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Complex analysis

Application of Chebyshev polynomials to classes of analytic functions

Application des polynômes de Chebyshev à des classes de fonctions analytiques

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ABSTRACT

Our objective in this paper is to consider some basic properties of the familiar Chebyshev polynomials in the theory of analytic functions. We investigate some basic useful characteristics for a class $\mathcal{H}(t)$, $t \in (1/2, 1]$, of functions f, with $f(0) = 0$, $f'(0) = 1$, analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ satisfying the condition that

$$1 + \frac{zf''(z)}{f'(z)} \prec H(z,t) = \frac{1}{1 - 2tz + z^2} \quad (z \in \mathbb{U}),$$

where $H(z,t)$ is the generating function of the second kind of Chebyshev polynomials. The Fekete–Szegö problem in the class is also solved.

RÉSUMÉ

Notre propos dans cette Note est d'étudier quelques propriétés de base des polynômes de Chebyshev habituels en théorie des fonctions analytiques. Nous considérons plusieurs caractéristiques fondamentales pour les classes $\mathcal{H}(t)$, $t \in (1/2, 1]$ de fonctions f satisfaisant $f(0) = 0$, $f'(0) = 1$, analytiques dans le disque unité ouvert $\mathbf{U} = \{z : |z| < 1\}$ et telles que pour $z \in \mathbf{U}$, on ait :

$$1 + \frac{zf''(z)}{f'(z)} \prec H(z,t),$$

où $H(z,t) = 1/(1 - 2tz + z^2)$ désigne la fonction génératrice des polynômes de Chebyshev de seconde espèce. Nous résolvons également le problème de Fekete–Szegö pour les classes considérées.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions that are univalent in $\mathbb{U}$. We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if $f(z) = F(\omega(z))$ for some holomorphic function ω such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. A function $f \in \mathcal{A}$ maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1-z}{1+z} \quad (z \in \mathbb{U}). \tag{1}$$

It is well known that if a function $f \in \mathcal{A}$ satisfies (1), then f is univalent and starlike in $\mathbb{U}$. Let $\beta \in [0, 1)$. A function $f \in \mathcal{H}$ is said to be convex of order β if

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1-(1-2\beta)z}{1+z} \quad (z \in \mathbb{U}). \tag{2}$$

The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{ST}$. We denote the set of all functions $f \in \mathcal{A}$ that are convex of order β by $\mathcal{CV}(\beta)$. In particular, $\mathcal{CV} := \mathcal{CV}(0)$ is the well-known class of convex functions.

Definition 1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{H}(t)$, $t \in (1/2, 1]$, if it satisfies the condition:

$$1 + \frac{zf''(z)}{f'(z)} \prec H(z,t) := \frac{1}{1-2tz+z^2} \quad (z \in \mathbb{U}). \tag{3}$$

We note that if $t = \cos\alpha$, $\alpha \in (-\pi/3, \pi/3)$, then

$$H(z,t) = \frac{1}{1-2\cos\alpha z+z^2} \tag{4}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\sin((n+1)\alpha)}{\sin\alpha} z^n \quad (z \in \mathbb{U}). \tag{5}$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \cdots \quad (z \in \mathbb{U}). \tag{6}$$

Following [16], we write $H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in \mathbb{U},\ t \in (-1,1))$, where $U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}}$ $(n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind. Also, it is known that $U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$,

$$\text{and } U_1(t) = 2t,\ \ U_2(t) = 4t^2 - 1,\ \ U_3(t) = 8t^3 - 4t,\ \ U_4(t) = 16t^4 - 12t^2 + 1, \ldots. \tag{7}$$

The Chebychev polynomials $T_n(t)$, $-1 \le t \le 1$, of the first kind have the generating function of the form:

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \mathbb{U}).$$

However, the Chebychev polynomials of the first kind $T_n(t)$ and of the second kind $U_n(t)$ are well connected by the following relationships:

$$\frac{\mathrm{d}T_n(t)}{\mathrm{d}t} = nU_{n-1}(t),\ T_n(t) = U_n(t) - tU_{n-1}(t),\ 2T_n(t) = U_n(t) - U_{n-2}(t). \tag{8}$$

In this paper, we aim at investigating the geometric properties of the class $\mathcal{H}(t)$. The inter-relationships between the Chebyshev polynomials of first and second kinds given by (8) above can be used to study the corresponding properties of the Chebychev polynomials of the first kind. The Fekete–Szegö problem in the class is also solved.

