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Numerical analysis

# Equilibrated tractions for the Hybrid High-Order method

## *Tractions équilibrées pour la méthode hybride d'ordre élevé*

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ABSTRACT

We show how to recover equilibrated face tractions for the Hybrid High-Order method for linear elasticity recently introduced in [1], and prove that these tractions are optimally convergent.



RÉSUMÉ

Nous montrons comment obtenir des tractions de face équilibrées pour la méthode hybride d'ordre élevé pour l'élasticité linéaire récemment introduite dans [1] et prouvons que ces tractions convergent de manière optimale.



## 1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a bounded connected polygonal or polyhedral domain. For $X \subset \overline{\Omega}$, we denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$, respectively, the standard inner product and norm of $L^2(X)$, and a similar notation is used for $L^2(X)^d$ and $L^2(X)^{d\times d}$. For a given external load $\boldsymbol{f} \in L^2(\Omega)^d$, we consider the linear elasticity problem: find $\boldsymbol{u} \in H_0^1(\Omega)^d$ such that

$$2\mu(\nabla_s \boldsymbol{u}, \nabla_s \boldsymbol{v})_\Omega + \lambda(\nabla\cdot \boldsymbol{u}, \nabla\cdot \boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega \tag{1}$$

with $\mu > 0$ and $\lambda \geq 0$ real numbers representing the scalar Lamé coefficients and $\nabla_s$ denoting the symmetric gradient operator. Classically, the solution to (1) satisfies $-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}$ a.e. in $\Omega$ with stress tensor $\boldsymbol{\sigma}(\boldsymbol{u}) := 2\mu\nabla_s\boldsymbol{u} + \lambda \boldsymbol{I}_d(\nabla\cdot\boldsymbol{u})$. Denoting by $T$ an open subset of $\Omega$ with non-zero Hausdorff measure ($T$ will represent a mesh element in what follows), partial integration yields the following local equilibrium property:

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}), \nabla_s \mathsf{v}_T\big)_T - \big(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}_T, \mathsf{v}_T\big)_{\partial T} = (\boldsymbol{f}, \mathsf{v}_T)_T \quad \forall \mathsf{v}_T \in \mathbb{P}_d^k(T)^d, \tag{2}$$

where $\partial T$ and $\boldsymbol{n}_T$ denote, respectively, the boundary and outward normal to $T$. Additionally, the normal interface tractions $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}_T$ are equilibrated across $\partial T \cap \Omega$. The goal of this work is (i) to devise a reformulation of the Hybrid High-Order method for linear elasticity introduced in [1] that identifies its local equilibrium properties expressed by a discrete

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counterpart of (2) and (ii) to show how the corresponding equilibrated face tractions can be obtained by element-wise post-processing. This is an important complement to the original analysis, as local equilibrium is an essential property in practice. The material is organized as follows: in Section 2 we outline the original formulation of the HHO method; in Section 3 we derive the local equilibrium formulation based on a new local displacement reconstruction.

## 2. The Hybrid High-Order method

We consider admissible mesh sequences in the sense of [2, Section 1.4]. Each mesh $\mathcal{T}_h$ in the sequence is a finite collection $\{T\}$ of nonempty, disjoint, open, polytopic elements such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ (with $h_T$ the diameter of $T$), and there is a matching simplicial submesh of $\mathcal{T}_h$ with locally equivalent mesh size, which is shape-regular in the usual sense. For all $T\in\mathcal{T}_h$, the faces of $T$ are collected in the set $\mathcal{F}_T$ and, for all $F\in\mathcal{F}_T$, $\boldsymbol{n}_{TF}$ is the unit normal to $F$ pointing out of $T$. Additionally, interfaces are collected in the set $\mathcal{F}_h^{\mathrm{i}}$ and boundary faces in $\mathcal{F}_h^{\mathrm{b}}$. The diameter of a face $F\in\mathcal{F}_h$ is denoted by $h_F$. For the sake of brevity, we abbreviate $a\lesssim b$ the inequality $a\le Cb$ for positive real numbers $a$ and $b$ and a generic constant $C$ that can depend on the mesh regularity, on $\mu$, $d$, and the polynomial degree, but is independent of $h$ and $\lambda$. We also introduce the notation $a\simeq b$ for the uniform equivalence $a\lesssim b\lesssim a$.

