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Differential geometry

The Atiyah class of a dg-vector bundle ☆

Classe d'Atiyah d'un fibré vectoriel différentiel gradué

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A B S T R A C T

We introduce the notions of Atiyah class and Todd class of a differential graded vector bundle with respect to a differential graded Lie algebroid. We prove that the space of vector fields $\mathfrak{X}(\mathcal{M})$ on a dg-manifold $\mathcal{M}$ with homological vector field Q admits a structure of $L_\infty[1]$-algebra with the Lie derivative L_Q as unary bracket λ_1, and the Atiyah cocycle $\mathrm{At}_\mathcal{M}$ corresponding to a torsion-free affine connection as binary bracket λ_2.

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R É S U M É

Nous introduisons les notions de classe d'Atiyah et de classe de Todd d'un fibré différentiel gradué relatives à un algébroïde de Lie différentiel gradué. Nous prouvons que l'espace des champs de vecteurs sur une variété différentielle graduée admet une structure d'algèbre $L_\infty[1]$ ayant la dérivée de Lie par rapport au champ de vecteur cohomologique pour crochet unaire et le cocycle d'Atiyah associé à une connexion affine sans torsion pour crochet binaire.

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Les variétés différentielles graduées jouent un rôle de plus en plus important en géométrie dérivée et en physique mathématique. Elles apparaissent aussi de façon naturelle en géométrie de Poisson. Notamment, Vaĭntrob a montré qu'un fibré vectoriel $A \to M$ admet une structure d'algébroïde de Lie si et seulement si la variété graduée $A[1]$ admet un champ de vecteurs cohomologique. Plus précisément, il existe une bijection entre les structures d'algébroïdes de Lie sur $A \to M$, d'une part, et les champs de vecteurs cohomologiques sur $A[1]$, d'autre part. Une variété $\mathbb{Z}$-graduée $\mathcal{M}$ est un faisceau d'algèbres commutatives $\mathbb{Z}$-graduées $\mathcal{R}$ sur une variété de base M dont tout point admet un voisinage ouvert U tel que $\mathcal{R}(U)$ est isomorphe en tant qu'algèbre à $C^\infty(U) \otimes \hat{S}(V^\vee)$, où V désigne un espace vectoriel $\mathbb{Z}$-gradué et $\hat{S}(V^\vee)$ l'algèbre graduée des séries de puissances formelles sur V. Par variété différentielle graduée, nous entendons une variété $\mathbb{Z}$-graduée munie d'un champ de vecteurs cohomologique, c'est-à-dire un champ de vecteurs Q de degré $+1$ satisfaisant $[Q, Q] = 0$.

Shoikhet [11] fut le premier à étudier la classe d'Atiyah d'une variété différentielle graduée – l'obstruction à l'existence de connexions affines dg-compatibles – en relation avec le théorème de formalité de Kontsevich et la formule de Duflo. La classe d'Atiyah d'un fibré vectoriel différentiel gradué est aussi apparue dans les travaux de Costello [1]. Dans la présente Note, nous introduisons les notions de classe d'Atiyah et de classe de Todd d'un fibré différentiel gradué relatives à un algébroïde de Lie différentiel gradué (c'est-à-dire un objet algébroïde de Lie dans la catégorie des variétés différentielles graduées), qui, nous le pensons, englobent de nombreux cas particuliers de ces notions apparus dans la littérature. En voici un exemple.

Proposition 0.1. *Soit* $\mathfrak{g}$ *une algèbre de Lie de dimension finie et soit* $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{CE})$ *la variété différentielle graduée qui lui est canoniquement associée. Alors la classe d'Atiyah* $\alpha_{\mathfrak{g}[1]}$ *de* $(\mathfrak{g}[1], d_{CE})$ *est précisément le crochet de Lie de* $\mathfrak{g}$ *vu comme élément de* $(\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}} \cong H^1\big(\Gamma(T^\vee(\mathfrak{g}[1]) \otimes \mathrm{End}\, T(\mathfrak{g}[1])), Q\big)$. *De plus, l'isomorphisme* $\bigoplus_k H^k(\Omega^k(\mathfrak{g}[1]), Q) \cong \hat{S}(\mathfrak{g}^\vee)$ *identifie la classe de Todd de* $(\mathfrak{g}[1], d_{CE})$ *à l'élément de Duflo de* $\mathfrak{g}$.