2. Basic results

In this section, we present some basic results for the class of analytic functions defined and introduced in Section 1 above.

Theorem 2. *For the function $H(z,t)$, the following inequality holds:*

$$|H(z,t)| > \frac{1}{2(t+1)} \quad (z \in \mathbb{U}). \tag{9}$$

Proof. It is sufficient to consider $|H(z,t)|$ on the boundary $\partial H(\mathbb{U},t) = \{H(e^{i\varphi},t) : \varphi \in [0,2\pi)\}$, $\cos\varphi \neq t$. Let us denote $\mathfrak{Re}\left\{H(e^{i\varphi},t)\right\}$ by $x(\varphi,t)$ and $\mathfrak{Im}\left\{H(e^{i\varphi},t)\right\}$ by $y(\varphi,t)$. Then after simple calculations, we get

$$x(\varphi,t) = \frac{\cos\varphi}{2(\cos\varphi - t)}, \quad y(\varphi,t) = -\frac{\sin\varphi}{2(\cos\varphi - t)}.$$

Thus, we can write

$$\left|H(e^{i\varphi},t)\right|^2 = \frac{\cos^2\varphi + \sin^2\varphi}{4(t-\cos\varphi)^2} > \frac{1}{4(t+1)^2},$$

which establishes (9). □

The following corollary is a simple consequence of Theorem 2.

Corollary 3. *If $f \in \mathcal{H}(t)$, then*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| > \frac{1}{2(t+1)} \quad (z \in \mathbb{U}). \tag{10}$$

Theorem 4. *The function (6) is univalent in $|z| < t$.*

Proof. We consider the equality

$$\frac{1}{1-2tz_1+z_1^2} = \frac{1}{1-2tz_2+z_2^2},$$

which yields $(z_1 - z_2)(z_1 + z_2 - 2t) = 0$. If now $|z_1| < t$ and $|z_2| < t$, then $z_1 + z_2 - 2t \neq 0$, and this proves the result. □

Usually, the new classes of analytic functions are defined by the condition of the type (3) with the univalent functions on the right-hand side. However, the function $H(z,t)$ is not univalent in the unit disc and this allows us only limited considerations of the class $\mathcal{H}(t)$. For similar other classes as defined by means of (3), with non-univalent functions occurring on the right-hand side, one may refer to [3,4,6,14,15]. It is worth observing that if $|z| < 1$ and $t \in (1/2, 1]$, then $H(z,t)$ takes its values in the set $\Omega(t) = \{x + \mathrm{i}\,y : t^2(x^2+y^2) > (x-1/2)^2\}$. On the other hand, if $t = 1$, then the boundary of $\Omega(1)$ becomes the parabola:

$$\partial\{\Omega(1)\} = \left\{x + \mathrm{i}\,y : y^2 = 1/4 - x\right\}, \tag{11}$$

in which the focus is at the origin with the vertex at 1/4 and the directrix given by $x = 1/2$, see Fig. 1. Consequently, $H(\mathbb{U},t)$ lies on the right-hand side of this parabola.

Now if $1/2 < t < 1$, then the boundary of $\Omega(t)$ is the left branch of the hyperbola of the form:

$$\partial\{\Omega(t)\} = \left\{x + iy : \left(2(t^2-1)x + 1\right)^2 + 4t^2(t^2-1)y^2 = t^2\right\}. \tag{12}$$

The hyperbola (12) has the right-hand side focus at $-1/(t^2-1)$ and the vertex of the left branch at $1/(2(t+1))$. If $k > 1$, then $H(z,t)$ takes its values on the right-hand side of this hyperbola (12), see Fig. 2.

Therefore, if $f \in \mathcal{H}(t)$, then $1 + zf''(z)/f'(z)$ lies in the set of points whose distance from the origin multiplied by t is greater than its distance from the directrix $w = 1/2$. In this way, we have proved the following theorem.