Let a polynomial degree $k\ge 1$ be fixed. The local and global spaces of degrees of freedom (DOFs) are:

$$\underline{\mathbf{U}}_T^k := \mathbb{P}_d^k(T)^d \times \Big\{ \bigtimes_{F\in\mathcal{F}_T} \mathbb{P}_{d-1}^k(F)^d \Big\} \quad \forall T\in\mathcal{T}_h, \qquad \underline{\mathbf{U}}_h^k := \Big\{ \bigtimes_{T\in\mathcal{T}_h} \mathbb{P}_d^k(T)^d \Big\} \times \Big\{ \bigtimes_{F\in\mathcal{F}_h} \mathbb{P}_{d-1}^k(F)^d \Big\}. \tag{3}$$

A generic collection of DOFs from $\underline{\mathbf{U}}_h^k$ is denoted by $\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h})$ and, for a given $T\in\mathcal{T}_h$, $\underline{\mathbf{v}}_T = (\mathbf{v}_T, (\mathbf{v}_F)_{F\in\mathcal{F}_T}) \in \underline{\mathbf{U}}_T^k$ indicates its restriction to $\underline{\mathbf{U}}_T^k$. For all $T\in\mathcal{T}_h$, we define a high-order local displacement reconstruction operator $\boldsymbol{p}_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$ by solving the following (well-posed) pure traction problem: For a given $\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}_T^k$, $\boldsymbol{p}_T^k\underline{\mathbf{v}}_T$ is such that

$$\left(\nabla_{\mathrm{s}}\boldsymbol{p}_T^k\underline{\mathbf{v}}_T, \nabla_{\mathrm{s}}\boldsymbol{w}\right)_T = (\nabla_{\mathrm{s}}\mathbf{v}_T, \nabla_{\mathrm{s}}\boldsymbol{w})_T + \sum_{F\in\mathcal{F}_T} (\mathbf{v}_F - \mathbf{v}_T, \nabla_{\mathrm{s}}\boldsymbol{w}\,\boldsymbol{n}_{TF})_F \quad \forall \boldsymbol{w}\in\mathbb{P}_d^{k+1}(T)^d, \tag{4}$$

and the rigid-body motion components of $\boldsymbol{p}_T^k\underline{\mathbf{v}}_T$ are prescribed so that $\int_T \boldsymbol{p}_T^k\underline{\mathbf{v}}_T = \int_T \mathbf{v}_T$ and $\int_T \nabla_{\mathrm{ss}}(\boldsymbol{p}_T^k\underline{\mathbf{v}}_T) = \sum_{F\in\mathcal{F}_T}\int_F \frac{1}{2}(\boldsymbol{n}_{TF}\otimes\mathbf{v}_F - \mathbf{v}_F\otimes\boldsymbol{n}_{TF})$ where $\nabla_{\mathrm{ss}}$ is the skew-symmetric gradient operator. Additionally, we define the divergence reconstruction $D_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{P}_d^k(T)$ such that, for a given $\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}_T^k$,

$$\left(D_T^k\underline{\mathbf{v}}_T, q\right)_T = (\nabla\cdot\mathbf{v}_T, q)_T + \sum_{F\in\mathcal{F}_T} (\mathbf{v}_F - \mathbf{v}_T, q\boldsymbol{n}_{TF})_F \quad \forall q\in\mathbb{P}_d^k(T). \tag{5}$$

We introduce the local bilinear form $a_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \to \mathbb{R}$ such that

$$a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := 2\mu\left\{\left(\nabla_{\mathrm{s}}\boldsymbol{p}_T^k\underline{\mathbf{w}}_T, \nabla_{\mathrm{s}}\boldsymbol{p}_T^k\underline{\mathbf{v}}_T\right)_T + s_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T)\right\} + \lambda\left(D_T^k\underline{\mathbf{w}}_T, D_T^k\underline{\mathbf{v}}_T\right)_T, \tag{6}$$

where the stabilizing bilinear form $s_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \to \mathbb{R}$ is such that

$$s_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := \sum_{F\in\mathcal{F}_T} h_F^{-1}\left(\pi_F^k\left(\boldsymbol{P}_T^k\underline{\mathbf{w}}_T - \mathbf{w}_F\right), \pi_F^k\left(\boldsymbol{P}_T^k\underline{\mathbf{v}}_T - \mathbf{v}_F\right)\right)_F, \tag{7}$$