Kapranov a montré que, dans le cadre classique, le 1-cocycle d'Atiyah est à l'origine d'une structure d'algèbre de Lie homotopique intéressante [3,5]. Nous étendons ici ce résultat au contexte différentiel gradué. Le résultat principal de la présente Note est le suivant.

Theorem 0.2. *À chaque connexion affine sans torsion sur une variété différentielle graduée* $(\mathcal{M}, Q)$ *correspond une suite* $\{\lambda_k\}_{k \geq 2}$ *d'applications* $\lambda_k \in \mathrm{Hom}(S^k T\mathcal{M}, T\mathcal{M}[-1])$ *débutant par* $\lambda_2 := \mathrm{At}_{\mathcal{M}} \in \mathrm{Hom}(T\mathcal{M} \otimes T\mathcal{M}, T\mathcal{M}[-1])$ *telle que la suite des multicrochets induits* $\lambda_k : \mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ *complétée par la dérivée de Lie* L_Q *considérée comme crochet unaire* $\lambda_1 : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ *définit une structure d'algèbre* $L_\infty[1]$ *sur* $\mathfrak{X}(\mathcal{M})$ *vu comme espace vectoriel sur* $\Bbbk$.

Le théorème ci-dessus est un corollaire du théorème suivant, lequel découle de l'existence d'isomorphismes de Poincaré–Birkhoff–Witt pour les variétés graduées.

Theorem 0.3. *À chaque connexion affine sans torsion* ∇ *sur une variété différentielle graduée* $(\mathcal{M}, Q)$ *correspond une dérivation* D *de* $\Gamma(\hat{S}(T^\vee \mathcal{M}))$ *de degré* $+1$ *telle que* $D^2 = 0$ *et* $D = L_Q + \sum_{k=2}^{\infty} \widetilde{R}_k$, *où chaque* $\widetilde{R}_k$ *est une transformation* $C^\infty(\mathcal{M})$-*linéaire de* $\Gamma(\hat{S}(T^\vee \mathcal{M}))$ *correspondant à un élément* $R_k \in \mathrm{Hom}(S^k T\mathcal{M}, T\mathcal{M}[-1])$ *et* R_2 *désigne le 1-cocycle d'Atiyah* $\mathrm{At}_{\mathcal{M}}$ *associé à la connexion* ∇.

1. dg-Manifolds and dg-vector bundles

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold M is a sheaf of $\mathbb{Z}$-graded, graded-commutative algebras $\{\mathcal{R}_U | U \subset M \text{ open}\}$ over M, locally isomorphic to $C^\infty(U) \otimes \hat{S}(V^\vee)$, where $U \subset M$ is an open submanifold, V is a $\mathbb{Z}$-graded vector space, and $\hat{S}(V^\vee)$ denotes the graded algebra of formal power series on V. By $C^\infty(\mathcal{M})$, we denote the $\mathbb{Z}$-graded, graded-commutative algebra of global sections of this sheaf. By a dg-manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a derivation Q of degree $+1$ of $C^\infty(\mathcal{M})$ satisfying $[Q, Q] = 0$.

Example 1.1. Let $A \to M$ be a Lie algebroid over $\mathbb{C}$. Then $A[1]$ is a dg-manifold with the Chevalley–Eilenberg differential d_{CE} as homological vector field. In fact, according to Vaĭntrob [12], there is a bijection between the Lie algebroid structures on the vector bundle $A \to M$ and the homological vector fields on the $\mathbb{Z}$-graded manifold $A[1]$.

Example 1.2. Let s be a smooth section of a vector bundle $E \to M$. Then $E[-1]$ is a dg-manifold with the contraction operator i_s as homological vector field.