Theorem 5. *If a function f belongs to the class $\mathcal{H}(t)$, $t \in (1/2, 1]$, then it satisfies the condition*

$$t\left|\frac{zf''(z)}{f'(z)} + 1\right| > \left|\mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} + \frac{1}{2}\right| \quad (z \in \mathbb{U}). \tag{13}$$

Let us recall here the following classes of k-uniformly convex and of k-starlike functions:

$$k\text{-}\mathcal{UCV} := \left\{f \in \mathcal{S} : k\left|\frac{zf''(z)}{f'(z)}\right| < \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \quad (z \in \mathbb{U};\ 0 \le k < \infty)\right\},$$

$$k\text{-}\mathcal{ST} := \left\{f \in \mathcal{S} : k\left|\frac{zf'(z)}{f(z)} - 1\right| < \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \quad (z \in \mathbb{U};\ 0 \le k < \infty)\right\}.$$

The class k-$\mathcal{UCV}$ was introduced by Kanas and Wiśniowska [9], where its geometric definition and connections with the conic domains were considered. The class k-$\mathcal{UCV}$ was defined geometrically as a subclass of univalent functions that

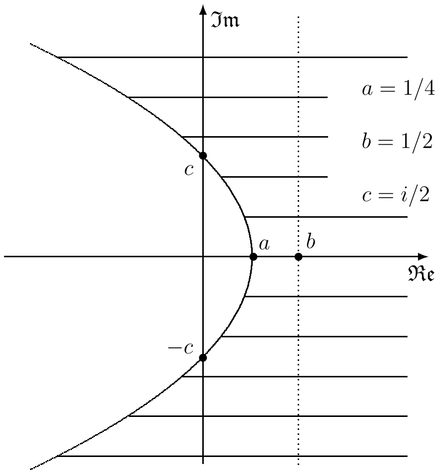

Fig. 1. $\Omega(1)$.

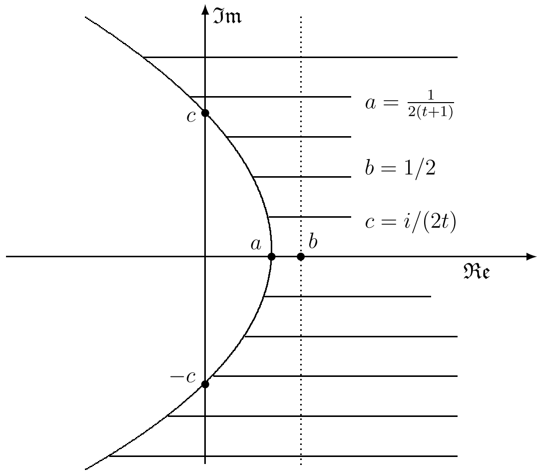

Fig. 2. $\Omega(t)$, $1/2 < t < 1$.

map each circular arc contained in the unit disk $\mathbb{U}$ with center $\xi, |\xi| \le k \quad (0 \le k < \infty)$ onto a convex arc. The notion of k-uniformly convex function is a natural extension of the classical convexity. Observe that, if $k = 0$, then the center ξ is the origin and the class k-$\mathcal{UCV}$ reduces to the class of convex univalent functions $\mathcal{CV}$. Moreover, this class for $k = 1$, corresponds to the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [5], which was studied extensively by Rønning [13], and independently also by Ma and Minda [12]. The class k-$\mathcal{ST}$ is related to the class k-$\mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex $\mathcal{CV}$ and starlike $\mathcal{ST}$ functions (see also the works [1,8,10,7,12,13] concerning further developments involving each one of the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$). The class k-$\mathcal{ST}$ has the geometric characterization (see [11]) that if $f \in k$-$\mathcal{ST}$, then it maps a lens-like domain $U(\zeta, r) \cap U(0, R)$ onto a starlike domain, where $U(\zeta, r)$ is a disk of radius r with center ζ, and $0 < R \le 1$, $|\zeta| \le k$, $r \ge \sqrt{|\zeta|^2 + R^2}$.

Theorem 6. *If $f \in \mathcal{H}(t)$ has the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \ (z \in \mathbb{U})$,*

then $|a_2| \le t, \quad |a_3| \le \dfrac{4}{3}t^2 - \dfrac{1}{6}.$ (14)

Moreover, $\left|a_3 - \mu a_2^2\right| \le \begin{cases} \left|8t^2 - 1 - 6\mu t^2\right|/6, & \mu \notin [\mu_1, \mu_2] \\ t/3, & \mu \in [\mu_1, \mu_2], \end{cases}$ (15)

where $\mu_1 = \dfrac{8t^2 - 2t - 1}{6t^2}, \qquad \mu_2 = \dfrac{8t^2 + 2t - 1}{6t^2}.$

All of the inequalities are sharp.

