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Upper bounds for dimensions of singularity categories

Bornes supérieures pour les dimensions des catégories de singularités

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ABSTRACT

This paper gives upper bounds for the dimension of the singularity category of a Cohen–Macaulay local ring with an isolated singularity. One of them recovers an upper bound given by Ballard, Favero and Katzarkov in the case of a hypersurface.

RÉSUMÉ

Cet article donne des bornes supérieures pour la dimension de la catégorie de singularité d'un anneau local Cohen–Macaulay à singularité isolée. L'une de nos estimations redonne une borne fournie par Ballard, Favero et Katzarkov dans le cas des hypersurfaces.

1. Results

The notion of the dimension of a triangulated category has been introduced by Bondal, Rouquier, and Van den Bergh [4,13]. Roughly speaking, it measures the number of extensions necessary to build the category out of a single object. The singularity category $\mathrm{D_{sg}}(R)$ of a Noetherian ring/scheme R is one of the most crucial triangulated categories. This has been introduced by Buchweitz [6] under the name of stable derived category. There are many studies on singularity categories by Orlov [9–12] in connection with the Homological Mirror Symmetry Conjecture.

It is a natural and fundamental problem to find upper bounds for the dimension of the singularity category of a Noetherian ring. In general, the dimension of the singularity category is known to be finite for large classes of excellent rings containing fields [1,13], but only a few explicit upper bounds have been found so far. The Loewy length is an upper bound for an Artinian ring [13], and so is the global dimension for a ring of finite global dimension [7,8]. Recently, an upper bound for an isolated hypersurface singularity has been given [2].

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The main purpose of this paper is to give upper bounds for a Cohen–Macaulay local ring with an isolated singularity. The main result of this paper is the following theorem.

Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a complete equicharacteristic Cohen–Macaulay local ring with k perfect.*

(1) *If R is an isolated singularity, then the sum $\mathfrak{N}^R$ of the Noether differents of R is $\mathfrak{m}$-primary.*
(2) *Let I be an $\mathfrak{m}$-primary ideal of R contained in $\mathfrak{N}^R$.*
 (a) *One has $\mathsf{D}_{\mathsf{sg}}(R) = \langle k\rangle_{(\nu(I)-\dim R+1)\ell\ell(R/I)}$. Hence there is an inequality*

$$\dim \mathsf{D}_{\mathsf{sg}}(R) \le \big(\nu(I) - \dim R + 1\big)\ell\ell(R/I) - 1.$$

 (b) *Assume that k is infinite. Then $\mathsf{D}_{\mathsf{sg}}(R) = \langle k\rangle_{\mathrm{e}(I)}$, and hence one has*

$$\dim \mathsf{D}_{\mathsf{sg}}(R) \le \mathrm{e}(I) - 1.$$

Here we explain the notation used in the above theorem. Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian complete equicharacteristic local ring. Let A be a Noether normalization of R, that is, a formal power series subring $k[[x_1, \dots, x_d]]$, where $x_1, \dots, x_d$ is a system of parameters of R. Let $R^e = R \otimes_A R$ be the enveloping algebra of R over A. Define a map $\mu : R^e \to R$ by $\mu(a \otimes b) = ab$ for $a, b \in R$. Then the ideal $\mathfrak{N}^R_A = \mu(\mathsf{Ann}_{R^e} \mathsf{Ker}\,\mu)$ of R is called the Noether different of R over A. We denote by $\mathfrak{N}^R$ the sum of $\mathfrak{N}^R_A$, where A runs through the Noether normalizations of R. For an $\mathfrak{m}$-primary ideal I of R, let $\nu(I) = \dim_k(I \otimes_R k)$ be the minimal number of generators of I and $\mathrm{e}(I) = \lim_{n\to\infty} \frac{d!}{n^d}\ell(R/I^{n+1})$ the multiplicity of I, where $\ell(R/I^{n+1})$ stands for the (usual) length of the Artinian ring R/I^{n+1}. The Loewy length of an Artinian ring Λ is denoted by $\ell\ell(\Lambda)$, that is, the minimum positive integer n with $(\mathsf{rad}\,\Lambda)^n = 0$.

Combining Theorem 1.1 with [3, Corollary 5.10], we obtain the following inequality for a complete intersection.