Example 1.3. Let $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-graded vector space of finite type, i.e. each $\mathfrak{g}_i$ is a finite-dimensional vector space. Then $\mathfrak{g}[1]$ is a dg-manifold if and only if $\mathfrak{g}$ is an L_∞-algebra.

A dg-vector bundle is a vector bundle in the category of dg-manifolds. We refer the reader to [10,4] for details on dg-vector bundles. The following example is essentially due to Kotov–Strobl [4].

Example 1.4. Let $A \to M$ be a gauge Lie algebroid with anchor ρ. Then $A[1] \to T[1]M$ is a dg-vector bundle, where the homological vector fields on $A[1]$ and $T[1]M$ are the Chevalley–Eilenberg differentials.

The example above is a special case of a general fact [10], i.e. that LA-vector bundles [6–8] (also known as VB-algebroids [2]) give rise to dg-vector bundles.

Given a vector bundle $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ of graded manifolds, its space of sections, denoted $\Gamma(\mathcal{E})$, is defined to be $\bigoplus_{j \in \mathbb{Z}} \Gamma_j(\mathcal{E})$, where $\Gamma_j(\mathcal{E})$ consists of degree preserving maps $s \in \mathrm{Hom}(\mathcal{M}, \mathcal{E}[-j])$ such that $(\pi[-j]) \circ s = \mathrm{id}_{\mathcal{M}}$, where $\pi[-j] : \mathcal{E}[-j] \to \mathcal{M}$ is the natural map induced from π; see [10] for more details. When $\mathcal{E} \to \mathcal{M}$ is a dg-vector bundle, the homological

vector fields on $\mathcal{E}$ and $\mathcal{M}$ naturally induce a degree-1 operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$, making $\Gamma(\mathcal{E})$ a dg-module over $C^\infty(\mathcal{M})$. Since the space $\Gamma(\mathcal{E}^\vee)$ of linear functions on $\mathcal{E}$ generates $C^\infty(\mathcal{E})$, the converse is also true.

Lemma 1.5. *Let $\mathcal{E} \to \mathcal{M}$ be a vector bundle object in the category of graded manifolds and suppose $\mathcal{M}$ is a dg-manifold. If $\Gamma(\mathcal{E})$ is a dg-module over $C^\infty(\mathcal{M})$, then $\mathcal{E}$ admits a natural dg-manifold structure such that $\mathcal{E} \to \mathcal{M}$ is a dg-vector bundle. In fact, the categories of dg-vector bundles and of locally free dg-modules are equivalent.*

In this case, the degree $+1$ operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$ gives rise to a cochain complex

$$\cdots \to \Gamma_{i-1}(\mathcal{E}) \xrightarrow{\mathcal{Q}} \Gamma_i(\mathcal{E}) \xrightarrow{\mathcal{Q}} \Gamma_{i+1}(\mathcal{E}) \to \cdots,$$

whose cohomology group will be denoted by $H^\bullet(\Gamma(\mathcal{E}), \mathcal{Q})$.

In particular, the space $\mathfrak{X}(\mathcal{M})$ of vector fields on a dg-manifold $(\mathcal{M}, Q)$ (i.e. graded derivations of $C^\infty(\mathcal{M})$), which can be regarded as the space of sections $\Gamma(T\mathcal{M})$, is naturally a dg-module over $C^\infty(\mathcal{M})$ with the Lie derivative $L_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ playing the role of the degree $+1$ operator $\mathcal{Q}$.

Thus we have the following.

Corollary 1.6. *For every dg-manifold $(\mathcal{M}, Q)$, the Lie derivative L_Q makes $\Gamma(T\mathcal{M})$ into a dg-module over $C^\infty(\mathcal{M})$ and therefore $T\mathcal{M} \to \mathcal{M}$ is naturally a dg-vector bundle.*

Following the classical case, the corresponding homological vector field on $T\mathcal{M}$ is called the *tangent lift* of Q.

Differential graded Lie algebroids are another useful notion. Roughly, a dg-Lie algebroid can be thought of as a Lie algebroid object in the category of dg-manifolds. For more details, we refer the reader to [10], where dg-Lie algebroids are called Q-algebroids.