and a second displacement reconstruction $\boldsymbol{P}_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)^d$ is defined such that, for all $\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}_T^k$, $\boldsymbol{P}_T^k\underline{\mathbf{v}}_T := \mathbf{v}_T + (\boldsymbol{p}_T^k\underline{\mathbf{v}}_T - \pi_T^k\boldsymbol{p}_T^k)$. Let $\underline{\boldsymbol{I}}_T^k : H^1(T)^d \to \underline{\mathbf{U}}_T^k$ be the reduction map such that, for all $T\in\mathcal{T}_h$ and all $\boldsymbol{v}\in H^1(T)^d$, $\underline{\boldsymbol{I}}_T^k\boldsymbol{v} = (\pi_T^k\boldsymbol{v}, (\pi_F^k\boldsymbol{v})_{F\in\mathcal{F}_{ThT}})$. The potential reconstruction $\boldsymbol{p}_T^k$ and the bilinear form $s_T$ are conceived so that they satisfy the following two key properties:

(i) *Stability.* For all $\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}_T^k$,

$$\left\|\nabla_{\mathrm{s}}\boldsymbol{p}_T^k\underline{\mathbf{v}}\right\|_T^2 + s_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T) \simeq \|\nabla_{\mathrm{s}}\mathbf{v}\|_T^2 + j_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T), \tag{8}$$

with bilinear form $j_T : \underline{\mathbf{U}}_T^k \times \underline{\mathbf{U}}_T^k \to \mathbb{R}$ such that $j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := \sum_{F\in\mathcal{F}_T} h_F^{-1}(\mathbf{w}_T - \mathbf{w}_F, \mathbf{v}_T - \mathbf{v}_F)_F$.

(ii) *Approximation.* For all $\boldsymbol{v}\in H^{k+2}(T)^d$,

$$\left\{\left\|\nabla_{\mathrm{s}}\left(\boldsymbol{v} - \boldsymbol{p}_T^k\underline{\boldsymbol{I}}_T^k\boldsymbol{v}\right)\right\|_T^2 + s_T\left(\underline{\boldsymbol{I}}_T^k\boldsymbol{v}, \underline{\boldsymbol{I}}_T^k\boldsymbol{v}\right)\right\}^{1/2} \lesssim h_T^{k+1}\|\boldsymbol{v}\|_{H^{k+2}(T)^d}. \tag{9}$$

We observe that, unlike $s_T$, the stabilization bilinear form $j_T$ only satisfies $j_T(\underline{\boldsymbol{I}}_T^k\boldsymbol{v}, \underline{\boldsymbol{I}}_T^k\boldsymbol{v}) \lesssim h^k\|\boldsymbol{v}\|_{H^{k+1}(T)^d}$. The discrete problem reads: find $\underline{\mathbf{u}}_h\in\underline{\mathbf{U}}_{h,0}^k := \{\underline{\mathbf{u}}_h\in\underline{\mathbf{U}}_h^k \mid \mathbf{u}_F\equiv\mathbf{0}\ \forall F\in\mathcal{F}_h^{\mathrm{b}}\}$ such that

$$a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) := \sum_{T\in\mathcal{T}_h} a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = \sum_{T\in\mathcal{T}_h} (\boldsymbol{f}, \mathbf{v}_T)_T \quad \forall\underline{\mathbf{v}}_h\in\underline{\mathbf{U}}_{h,0}^k. \tag{10}$$

The following convergence result was proved in [1]:

**Theorem 1** *(Energy error estimate). Let* $\boldsymbol{u} \in H_0^1(\Omega)^d$ *and* $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k$ *denote the unique solutions to (1) and (10), respectively, and assume* $\boldsymbol{u} \in H^{k+2}(\Omega)^d$ *and* $\nabla\cdot\boldsymbol{u} \in H^{k+1}(\Omega)$*. Then, letting* $\widehat{\underline{\mathbf{u}}}_h \in \underline{\mathbf{U}}_{h,0}^k$ *be such that* $\widehat{\underline{\mathbf{u}}}_T := \underline{\boldsymbol{I}}_T^k \boldsymbol{u}$ *for all* $T \in \mathcal{T}_h$*, the following holds (with* $\|\underline{\mathbf{v}}\|_{a,T}^2 = a_T(\underline{\mathbf{v}}_T, \underline{\mathbf{v}}_T)$ *for all* $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$*):*

$$\sum_{T\in\mathcal{T}_h} \|\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T\|_{a,T}^2 \lesssim h^{2(k+1)}\big(\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \lambda\|\nabla\cdot\boldsymbol{u}\|_{H^{k+1}(\Omega)}\big)^2. \tag{11}$$