Proof. From (3), we have

$$f'(z) + zf''(z) = f'(z)\left(1 + U_1(t)\omega(z) + U_2(t)\omega^2(z) + U_3(t)\omega^3(z) + \cdots\right), \tag{16}$$

for some holomorphic function ω such that $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in \mathbb{U}$. The equality (16) with $\omega(z) = z$ defines the function $\widetilde{f}$ such that $\widetilde{f}'(z) + z\widetilde{f}''(z) = \widetilde{f}'(z)(1 + U_1(t)z + U_2(t)z^2 + U_3(t)z^3 + \cdots)$, and by using (7), we obtain that

$$\widetilde{f}(z) = z + \frac{1}{2}U_1(t)z^2 + \frac{1}{6}\left(U_2(t) + U_1^2(t)\right)z^3 + \cdots = z + tz^2 + \left(4t^2/3 - 1/6\right)z^3 + \cdots \tag{17}$$

is in the class $\mathcal{H}(t)$. It is fairly well-known that if $|\omega(z)| = |c_1 z + c_2 z^2 + c_3 z^3 + \cdots| < 1, \quad z \in \mathbb{U}$,

then $\left|c_j\right| \le 1, \quad$ for all $j \in \mathbb{N}$. (18)

and $\left|c_2 - \mu c_1^2\right| \le \max\{1, |\mu|\}, \quad$ for all $\mu \in \mathbb{C}$. (19)

It follows from (16) that

$$a_2 = \frac{1}{2}U_1(t)c_1, \ \ a_3 = \frac{1}{6}\left(U_1(t)c_2 + \left\{U_2(t) + U_1^2(t)\right\}c_1^2\right). \tag{20}$$

Applying (7), (18) and (20), we obtain the first inequality in (16). Furthermore, from (19) and (20), we have

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \frac{1}{6}U_1(t)\left|c_2+\left(\frac{U_2(t)}{U_1(t)}+U_1(t)-\frac{3\mu}{2}U_1(t)\right)c_1^2\right| \\ &\le \frac{U_1(t)}{6}\max\left\{1,\frac{1}{U_1(t)}\left|U_2(t)+U_1^2(t)-\frac{3\mu}{2}U_1(t)\right|\right\} \\ &= \frac{t}{3}\max\left\{1,\left|\frac{8t^2-1-6\mu t^2}{2t}\right|\right\}.\end{aligned}$$

Because $t>0$, we have

$$\left|\frac{8t^2-1-6\mu t^2}{2t}\right|\le 1 \Leftrightarrow \left\{\frac{8t^2-2t-1}{6t^2}\le\mu\le\frac{8t^2+2t-1}{6t^2}\right\} \Leftrightarrow \{\mu_1\le\mu\le\mu_2\}$$

and so we obtain (15). If we take $\mu=0$ in (15), then we obtain the second inequality (14). For function $\widetilde{f}$ given by (17), we have $|a_2|=t$, $|a_3|=4t^2/3-1/6$, which shows that the inequalities (14) are sharp. Moreover, in this case

$$\left|a_3-\mu\, a_2^2\right| = \left|\frac{8t^2-1-6\mu t^2}{6}\right|,$$

which shows the sharpness of (15) for $\mu\notin[\mu_1,\mu_2]$. Furthermore, (16) with $\omega(z)=z^2$ generate the function $\widehat{f}\in\mathcal{H}(t)$ such that $\widehat{f}(z)=z+\frac{t}{3}z^3+\cdots$. It shows the sharpness of (15) for $\mu\in[\mu_1,\mu_2]$. □

Theorem 6 is the solution of the Fekete–Szegö problem in $\mathcal{H}(t)$. For a general solution of the Fekete–Szegö problem in some classes, see [2].

Theorem 7. *A function f belongs to the class $\mathcal{H}(t)$ if and only if there exists a function $q\in\mathcal{H}$, $q(z)\prec H(z,t)$ such that*

$$f(z)=\int_0^z\left\{\exp\int_0^w\frac{q(u)-1}{u}\,\mathrm{d}u\right\}\mathrm{d}w. \tag{21}$$

Proof. If a function f satisfies (3), then there exists a function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$, for all $z\in\mathbb{U}$ such that

$$1+\frac{zf''(z)}{f'(z)}=H(\omega(z),t):=q(z). \tag{22}$$

Now $q(z)\prec H(z,t)$ and the equality (22) readily gives

$$\left\{\log f'(z)\right\}'=\frac{q(z)-1}{z}, \tag{23}$$

which upon integration yields (21). Conversely, some easy calculations show that if f satisfies (21), then f belongs to the class $\mathcal{H}(t)$, and this completes the proof. □