Differential graded foliations constitute an important class of examples of dg-Lie algebroids.

Lemma 1.7. *Let $\mathcal{D} \subset T\mathcal{M}$ be an integrable distribution on a graded manifold $\mathcal{M}$. Suppose that there exists a homological vector field Q on $\mathcal{M}$ such that $\Gamma(\mathcal{D})$ is stable under L_Q. Then $\mathcal{D} \to \mathcal{M}$ is a dg-Lie algebroid with the inclusion $\rho : \mathcal{D} \to T\mathcal{M}$ as its anchor map.*

2. Atiyah class and Todd class of a dg-vector bundle

Let $\mathcal{E} \to \mathcal{M}$ be a dg-vector bundle and let $\mathcal{A} \to \mathcal{M}$ be a dg-Lie algebroid with anchor $\rho : \mathcal{A} \to T\mathcal{M}$. An $\mathcal{A}$-connection on $\mathcal{E} \to \mathcal{M}$ is a degree 0 map $\nabla : \Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that

$$\nabla_{fX}s = f\nabla_X s$$

and

$$\nabla_X(fs) = \rho(X)(f)s + (-1)^{|X||f|}f\nabla_X s$$

for all $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(\mathcal{A})$, and $s \in \Gamma(\mathcal{E})$. Here we use the 'absolute value' notation to denote the degree of the argument. When we say that ∇ is of degree 0, we actually mean that $|\nabla_X s| = |X| + |s|$ for every pair of homogeneous elements X and s. Such connections always exist since the standard partition of unity argument holds in the context of graded manifolds. Given a dg-vector bundle $\mathcal{E} \to \mathcal{M}$ and an $\mathcal{A}$-connection ∇ on it, we can consider the bundle map $\mathrm{At}_\mathcal{E} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ defined by

$$\mathrm{At}_\mathcal{E}(X, s) := \mathcal{Q}(\nabla_X s) - \nabla_{\mathcal{Q}(X)}s - (-1)^{|X|}\nabla_X(\mathcal{Q}(s)), \quad \forall X \in \Gamma(\mathcal{A}), s \in \Gamma(\mathcal{E}). \tag{1}$$

Proposition 2.1.

(i) $\mathrm{At}_\mathcal{E} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ *is a degree $+1$ bundle map and therefore can also be regarded as a degree $+1$ section of $\mathcal{A}^\vee \otimes \mathrm{End}\,\mathcal{E}$.*
(ii) $\mathrm{At}_\mathcal{E}$ *is a cocycle: $\mathcal{Q}(\mathrm{At}_\mathcal{E}) = 0$.*
(iii) *The cohomology class of $\mathrm{At}_\mathcal{E}$ is independent of the choice of the connection ∇.*

Thus there is a natural cohomology class $\alpha_\mathcal{E} := [\mathrm{At}_\mathcal{E}]$ in $H^1\big(\Gamma(\mathcal{A}^\vee \otimes \mathrm{End}\,\mathcal{E}), Q\big)$. The class $\alpha_\mathcal{E}$ is called the *Atiyah class* of the dg-vector bundle $\mathcal{E} \to \mathcal{M}$ relative to the dg-Lie algebroid $\mathcal{A} \to \mathcal{M}$.

The Atiyah class of a dg-manifold, which is the obstruction to the existence of connections compatible with the differential, was first investigated by Shoikhet [11] in relation with Kontsevich's formality theorem and Duflo formula. More recently, the Atiyah class of a dg-vector bundle appeared in Costello's work [1].