*Moreover, assuming elliptic regularity,* $\sum_{T\in\mathcal{T}_h} \|\boldsymbol{u} - \boldsymbol{p}_T^k \underline{\mathbf{u}}_T\|_{a,T}^2 \lesssim h^{2(k+2)}(\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \lambda\|\nabla\cdot\boldsymbol{u}\|_{H^{k+1}(\Omega)})^2$*.*

## 3. Local equilibrium formulation

The difficulty in devising an equivalent local equilibrium formulation for problem (10) comes from the stabilization term $s_T$, which introduces a non-trivial coupling of interface DOFs inside each element. In this section, we introduce post-processed discrete displacement and stress reconstructions that allow us to circumvent this difficulty. For a given element $T \in \mathcal{T}_h$, define the following bilinear form on $\underline{\mathbf{U}}_T^k$:

$$\widetilde{a}_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) := 2\mu\big\{\big(\nabla_s \boldsymbol{p}_T^k \underline{\mathbf{w}}_T, \nabla_s \boldsymbol{p}_T^k \underline{\mathbf{v}}_T\big)_T + j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T)\big\} + \lambda\big(D_T^k \underline{\mathbf{w}}_T, D_T^k \underline{\mathbf{v}}_T\big)_T, \tag{12}$$

where the only difference with respect to the bilinear form $a_T$ defined by (6) is that we have stabilized using $j_T$ instead of $s_T$. We observe that, while proving a discrete local equilibrium relation for the method based on $\widetilde{a}_T$ would not require any local post-processing, the suboptimal consistency properties of $j_T$ would only yield $h^{2k}$ in the right-hand side of (11). Denoting by $\|\cdot\|_{\widetilde{a},T}$ the local seminorm induced by $\widetilde{a}_T$ on $\underline{\mathbf{U}}_T^k$, one can prove that, for all $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$,

$$\|\underline{\mathbf{v}}_T\|_{\widetilde{a},T} \simeq \|\underline{\mathbf{v}}_T\|_{a,T}. \tag{13}$$

We next define the isomorphism $\underline{\mathbf{c}}_T^k : \underline{\mathbf{U}}_T^k \to \underline{\mathbf{U}}_T^k$ such that

$$\widetilde{a}_T\big(\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T\big) = a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) + (2\mu) j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \quad \forall \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k, \tag{14}$$

and rigid-body motion components prescribed as above. We also introduce the stress reconstruction $\boldsymbol{S}_T^k : \underline{\mathbf{U}}_T^k \to \mathbb{P}_d^k(T)^{d\times d}$ such that

$$\boldsymbol{S}_T^k := \big(2\mu\nabla_s \boldsymbol{p}_T^k + \lambda \boldsymbol{I}_d D_T^k\big) \circ \underline{\mathbf{c}}_T^k. \tag{15}$$

**Lemma 2** *(Equilibrium formulation). The bilinear form* $a_T$ *defined by (6) is such that, for all* $\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$*,*

$$a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) = \big(\boldsymbol{S}_T^k \underline{\mathbf{w}}_T, \nabla_s \mathbf{v}_T\big)_T + \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{\tau}_{TF}(\underline{\mathbf{w}}_T), \mathbf{v}_F - \mathbf{v}_T\big)_F, \tag{16}$$

*with interface traction* $\boldsymbol{\tau}_{TF} : \underline{\mathbf{U}}_T^k \to \mathbb{P}_{d-1}^k(F)^d$ *such that*

$$\boldsymbol{\tau}_{TF}(\underline{\mathbf{w}}_T) = \boldsymbol{S}_T^k \underline{\mathbf{w}}_T\, \boldsymbol{n}_{TF} + h_F^{-1}\big[\big((\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T)_F - \mathbf{w}_F\big) - \big((\underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T)_T - \mathbf{w}_T\big)\big]. \tag{17}$$

**Proof.** Let $\widetilde{\underline{\mathbf{w}}}_T := \underline{\mathbf{c}}_T^k \underline{\mathbf{w}}_T$. We have, using the definitions (14) of $\underline{\mathbf{c}}_T^k$ and (12) of the bilinear form $\widetilde{a}_T$,