Theorem 7 provides us a method of finding the members of the class $\mathcal{H}(t)$. Let us find a function in $\mathcal{H}(t)$ that corresponds to the function $H(z,t)$ in the formula (21). Applying the formula (4) on $H(z,t)$, with $t=\cos\alpha$, we obtain from (21):

$$\begin{aligned}\widetilde{f}(z)\{&\int_0^z\left\{\exp\int_0^w\frac{H(u,t)-1}{u}\,\mathrm{d}u\right\}\mathrm{d}w \\ &=\int_0^z\left\{\exp\int_0^w\frac{2\cos\alpha-u}{1-2\cos\alpha u+u^2}\,\mathrm{d}u\right\}\mathrm{d}w \\ &=\int_0^z\left\{\exp\int_0^w\frac{1}{2\mathrm{i}\sin\alpha}\left(\frac{\mathrm{e}^{2\mathrm{i}\alpha}}{1-\mathrm{e}^{\mathrm{i}\alpha}u}-\frac{\mathrm{e}^{-2\mathrm{i}\alpha}}{1-\mathrm{e}^{-\mathrm{i}\alpha}u}\right)\mathrm{d}u\right\}\mathrm{d}w \\ &=\int_0^z\exp\left\{\frac{\mathrm{e}^{-\mathrm{i}\alpha}}{2\mathrm{i}\sin\alpha}\log\left(1-e^{-\mathrm{i}\alpha}w\right)-\frac{e^{\mathrm{i}\alpha}}{2\mathrm{i}\sin\alpha}\log\left(1-\mathrm{e}^{\mathrm{i}\alpha}w\right)\right\}\mathrm{d}w\end{aligned}$$

$$= \int_0^z \frac{\left(1-e^{-i\alpha}w\right)^{e^{-i\alpha}/(2i\sin\alpha)}}{\left(1-e^{i\alpha}w\right)^{e^{i\alpha}/(2i\sin\alpha)}}\,dw. \tag{24}$$

On the other hand, from (21) and (5) we also have the representation given by

$$\log\left\{\widetilde{f}'(z)\right\} = \sum_{n=1}^{\infty} \frac{\sin((n+1)\alpha)}{n\sin\alpha} z^n.$$

The few first coefficients of $\widetilde{f}'$ are given in (17).

Theorem 8. *Let $n \geq 2$ be a positive integer. If the function $g(z) = z + cz^n$ $(z \in \mathbb{U})$ is in the class $\mathcal{H}(t)$, then*

$$|c| \leq \frac{2t+1}{2(t+1)n^2 - n}. \tag{25}$$

Proof. Let a function G be defined by

$$G(z) := 1 + \frac{zg''(z)}{g'(z)} = \frac{1+n^2 c\, z^{n-1}}{1+n\, c\, z^{n-1}} \quad (z \in \mathbb{U}).$$

The function G is analytic in $\mathbb{U}$ for $n\,|c| \leq 1$, but if $n\,|c| = 1$, then $G(z)$ becomes 0 in the unit disc, so we may assume that $n\,|c| < 1$, and thus G maps the unit disc onto the disc symmetric with respect to the real axis lying between the points:

$$x_1 = \frac{1-n^2|c|}{1-n|c|}, \quad x_2 = \frac{1+n^2|c|}{1+n|c|}.$$

If the function g is in the class $\mathcal{H}(t)$, then $G(\mathbb{U})$ lies in the domain indicated in Fig. 1, or in Fig. 2. Therefore, in view of Theorem 2, if g is in the class $\mathcal{H}(t)$, then

$$\frac{1}{2(t+1)} \leq \frac{1-n^2|c|}{1-n|c|}.$$

Since $n\,|c| < 1$, the inequality gives the desired assertion (25). □

It easy to observe from Corollary 3 and Figs. 1 and 2 that if f is convex of order at least $1/(2t+2)$, then f is in the class $\mathcal{H}(t)$. However, the convex function $h(z) = z + z^n/n^2$ is not in the class $\mathcal{H}(t)$ because it does not satisfy (25). Therefore, $\mathcal{CV} \not\subset \mathcal{H}(t)$ for all $t \in (1/2, 1]$. Also, $\mathcal{ST} \not\subset \mathcal{H}(t)$ for all $t \in (1/2, 1]$, because the Koebe function $z/(1-z)^2$ does not satisfy the condition (3).

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