We define the *Todd class* $\mathrm{Td}_{\mathcal{E}}$ of a dg-vector bundle $\mathcal{E} \to \mathcal{M}$ relative to a dg-Lie algebroid $\mathcal{A} \to \mathcal{M}$ by

$$\mathrm{Td}_{\mathcal{E}} := \mathrm{Ber}\left(\frac{1 - e^{-\alpha_{\mathcal{E}}}}{\alpha_{\mathcal{E}}} \right) \in \prod_{k \geq 0} H^k\big(\Gamma(\wedge^k \mathcal{A}^\vee), Q\big), \tag{2}$$

where Ber denotes the Berezinian [9] and $\wedge^k \mathcal{A}^\vee$ denotes the dg vector bundle $S^k(\mathcal{A}^\vee[-1])[k] \to \mathcal{M}$. One checks that $\mathrm{Td}_{\mathcal{E}}$ can be expressed in terms of scalar Atiyah classes $c_k = \frac{1}{k!}(\frac{i}{2\pi})^k \mathrm{str}\, \alpha_{\mathcal{E}}^k \in H^k\big(\Gamma(\wedge^k \mathcal{A}^\vee), Q\big)$. Here $\mathrm{str} : \mathrm{End}\,\mathcal{E} \to C^\infty(\mathcal{M})$ denotes the supertrace. Note that $\mathrm{str}\, \alpha_{\mathcal{E}}^k \in \Gamma(\wedge^k \mathcal{A}^\vee)$ since $\alpha_{\mathcal{E}}^k \in \Gamma(\wedge^k \mathcal{A}^\vee) \otimes_{C^\infty(\mathcal{M})} \mathrm{End}\,\mathcal{E}$. If $\mathcal{A} = T\mathcal{M}$, we write $\Omega^k(\mathcal{M})$ instead of $\Gamma(\wedge^k T^\vee \mathcal{M})$.

3. Atiyah class and Todd class of a dg-manifold

Consider a dg-manifold $(\mathcal{M}, Q)$. According to Lemma 1.7, its tangent bundle $T\mathcal{M}$ is indeed a dg-Lie algebroid. By the *Atiyah class of a dg-manifold* $(\mathcal{M}, Q)$, denoted $\alpha_{\mathcal{M}}$, we mean the Atiyah class of the tangent dg-vector bundle $T\mathcal{M} \to \mathcal{M}$ with respect to the dg-Lie algebroid $T\mathcal{M}$. Similarly, the Atiyah 1-cocycle of a dg manifold $\mathcal{M}$ corresponding to an affine connection on $\mathcal{M}$ (i.e. a $T\mathcal{M}$-connection on $T\mathcal{M} \to \mathcal{M}$) is the 1-cocycle defined as in Eq. (1).

Lemma 3.1. *Suppose V is a vector space. The only connection on the graded manifold $V[1]$ is the trivial connection.*

Proof. Since the graded algebra of functions on $V[1]$ is $\wedge(V^\vee)$, every vector $v \in V$ determines a degree -1 vector field ι_v on $V[1]$, which acts as a contraction operator on $\wedge(V^\vee)$. The $C^\infty(V[1])$-module of all vector fields on $V[1]$ is generated by its subset $\{\iota_v\}_{v \in V}$. It follows that a connection ∇ on $V[1]$ is completely determined by the knowledge of $\nabla_{\iota_v} \iota_w$ for all $v, w \in V$. Since the degree of every vector field $\nabla_{\iota_v} \iota_w$ must be -2 and there are no nonzero vector fields of degree -2, it follows that $\nabla_{\iota_v} \iota_w = 0$. $\square$

Given a finite-dimensional Lie algebra $\mathfrak{g}$, consider the dg-manifold $(\mathcal{M}, Q)$, where $\mathcal{M} = \mathfrak{g}[1]$ and Q is the Chevalley–Eilenberg differential d_{CE}. The following result can be easily verified using the canonical trivialization $T\mathcal{M} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$.