$$\begin{aligned}
a_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) &= \widetilde{a}_T(\widetilde{\underline{\mathbf{w}}}_T, \underline{\mathbf{v}}_T) - (2\mu) j_T(\underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T) \\
&= 2\mu\big\{\big(\nabla_s \boldsymbol{p}_T^k \widetilde{\underline{\mathbf{w}}}_T, \nabla_s \boldsymbol{p}_T^k \underline{\mathbf{v}}_T\big)_T + j_T(\widetilde{\underline{\mathbf{w}}}_T - \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T)\big\} + \lambda\big(D_T^k \widetilde{\underline{\mathbf{w}}}_T, D_T^k \underline{\mathbf{v}}_T\big)_T \\
&= \big(\boldsymbol{S}_T^k \underline{\mathbf{w}}_T, \nabla_s \mathbf{v}_T\big)_T + \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{S}_T^k \underline{\mathbf{w}}_T\, \boldsymbol{n}_{TF}, \mathbf{v}_F - \mathbf{v}_T\big)_F + (2\mu) j_T(\widetilde{\underline{\mathbf{w}}}_T - \underline{\mathbf{w}}_T, \underline{\mathbf{v}}_T),
\end{aligned}$$

where we have concluded using (4) with $\boldsymbol{w} = \boldsymbol{p}_T^k \widetilde{\underline{\mathbf{w}}}_T$, (5) with $q = D_T^k \widetilde{\underline{\mathbf{w}}}_T$, and recalling the definition (15) of $\boldsymbol{S}_T^k$. To obtain (16), it suffices to use the definition of $j_T$. □

**Lemma 3** *(Local equilibrium). Let* $\underline{\mathbf{u}}_h \in \underline{\mathbf{U}}_{h,0}^k$ *denote the unique solution to (10). Then, for all* $T \in \mathcal{T}_h$*, the following discrete counterpart of the local equilibrium relation (2) holds:*

$$\big(\boldsymbol{S}_T^k \underline{\mathbf{u}}_T, \nabla_s \mathbf{v}_T\big)_T - \sum_{F\in\mathcal{F}_T} \big(\boldsymbol{\tau}_{TF}(\underline{\mathbf{u}}_T), \mathbf{v}_T\big)_F = (\boldsymbol{f}, \mathbf{v}_T)_T \quad \forall \mathbf{v}_T \in \mathbb{P}_d^k(T)^d, \tag{18}$$

*and the numerical flux are equilibrated in the following sense: for all* $F \in \mathcal{F}_h^{\mathrm{i}}$ *such that* $F \subset \partial T_1 \cap \partial T_2$*,*

$$\boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_1 F}(\underline{\mathbf{u}}_{T_2}) = \mathbf{0}. \tag{19}$$

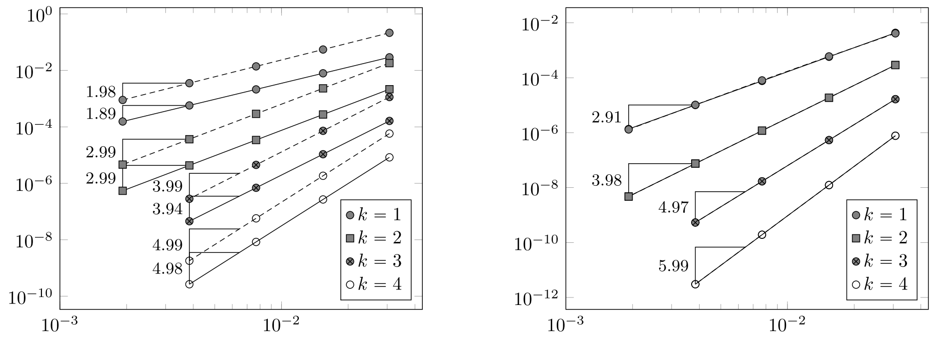


**Fig. 1.** Convergence results in the energy-norm (left) and $L^2$-norm (right) for the solution to (10) (solid lines) and its post-processing based on $\underline{\mathbf{c}}_T^k$ (dashed lines). The right panel shows that the post-processing has no sizable effect on element unknowns.