Corollary 1.2. *Let $(R, \mathfrak{m}, k)$ be a complete equicharacteristic local complete intersection with k perfect. Let I be an $\mathfrak{m}$-primary ideal of R contained in $\mathfrak{N}^R$. Then one has*

$$\operatorname{codim} R \le \min\big\{\big(\nu(I) - \dim R + 1\big)\ell\ell(R/I), \mathrm{e}(I)\big\}.$$

Our Theorem 1.1 yields the following result.

Corollary 1.3. *Let k be a perfect field, and let $R = k[[x_1, \dots, x_n]]/(f_1, \dots, f_m)$ be a Cohen–Macaulay ring having an isolated singularity. Let J be the Jacobian ideal of R, namely, the ideal generated by the h-minors of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$, where $h = \mathrm{ht}(f_1, \dots, f_m)$.*

(1) *One has $\mathsf{D}_{\mathsf{sg}}(R) = \langle k\rangle_{(\nu(J)-\dim R+1)\ell\ell(R/J)}$. Hence there is an inequality $\dim \mathsf{D}_{\mathsf{sg}}(R) \le (\nu(J) - \dim R + 1)\ell\ell(R/J) - 1$.*
(2) *If k is infinite, then $\mathsf{D}_{\mathsf{sg}}(R) = \langle k\rangle_{\mathrm{e}(J)}$, and it holds that $\dim \mathsf{D}_{\mathsf{sg}}(R) \le \mathrm{e}(J) - 1$.*

Corollary 1.3 immediately recovers the following result, which is stated in [2].

Corollary 1.4 *(Ballard–Favero–Katzarkov). Let k be an algebraically closed field of characteristic zero. Let $R = k[[x_1, \dots, x_n]]/(f)$ be an isolated hypersurface singularity. Then $\mathsf{D}_{\mathsf{sg}}(R) = \langle k\rangle_{2\ell\ell(R/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})R)}$, and hence $\dim \mathsf{D}_{\mathsf{sg}}(R) \le 2\ell\ell(R/(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})R) - 1$.*

As another application of Theorem 1.1, we obtain upper bounds for the dimension of the stable category $\underline{\mathsf{CM}}(R)$ of maximal Cohen–Macaulay modules over an excellent Gorenstein ring R:

Corollary 1.5. *Let R be an excellent Gorenstein equicharacteristic local ring with perfect residue field k, and assume that R is an isolated singularity. Then $\mathfrak{N}^{\widehat{R}}$ is an $\widehat{\mathfrak{m}}$-primary ideal of the completion $\widehat{R}$ of R. Let I be an $\widehat{\mathfrak{m}}$-primary ideal contained in $\mathfrak{N}^{\widehat{R}}$. Put $d = \dim R$, $n = \nu(I)$, $l = \ell\ell(\widehat{R}/I)$ and $e = \mathrm{e}(I)$.*

(1) *One has $\underline{\mathsf{CM}}(R) = \langle \Omega^d k\rangle_{(n-d+1)l}$, and $\dim \underline{\mathsf{CM}}(R) \le (n - d + 1)l - 1$.*
(2) *If k is infinite, then $\underline{\mathsf{CM}}(R) = \langle \Omega^d k\rangle_e$, and one has $\dim \underline{\mathsf{CM}}(R) \le e - 1$.*

2. Proofs

This section is devoted to proving our results stated in the previous section. For the definition of the dimension of a triangulated category and related notation, we refer the reader to [13, Definition 3.2]. We denote by $\mathsf{D}(\mathcal{A})$ the derived category of an Abelian category $\mathcal{A}$. Let $\mathsf{H}^i X$ (respectively, $\mathsf{Z}^i X$, $\mathsf{B}^i X$) denote the i-th homology (respectively, cycle, boundary) of a complex X of objects of $\mathcal{A}$, and set $\mathsf{H}X = \bigoplus_{i\in\mathbb{Z}} \mathsf{H}^i X$.