Lemma 3.2. *Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra $\mathfrak{g}$. Then,*

$$H^k(\Gamma(T^\vee \mathcal{M} \otimes \mathrm{End}\, T\mathcal{M}), Q) \cong H^{k-1}_{\mathrm{CE}}(\mathfrak{g}, \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g}),$$

and

$$H^k(\Omega^k(\mathcal{M}), Q) \cong (S^k \mathfrak{g}^\vee)^{\mathfrak{g}}.$$

Proposition 3.3. *Let $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ be the canonical dg-manifold corresponding to a finite-dimensional Lie algebra $\mathfrak{g}$. Then the Atiyah class $\alpha_{\mathfrak{g}[1]}$ is precisely the Lie bracket of $\mathfrak{g}$ regarded as an element of $(\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}} \cong H^1\big(\Gamma(T^\vee \mathcal{M} \otimes \mathrm{End}\, T\mathcal{M}), Q\big)$. Consequently, the isomorphism*

$$\prod_k H^k\big(\Omega^k(\mathcal{M}), Q\big) \xrightarrow{\cong} \big(\widehat{S}(\mathfrak{g}^\vee)\big)^{\mathfrak{g}}$$

maps the Todd class $\mathrm{Td}_{\mathfrak{g}[1]}$ onto the Duflo element of $\mathfrak{g}$.

Example 3.4. Consider a dg-manifold of the form $\mathcal{M} = (\mathbb{R}^{m|n}, Q)$. Let $(x_1, \cdots, x_m; x_{m+1} \cdots x_{m+n})$ be coordinate functions on $\mathbb{R}^{m|n}$, and write $Q = \sum_k Q_k(x) \frac{\partial}{\partial x_k}$. Then the Atiyah 1-cocycle associated with the trivial connection $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$ is given by

$$\mathrm{At}_{\mathcal{M}}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = (-1)^{|x_i| + |x_j|} \sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j} \frac{\partial}{\partial x_k} \tag{3}$$

As we can see from (3), the Atiyah 1-cocycle $\mathrm{At}_{\mathcal{M}}$ includes the information about the homological vector field of second order and higher.

4. Atiyah class and homotopy Lie algebras

Let $\mathcal{M}$ be a graded manifold. A $(1,2)$-tensor of degree k on $\mathcal{M}$ is a $\mathbb{C}$-linear map $\alpha : \mathfrak{X}(\mathcal{M}) \otimes_{\mathbb{C}} \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ such that $|\alpha(X, Y)| = |X| + |Y| + k$ and

$$\alpha(fX, Y) = (-1)^{k|f|} f\alpha(X, Y) = (-1)^{|f||X|} \alpha(X, fY).$$

We denote the space of $(1, 2)$-tensors of degree k by $\mathcal{T}_k^{1,2}(\mathcal{M})$, and the space of all $(1, 2)$-tensors by $\mathcal{T}^{1,2}(\mathcal{M}) = \bigoplus_k \mathcal{T}_k^{1,2}(\mathcal{M})$.

The torsion of an affine connection ∇ is given by

$$T(X, Y) = \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y]. \tag{4}$$

The torsion is an element in $\mathcal{T}_0^{1,2}(\mathcal{M})$. Given any affine connection, one can define its opposite affine connection ∇^{op}, given by

$$\nabla_X^{\mathrm{op}} Y = \nabla_X Y - T(X, Y) = [X, Y] + (-1)^{|X||Y|} \nabla_Y X. \tag{5}$$

The average $\frac{1}{2}(\nabla + \nabla^{\mathrm{op}})$ is a torsion-free affine connection. This shows that torsion-free affine connections always exist on graded manifolds.

In this section, we show that, as in the classical situation considered by Kapranov in [3,5], the Atiyah 1-cocycle of a dg-manifold gives rise to an interesting homotopy Lie algebra. As in the last section, let $(\mathcal{M}, Q)$ be a dg-manifold and let ∇ be an affine connection on $\mathcal{M}$. The following can be easily verified by direct computation.

