sigma}_{\mathrm{per}}(\Omega)\big),$$

$$\{\boldsymbol{u}'_\varepsilon, \omega'_\varepsilon, k'_\varepsilon\} \in L^{\rho'}\big(0, T; \big(\boldsymbol{W}^{1,\rho}_{\mathrm{per,div}}(\Omega)\big)^*\big) \times L^{4/3}\big(0, T; \big(W^{1,4}_{\mathrm{per}}(\Omega)\big)^*\big) \times L^{\sigma'}\big(0, T; \big(lW^{1,\sigma}_{\mathrm{per}}(\Omega)\big)^*\big),$$

$$\left.\begin{aligned} &\big\langle \boldsymbol{u}'_\varepsilon(t), \boldsymbol{v}\big\rangle_{W^{1,\rho}_{\mathrm{per,div}}} - \int_\Omega \Phi_\varepsilon\big(|\boldsymbol{u}_\varepsilon|^2\big)(\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \nabla \boldsymbol{v} + \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\, \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^{\rho-2}\Big)\boldsymbol{D}(\boldsymbol{u}_\varepsilon) : \boldsymbol{D}(\boldsymbol{v}) \\ &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \ \text{ for a.e. } t \in]0, T[, \ \ \forall\, \boldsymbol{v} \in \boldsymbol{W}^{1,\rho}_{\mathrm{per}}(\Omega); \ \ \boldsymbol{u}_\varepsilon(\cdot, 0) = \boldsymbol{u}_0, \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &\big\langle \omega'_\varepsilon(t), \varphi\big\rangle_{W^{1,4}_{\mathrm{per}}} - \int_\Omega \omega_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla\varphi + \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\, |\nabla\omega_\varepsilon|^2\Big)\nabla\omega_\varepsilon \cdot \nabla\varphi \\ &= -\int_\Omega \omega_\varepsilon^2 \varphi \ \text{ for a.e. } t \in]0, T[, \ \ \forall\, \varphi \in W^{1,4}_{\mathrm{per}}(\Omega); \ \ \omega_\varepsilon(\cdot, 0) = \omega_0, \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} &\big\langle k'_\varepsilon(t), z\big\rangle_{W^{1,\sigma}_{\mathrm{per}}} - \int_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla z + \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\, |\nabla k_\varepsilon|^{\sigma-2}\Big)\nabla k_\varepsilon \cdot \nabla z \\ &= \int_\Omega \Big(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon}\, \big|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\big|^2 - k_\varepsilon\omega_\varepsilon\Big) z \ \text{ for a.e. } t \in]0, T[, \ \ \forall\, z \in W^{1,\sigma}_{\mathrm{per}}(\Omega); \ \ k_\varepsilon(\cdot, 0) = k_0. \end{aligned}\right\} \tag{19}$$

This result can be proved by reformulating (17)–(19) in terms of an abstract operator equation and applying [11, Chap. 3.1.4, Théorème 1.2]. For this we have to pass from the data $\{\boldsymbol{u}_0, \omega_0, k_0\}$ to zero initial data. With regard to $\boldsymbol{u}_0$, this is easily done by (5) and with regard to ω_0, k_0 by routine arguments.

2° *A-priori estimates a.e. in Q_T for ω_ε and k_ε. For every $\varepsilon > 0$ there holds*

$$\frac{\omega_*}{1 + t\omega_*} \le \omega_\varepsilon(x, t) \le \frac{\omega^*}{1 + t\omega^*}, \quad k_\varepsilon(x, t) \ge \frac{k_*}{1 + t\omega^*} \ \textit{for a.e.}\ (x, t) \in Q_T \ \ (\omega_*, \omega^* \textit{ and } k_* \textit{ as in } (6)). \tag{20}$$

We establish the estimate from below for ω_ε. Set $\underline{\omega}(t) := \frac{\omega_*}{1+t\omega_*}$, $0 \le t \le T$. Then

$$\big(\omega_\varepsilon(\cdot, t) - \underline{\omega}(t)\big) \in W^{1,4}_{\mathrm{per}}(\Omega) \ \text{ for a.e. } t \in]0, T[, \quad \big(\omega_\varepsilon(x, 0) - \underline{\omega}(0)\big)^- = 0 \ \text{ for a.e. } x \in \Omega.$$

We take $\varphi = (\omega_\varepsilon(\cdot, t) - \underline{\omega}(t))^-$ in (18), add the term $-\dot{\underline{\omega}}(t)\int_\Omega(\omega_\varepsilon(x, t) - \underline{\omega}(t))^-\mathrm{d}x$ ($\dot{\underline{\omega}}$ = derivative of ω) to both sides and integrate over the interval $[0, t]$. It follows that

$$\frac{1}{2}\int_\Omega \big((\omega_\varepsilon(x, t) - \underline{\omega}(t))^-\big)^2 \mathrm{d}x - \int_0^t \int_\Omega \omega_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla(\omega_\varepsilon - \underline{\omega})^- \le \int_0^t \int_\Omega \big(\underline{\omega}^2 - \omega_\varepsilon^2\big)(\omega_\varepsilon - \underline{\omega})^- \le 0$$

for a.e. $t \in]0, T[$ (notice that $\dot{\underline{\omega}} = -\underline{\omega}^2$). Since

$$\begin{aligned} &\int_\Omega \omega_\varepsilon(x, s)\boldsymbol{u}_\varepsilon(x, s) \cdot \nabla\big(\omega_\varepsilon(x, s) - \underline{\omega}(s)\big)^- \mathrm{d}x \\ &= \int_\Omega \big(\omega_\varepsilon(x, s) - \underline{\omega}(s)\big)^- \boldsymbol{u}_\varepsilon(x, s) \cdot \nabla\big(\omega_\varepsilon(x, s) - \underline{\omega}(s)\big)^- \mathrm{d}x + \underline{\omega}(s)\int_\Omega \boldsymbol{u}_\varepsilon(x, s) \cdot \nabla\big(\omega_\varepsilon(x, s) - \underline{\omega}(s)\big)^- \mathrm{d}x = 0 \end{aligned}$$

for a.e. $s \in]0, t[$, the estimate from below for ω_ε follows.