Lemma 2.1. *Let $\mathcal{A}$ be an Abelian category and X a complex of objects of $\mathcal{A}$.*

(1) *Let n be an integer. If $\mathrm{H}^i X = 0$ for all $i > n$, then there exists an exact triangle*

$$Y \to X \to \mathrm{H}^n X[-n] \rightsquigarrow$$

in $\mathrm{D}(\mathcal{A})$ such that $\mathrm{H}^i Y \cong \begin{cases} 0 & (i \geq n) \\ \mathrm{H}^i X & (i<n). \end{cases}$

(2) *Let $n \geq m$ be integers. If $\mathrm{H}^i X = 0$ for all $i > n$ and $i < m$, then $X \in \langle \mathrm{H} X \rangle_{n-m+1}^{\mathrm{D}(\mathcal{A})}$.*

Proof. (1) Truncating $X = (\cdots \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots)$, we get complexes

$$X' = \big(\cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} \mathrm{Z}^n X \to 0\big),$$
$$Y = \big(\cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} \mathrm{B}^n X \to 0\big).$$

There are natural morphisms $Y \xrightarrow{f} X' \xrightarrow{g} X$, where f is a monomorphism and g is a quasi-isomorphism. We have a short exact sequence $0 \to Y \xrightarrow{f} X' \to \mathrm{H}^n X[-n] \to 0$ of complexes, which induces an exact triangle as in the assertion.

(2) Applying (1) repeatedly, for each $0 \leq j \leq n-m$, we obtain an exact triangle

$$X_{j+1} \to X_j \to \mathrm{H}^{n-j} X\big[-(n-j)\big] \rightsquigarrow$$

in $\mathrm{D}(\mathcal{A})$ with $X_0 = X$ such that $\mathrm{H}^i X_j \cong 0$ for $i > n-j$ and $\mathrm{H}^i X \cong \mathrm{H}^i X$ for $i \leq n-j$. Hence $X_{n-m+1} \cong 0$ in $\mathrm{D}(\mathcal{A})$, which implies that X_{n-m} is in $\langle \mathrm{H}^m X \rangle$. Inductively, we observe that $X = X_0$ belongs to $\langle \mathrm{H}^m X \oplus \mathrm{H}^{m+1} X \oplus \cdots \oplus \mathrm{H}^n X \rangle_{n-m+1} = \langle \mathrm{H} X \rangle_{n-m+1}$. $\square$

For a commutative Noetherian ring R, we denote by $\operatorname{mod} R$ the category of finitely generated R-modules, and by $\mathrm{D}^b(\operatorname{mod} R)$ the bounded derived category of $\operatorname{mod} R$. For a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ of elements of R and an R-module M, let $\mathrm{K}(\boldsymbol{x}, M)$ denote the Koszul complex of $\boldsymbol{x}$ on M.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and I an $\mathfrak{m}$-primary ideal of R. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements of R that generates I. Then for any finitely generated R-module M one has $\mathrm{K}(\boldsymbol{x}, M) \in \langle k \rangle_{(n-t+1)l}$ in $\mathrm{D}^b(\operatorname{mod} R)$, where $t = \operatorname{depth} M$ and $l = \ell\ell(R/I)$.*

Proof. Set $\mathrm{K}(\boldsymbol{x}, M) = K = (0 \to K^{-n} \to \cdots \to K^0 \to 0)$. By [5, Proposition 1.6.5(b)], each homology $H^i = \mathrm{H}^i K$ is annihilated by I, and H^i is regarded as a module over R/I. There is a filtration $0 = \mathfrak{m}^l(R/I) \subsetneq \cdots \subsetneq \mathfrak{m}(R/I) \subsetneq R/I$ of ideals of R/I. For each integer i, we have a filtration

$$0 = \mathfrak{m}^l H^i \subseteq \cdots \subseteq \mathfrak{m} H^i \subseteq H^i$$

of submodules of H^i, which shows $H^i \in \langle k \rangle_l$ in $\mathrm{D}^b(\operatorname{mod} R)$. We see from [5, Theorem 1.6.17(b)] that $H^i = 0$ for all $i < t-n$ and $i > 0$. It follows from Lemma 2.1(2) that K is in $\langle \bigoplus_{i=t-n}^{0} H^i \rangle_{n-t+1}$ in $\mathrm{D}^b(\operatorname{mod} R)$, which is contained in $\langle k \rangle_{(n-t+1)l}$. $\square$

Recall that the singularity category $\mathrm{D_{sg}}(R)$ of a (commutative) Noetherian ring R is defined as the Verdier quotient of $\mathrm{D}^b(\operatorname{mod} R)$ by the full subcategory of perfect complexes. (A perfect complex is by definition a bounded complex of finitely generated modules.)