(i) The anti-symmetrization of the Atiyah 1-cocycle $\mathrm{At}_{\mathcal{M}}$ is equal to $L_Q T$, so $\mathrm{At}_{\mathcal{M}}$ is graded symmetric up to an exact term. In particular, if ∇ is torsion-free, we have

$$\mathrm{At}_{\mathcal{M}}(X, Y) = (-1)^{|X||Y|} \mathrm{At}_{\mathcal{M}}(Y, X).$$

(ii) The degree $1 + |X|$ operator $\mathrm{At}_{\mathcal{M}}(X, -)$ need not be a derivation of the degree $+1$ 'product' $\mathfrak{X}(\mathcal{M}) \otimes_{\mathbb{C}} \mathfrak{X}(\mathcal{M}) \xrightarrow{\mathrm{At}_{\mathcal{M}}} \mathfrak{X}(\mathcal{M})$. However, the Jacobiator

$$\begin{aligned}(X, Y, Z) \mapsto \mathrm{At}_{\mathcal{M}}\left(X, \mathrm{At}_{\mathcal{M}}(Y, Z)\right) - \big\{(-1)^{|X|+1} \mathrm{At}_{\mathcal{M}}\left(\mathrm{At}_{\mathcal{M}}(X, Y), Z\right) \\ + (-1)^{(|X|+1)(|Y|+1)} \mathrm{At}_{\mathcal{M}}\left(Y, \mathrm{At}_{\mathcal{M}}(X, Z)\right)\big\},\end{aligned}$$

of $\mathrm{At}_{\mathcal{M}}$, which vanishes precisely when $\mathrm{At}_{\mathcal{M}}(X, -)$, is a derivation of $\mathrm{At}_{\mathcal{M}}$, is equal to $\pm L_Q(\nabla \mathrm{At}_{\mathcal{M}})$. Hence $\mathrm{At}_{\mathcal{M}}$ satisfies the graded Jacobi identity up to the exact term $L_Q(\nabla \mathrm{At}_{\mathcal{M}})$.

Armed with this motivation, we can now state the main result of this note.

Theorem 4.1. *Let $(\mathcal{M}, Q)$ be a dg-manifold and let ∇ be a torsion-free affine connection on $\mathcal{M}$. There exists a sequence $(\lambda_k)_{k \geq 2}$ of maps $\lambda_k \in \mathrm{Hom}(S^k(T\mathcal{M}), T\mathcal{M}[-1])$ starting with $\lambda_2 := \mathrm{At}_{\mathcal{M}} \in \mathrm{Hom}(S^2(T\mathcal{M}), T\mathcal{M}[-1])$ which, together with $\lambda_1 := L_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$, satisfy the $L_\infty[1]$-algebra axioms. As a consequence, the space of vector fields $\mathfrak{X}(\mathcal{M})$ on a dg-manifold $(\mathcal{M}, Q)$ admits an $L_\infty[1]$-algebra structure with the Lie derivative L_Q as unary bracket λ_1 and the Atiyah cocycle $\mathrm{At}_{\mathcal{M}}$ as binary bracket λ_2.*

To prove Theorem 4.1, we introduce a Poincaré–Birkhoff–Witt map for graded manifolds.

It was shown in [5] that every torsion-free affine connection ∇ on a smooth manifold M determines an isomorphism of coalgebras (over $C^\infty(M)$)

$$\mathrm{pbw}^\nabla : \Gamma(S(TM)) \xrightarrow{\cong} D(M), \tag{6}$$

called the Poincaré–Birkhoff–Witt (PBW) map. Here $D(M)$ denotes the space of differential operators on M.

Geometrically, an affine connection ∇ induces an exponential map $TM \to M \times M$, which is a well-defined diffeomorphism from a neighborhood of the zero section of TM to a neighborhood of the diagonal $\Delta(M)$ of $M \times M$. Sections of $S(TM)$ can be viewed as fiberwise distributions on TM supported on the zero section, while $D(M)$ can be viewed as the space of source-fiberwise distributions on $M \times M$ supported on the diagonal $\Delta(M)$. The map $\mathrm{pbw}^\nabla : \Gamma(S(TM)) \to D(M)$ is simply the push-forward of fiberwise distributions through the exponential map $\exp^\nabla : TM \to M \times M$ and is clearly an isomorphism of coalgebras over $C^\infty(M)$.

Even though, for a *graded* manifold $\mathcal{M}$ endowed with a torsion-free affine connection ∇, we lack an exponential map $\exp^\nabla : \mathcal{T}\mathcal{M} \to \mathcal{M} \times \mathcal{M}$, a PBW map can still be defined purely algebraically thanks to the iteration formula introduced in [5].