The estimate for ω_ε from above by $\overline{\omega}(t) := \frac{\omega^*}{1+t\omega^*}$ $(0 \le t \le T)$ can be proved by testing (18) with $\varphi = (\omega_\varepsilon(\cdot, t) - \overline{\omega}(t))^+$. To prove the estimate from below for k_ε, set $\kappa(t) := \frac{k_*}{1+t\omega_*}$, $0 \le t \le T$. We insert $z = (k_\varepsilon(\cdot, t) - \kappa(t))^-$ into (19), make use of $\dot{\kappa} = -\kappa\overline{\omega}$ and obtain

$$\frac{1}{2}\int_\Omega \big(\big(k_\varepsilon(x, t) - \kappa(t)\big)^-\big)^2 \mathrm{d}x - \int_0^t \int_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla(k_\varepsilon - \kappa)^- \le \int_0^t \int_\Omega (\kappa\overline{\omega} - k_\varepsilon\omega_\varepsilon)(k_\varepsilon - \kappa)^- \le 0$$

for a.e. $t \in]0, T[$. By an analogous reasoning as above, $\int_0^t \int_\Omega k_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla(k_\varepsilon - \kappa)^- = 0$. Thus, $k_\varepsilon \ge \kappa$ a.e. in Q_T.

3° *Integral estimates*. We insert $\boldsymbol{v} = \boldsymbol{u}(\cdot, t)$ into (17) and $z = 1$ into (19). This gives

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(0,T;\boldsymbol{L}^2)} + \int\limits_{Q_T} \left(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon \left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^{\rho-2}\right) \left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^2 \le c, \quad \|k_\varepsilon\|_{L^\infty(0,T;L^1)} \le c \tag{21}$$

(by c we denote different positive constants which do not depend on ε).

Next, define $\Psi(\xi) = \int_0^\xi \left(1 - \frac{1}{(1+s)^\delta}\right) \mathrm{d}s$ $(0 \le \xi < +\infty,\ 0 < \delta < 1)$. We take $z = \Psi'\big(k_\varepsilon(\cdot, t)\big)$ in (19). Using (21) we obtain

$$\delta \int\limits_{Q_T} \left(\frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} + \varepsilon\, |\nabla k_\varepsilon|^{\sigma-2}\right) \frac{|\nabla k_\varepsilon|^2}{(1 + k_\varepsilon)^{1+\delta}} \le c.$$

From this estimate it follows that

$$\left.\begin{aligned} &\int\limits_{Q_T} \left(k_\varepsilon^{4p/3} + |\nabla k_\varepsilon|^p\right) \le c \quad \forall\, 1 \le p < 2, \qquad \int\limits_{Q_T} \left(k_\varepsilon\, |\nabla k_\varepsilon|\right)^q \le c \quad \forall\, 1 \le q < \frac{8}{7}, \\ &\varepsilon \int\limits_{Q_T} |\nabla k_\varepsilon|^{\sigma-1} |\nabla z| \le \varepsilon^{1/\sigma} \|\nabla z\|_{\boldsymbol{L}^r(Q_T)} \quad \forall\, z \in L^r\big(0, T; W^{1,r}_{\mathrm{per}}(\Omega)\big), \end{aligned}\right\} \tag{22}$$

where $r = \frac{\sigma\kappa}{\kappa - 1}$, $\kappa = \frac{4p}{3} \cdot \frac{1}{(1+\delta)(\sigma-1)}$ $\left(0 < \delta < \frac{11-3\sigma}{3(\sigma-1)}\right)$. The integral estimates for ω_ε are straightforward.

Estimates for $\boldsymbol{u}'_\varepsilon$ and ω'_ε with respect to appropriate dual norms are easily derived. Finally, given $8 < s < +\infty$, there exists a constant $c(s)$ such that $\|k'_\varepsilon\|_{L^1(0,T;(W^{1,s}_{\mathrm{per}})^*)} \le c(s)$.

4° *Passage to the limit* $\varepsilon \to 0$. From (21), (22), the estimates for ω_ε and the estimates for $\boldsymbol{u}'_\varepsilon$, ω'_ε and k'_ε, we obtain the existence of a subsequence of $\{\boldsymbol{u}_\varepsilon, \omega_\varepsilon, k_\varepsilon\}$ that converges weakly [or weakly*] to a triple $\{\boldsymbol{u}, \omega, k\}$ in the respective spaces as well as a.e. in Q_T. Then (8)–(13), (15) and (16) are readily seen.

Finally, there exists a bounded Radon measure μ on the Borel σ-algebra of $\overline{Q}_T$ such that, for all $z \in C(\overline{Q}_T)$,

$$\int\limits_{Q_T} \frac{k_\varepsilon}{\varepsilon + \omega_\varepsilon} \left|\boldsymbol{D}(\boldsymbol{u}_\varepsilon)\right|^2 z \longrightarrow \int\limits_{Q_T} \frac{k}{\omega} \left|\boldsymbol{D}(\boldsymbol{u})\right|^2 z + \int\limits_{\overline{Q}_T} z\, \mathrm{d}\mu \quad \text{as } \varepsilon \to 0.$$

The passage to the limit $\varepsilon \to 0$ in (19) is now easily done by routine arguments.

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