Proposition 2.3. *Let R be a commutative Noetherian ring, and let M be a finitely generated R-module. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements of R such that the multiplication map $M \xrightarrow{x_i} M$ is a zero morphism in $\mathrm{D_{sg}}(R)$ for every $1 \leq i \leq n$. Then M is isomorphic to a direct summand of $\mathrm{K}(\boldsymbol{x}, M)$ in $\mathrm{D_{sg}}(R)$.*

Proof. By definition the Koszul complex $\mathrm{K}(x_i, M) = (0 \to M \xrightarrow{x_i} M \to 0)$ is the mapping cone of the multiplication map $M \xrightarrow{x_i} M$, and there is an exact triangle $M \xrightarrow{x_i} M \to \mathrm{K}(x_i, M) \rightsquigarrow$ in $\mathrm{D_{sg}}(R)$. By assumption, we have an isomorphism $M \oplus M[1] \cong \mathrm{K}(x_i, M) = \mathrm{K}(x_i, R) \otimes_R M$ in $\mathrm{D_{sg}}(R)$. We observe that

$$\begin{aligned} \mathrm{K}(\boldsymbol{x}, M) &= \mathrm{K}(x_1, R) \otimes_R \cdots \otimes_R \mathrm{K}(x_{n-1}, R) \otimes_R \big(\mathrm{K}(x_n, R) \otimes_R M\big) \\ &\gtrdot \mathrm{K}(x_1, R) \otimes_R \cdots \otimes_R \mathrm{K}(x_{n-2}, R) \otimes_R \big(\mathrm{K}(x_{n-1}, R) \otimes_R M\big) \\ &\cdots \\ &\gtrdot \mathrm{K}(x_1, R) \otimes_R M \gtrdot M, \end{aligned}$$

where $A \gtrdot B$ means that A has a direct summand isomorphic to B in $\mathrm{D_{sg}}(R)$. $\square$

Lemma 2.4.

(1) *Let $\mathcal{A}$ be an Abelian category. Let $P = (\cdots \xrightarrow{d^{b-1}} P^b \xrightarrow{d^b} \cdots \xrightarrow{d^{a-1}} P^a \to 0)$ be a complex of projective objects of $\mathcal{A}$ with $\mathrm{H}^i P = 0$ for all $i < b$. Then one has an exact triangle*

$$F \to P \to C[-b] \rightsquigarrow$$

in $\mathrm{D}(\mathcal{A})$, where $F = (0 \to P^{b+1} \xrightarrow{d^{b+1}} \cdots \xrightarrow{d^{a-1}} P^a \to 0)$ and $C = \mathrm{Coker}\, d^{b-1}$.

(2) *Let R be a commutative Noetherian ring.*
  (a) *For any $X \in \mathrm{D_{sg}}(R)$ there exist $M \in \mathrm{mod}\, R$ and $n \in \mathbb{Z}$ such that $X \cong M[n]$ in $\mathrm{D_{sg}}(R)$.*
  (b) *Let M be a finitely generated R-module. Then for an integer $n \ge 0$ there exists an exact triangle*

$$F \to M \to \Omega^n M[n] \rightsquigarrow$$

in $\mathrm{D}^b(\mathrm{mod}\, R)$, where $F = (0 \to F^{-(n-1)} \to \cdots \to F^0 \to 0)$ is a perfect complex.

Proof. (1) There is a short exact sequence $0 \to F \to P \to Q \to 0$ of complexes, where $Q = (\cdots \xrightarrow{d^{b-2}} P^{b-1} \xrightarrow{d^{b-1}} P^b \to 0)$. Then $Q \cong C[-b]$ in $\mathrm{D}(\mathcal{A})$.

(2) The assertion (a) is immediate from (1). Setting $a = 0 \ge -n = b$ and letting P be a projective resolution of M in (1) implies (b). $\square$

Recall that a commutative Noetherian ring R is called an isolated singularity if the local ring $R_\mathfrak{p}$ is regular for every nonmaximal prime ideal $\mathfrak{p}$ of R.