Lemma 4.2. *Let $\mathcal{M}$ be a $\mathbb{Z}$-graded manifold and let ∇ be a torsion-free affine connection on $\mathcal{M}$. The Poincaré–Birkhoff–Witt map inductively defined by the relations*[1]

$$\mathrm{pbw}^\nabla(f) = f, \quad \forall f \in C^\infty(\mathcal{M});$$
$$\mathrm{pbw}^\nabla(X) = X, \quad \forall X \in \mathfrak{X}(\mathcal{M});$$

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[1] We would like to thank Hsuan-Yi Liao for correcting a sign error in the inductive formula defining the map pbw^∇.

and

$$\mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^n (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)} \big\{ X_k \cdot \mathrm{pbw}^\nabla(X_0 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n)$$

$$- \mathrm{pbw}^\nabla\big(\nabla_{X_k}(X_0 \odot \cdots \odot \widehat{X}_k \odot \cdots \odot X_n)\big)\big\},$$

for all $n \in \mathbb{N}$ and $X_0, \ldots, X_n \in \mathfrak{X}(\mathcal{M})$, establishes an isomorphism

$$\mathrm{pbw}^\nabla : \Gamma(S(T\mathcal{M})) \xrightarrow{\cong} D(\mathcal{M}). \tag{7}$$

of coalgebras over $C^\infty(\mathcal{M})$.

Now assume that $(\mathcal{M}, Q)$ is a dg-manifold. The homological vector field Q induces a degree $+1$ coderivation of $D(\mathcal{M})$ defined by the Lie derivative:

$$L_Q(X_1 \cdots X_n) = \sum_{k=1}^n (-1)^{|X_1|+\cdots+|X_{k-1}|} X_1 \cdots X_{k-1}[Q, X_k]X_{k+1} \cdots X_n. \tag{8}$$

Now using the isomorphism of coalgebras pbw^∇ as in Eq. (7) to transfer L_Q from $D(\mathcal{M})$ to $\Gamma(S(T\mathcal{M}))$, we obtain $\delta := (\mathrm{pbw}^\nabla)^{-1} \circ L_Q \circ \mathrm{pbw}^\nabla$, a degree 1 coderivation of $\Gamma(S(T\mathcal{M}))$. Finally, dualizing δ, we obtain an operator

$$D : \Gamma(\hat{S}(T^\vee \mathcal{M})) \to \Gamma(\hat{S}(T^\vee \mathcal{M}))$$

as

$$\Gamma(\hat{S}(T^\vee \mathcal{M})) \cong \mathrm{Hom}_{C^\infty(\mathcal{M})}(\Gamma(S(T\mathcal{M})), C^\infty(\mathcal{M})).$$

Theorem 4.3. *Let $(\mathcal{M}, Q)$ be a dg-manifold and let ∇ be a torsion-free affine connection on $\mathcal{M}$.*

(i) *The operator D, dual to $(\mathrm{pbw}^\nabla)^{-1} \circ L_Q \circ \mathrm{pbw}^\nabla$, is a degree $+1$ derivation of the graded algebra $\Gamma(\widehat{S}(T^\vee \mathcal{M}))$ satisfying $D^2 = 0$.*
(ii) *There exists a sequence $\{R_k\}_{k\geq 2}$ of homomorphisms $R_k \in \mathrm{Hom}(S^k T\mathcal{M}, T\mathcal{M}[-1])$, whose first term R_2 is precisely the Atiyah 1-cocycle $\mathrm{At}_\mathcal{M}$, such that $D = L_Q + \sum_{k=2}^\infty \widetilde{R}_k$, where $\widetilde{R}_k$ denotes the $C^\infty(\mathcal{M})$-linear operator on $\Gamma(\widehat{S}(T^\vee \mathcal{M}))$ corresponding to R_k.*

Finally we note that Theorem 4.1 is a consequence of Theorem 4.3.

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