Proposition 2.5. *Let R be a complete equicharacteristic Cohen–Macaulay local commutative ring. Then for an element $x \in \mathfrak{N}^R$ and a maximal Cohen–Macaulay R-module M, the multiplication map $M \xrightarrow{x} M$ is a zero morphism in $\mathrm{D_{sg}}(R)$.*

Proof. Lemma 2.4(2) implies that there is an exact triangle

$$F \xrightarrow{f} M \xrightarrow{g} \Omega M[1] \rightsquigarrow$$

in $\mathrm{D}^b(\mathrm{mod}\, R)$, where F is a finitely generated free R-module. By virtue of [14, Corollary 5.13], the ideal $\mathfrak{N}^R$ annihilates $\mathrm{Ext}^1_R(M, \Omega M) = \mathrm{Hom}_{\mathrm{D}^b(\mathrm{mod}\, R)}(M, \Omega M[1])$. Hence $gx = xg = 0$ in $\mathrm{D}^b(\mathrm{mod}\, R)$, and there exists a morphism $h : M \to F$ such that $fh = (M \xrightarrow{x} M)$ in $\mathrm{D}^b(\mathrm{mod}\, R)$. Send this equality by the localization functor $\mathrm{D}^b(\mathrm{mod}\, R) \to \mathrm{D_{sg}}(R)$, and note that $F \cong 0$ in $\mathrm{D_{sg}}(R)$. Thus the multiplication map $M \xrightarrow{x} M$ is zero in $\mathrm{D_{sg}}(R)$. $\square$

Now we can prove the results given in the previous section.

Proof of Theorem 1.1. (1) As k is a perfect field and R is an isolated singularity, $\mathfrak{N}^R$ is $\mathfrak{m}$-primary by [15, Lemma (6.12)].

(2) (a) Put $d = \dim R$, $n = \nu(I)$, $l = \ell\ell(R/I)$ and $e = \mathrm{e}(I)$. We have $I = (\boldsymbol{x})$ for some sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements in I. Let $X \in \mathrm{D_{sg}}(R)$. Then, using Lemma 2.4(2), we see that $X \cong \Omega^d N[n]$ for some $N \in \mathrm{mod}\, R$ and $n \in \mathbb{Z}$. Note that $M := \Omega^d N$ is a maximal Cohen–Macaulay R-module. Proposition 2.2 implies that $\mathrm{K}(\boldsymbol{x}, M)$ belongs to $\langle k \rangle_{(n-d+1)l}$ in $\mathrm{D}^b(\mathrm{mod}\, R)$. Applying the localization functor $\mathrm{D}^b(\mathrm{mod}\, R) \to \mathrm{D_{sg}}(R)$, we have $\mathrm{K}(\boldsymbol{x}, M) \in \langle k \rangle_{(n-d+1)l}$ in $\mathrm{D_{sg}}(R)$. Since M is isomorphic to a direct summand of $\mathrm{K}(\boldsymbol{x}, M)$ in $\mathrm{D_{sg}}(R)$ by Propositions 2.3 and 2.5, we get $M \in \langle k \rangle_{(n-d+1)l}$ in $\mathrm{D_{sg}}(R)$. Therefore $\mathrm{D_{sg}}(R) = \langle k \rangle_{(n-d+1)l}$ follows.

(b) Since k is infinite, there exists a parameter ideal Q of R that is a reduction of I (cf. [5, Corollary 4.6.10]). Then we have $\nu(Q) = \dim R$, and it holds that

$$\big(\nu(Q) - \dim R + 1\big)\ell\ell(R/Q) = \ell\ell(R/Q) \le \ell(R/Q) = \mathrm{e}(Q) = \mathrm{e}(I).$$

The assertion is a consequence of (a). $\square$

Proof of Corollary 1.3. We see from [14, Lemmas 4.3, 5.8 and Propositions 4.4, 4.5] that J is contained in $\mathfrak{N}^R$ and defines the singular locus of R. Hence the assertion follows from Theorem 1.1. $\square$

Proof of Corollary 1.5. We notice that $\widehat{R}$ is an isolated singularity. Suppose that $\mathrm{D_{sg}}(\widehat{R}) = \langle k \rangle_r$ holds for some $r \ge 0$. Then it follows from [6, Theorem 4.4.1] that $\underline{\mathrm{CM}}(\widehat{R}) = \langle \Omega^d_{\widehat{R}} k \rangle_r = \widehat{\langle \Omega^d_R k \rangle_r}$. The proof of [1, Theorem 5.8] shows that $\underline{\mathrm{CM}}(R) = \langle \Omega^d_R k \rangle_r$. Thus, Theorem 1.1 completes the proof. $\square$

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