**Proof.** To prove (18), let an element $T \in \mathcal{T}_h$ be fixed, take in (10) $\underline{\mathbf{v}}_h = ((\mathbf{v}_T)_{T\in\mathcal{T}_h}, (\mathbf{0})_{F\in\mathcal{F}_h})$ with $\mathbf{v}_T$ in $\mathbb{P}_d^k(T)^d$ and $\mathbf{v}_{T'} \equiv \mathbf{0}$ for all $T' \in \mathcal{T}_h \setminus \{T\}$, and use (16) with $\underline{\mathbf{w}}_T = \underline{\mathbf{u}}_T$ to conclude that $a_T(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T)$ corresponds to the left-hand side of (18). Similarly, to prove (19), let an interface $F \in \mathcal{F}_h^{\mathrm{i}}$ be fixed and take in (10) $\underline{\mathbf{v}}_h = ((\mathbf{0})_{T\in\mathcal{T}_h}, (\mathbf{v}_F)_{F\in\mathcal{F}_h}) \in \underline{\mathbf{U}}_{h,0}^k$ with $\mathbf{v}_F$ in $\mathbb{P}_{d-1}^k(F)^d$ and $\mathbf{v}_{F'} \equiv \mathbf{0}$ for all $F' \in \mathcal{F}_h \setminus \{F\}$. Then, using (16) with $\underline{\mathbf{w}}_T = \underline{\mathbf{u}}_T$ in (10), it is inferred that $a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) = (\boldsymbol{\tau}_{T_1F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2,F}(\underline{\mathbf{u}}_{T_2}), \mathbf{v}_F)_F = 0$, which proves the desired result since $\boldsymbol{\tau}_{T_1F}(\underline{\mathbf{u}}_{T_1}) + \boldsymbol{\tau}_{T_2F}(\underline{\mathbf{u}}_{T_2}) \in \mathbb{P}_{d-1}^k(F)^d$. □

To conclude, we show that the locally post-processed solution yields a new collection of DOFs that is an equally good approximation of the exact solution as is the discrete solution $\underline{\mathbf{u}}_h$. Consequently, the equilibrated face numerical tractions defined in (17) optimally converge to the exact tractions.

**Proposition 4** *(Convergence for $\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T$). Using the notation of Theorem 1, the following holds:*

$$\sum_{T\in\mathcal{T}_h} \|\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T\|_{a,T}^2 \lesssim h^{2(k+1)}\left(\|\boldsymbol{u}\|_{H^{k+2}(\Omega)^d} + \lambda\|\nabla\cdot\boldsymbol{u}\|_{H^{k+1}(\Omega)}\right)^2. \tag{20}$$

**Proof.** Let $T \in \mathcal{T}_h$. Recalling (14), we have

$$\begin{aligned}\widetilde{a}_T\left(\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T\right) &= a_T(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) + (2\mu)j_T(\underline{\mathbf{u}}_T, \underline{\mathbf{v}}_T) - \widetilde{a}_T(\widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) \\ &= a_T(\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) + (2\mu)s_T(\widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T) + (2\mu)j_T(\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T).\end{aligned}$$

Hence, using the Cauchy–Schwarz inequality followed by the stability property (8) and multiple applications of the norm equivalence (13), we infer that

$$\begin{aligned}\left|\widetilde{a}_T\left(\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T\right)\right| &\le \left\{\|\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}\|_{a,T}^2 + (2\mu)s_T(\widehat{\underline{\mathbf{u}}}_T, \widehat{\underline{\mathbf{u}}}_T) + (2\mu)j_T(\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T)\right\}^{1/2}\|\underline{\mathbf{v}}_T\|_{\widetilde{a},T} \\ &\lesssim \left\{\|\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T\|_{a,T}^2 + (2\mu)s_T(\widehat{\underline{\mathbf{u}}}_T, \widehat{\underline{\mathbf{u}}}_T)\right\}^{1/2}\|\mathbf{v}_T\|_{\widetilde{a},T}.\end{aligned}$$

Using again (13) followed by the latter inequality, we infer that

$$\|\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T\|_{a,T} \lesssim \|\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T\|_{\widetilde{a},T} = \sup_{\underline{\mathbf{v}}_T\in\underline{\mathbf{U}}_T^k\setminus\{\mathbf{0}\}} \frac{\widetilde{a}_T(\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\underline{\mathbf{u}}}_T, \underline{\mathbf{v}}_T)}{\|\underline{\mathbf{v}}_T\|_{\widetilde{a},T}} \lesssim \left\{\|\mathbf{u}_T - \widehat{\underline{\mathbf{u}}}_T\|_{a,T}^2 + (2\mu)s_T(\widehat{\underline{\mathbf{u}}}_T, \widehat{\underline{\mathbf{u}}}_T)\right\}^{1/2}.$$

The estimate (20) then follows squaring the above inequality, summing over $T \in \mathcal{T}_h$, and using (11) and (9), respectively, to bound the terms in the right-hand side. □

To assess the estimate (20), we have numerically solved the pure displacement problem with exact solution $\boldsymbol{u} = (\sin(\pi x_1)\sin(\pi x_2) + x_1/2, \cos(\pi x_1)\cos(\pi x_2) + x_2/2)$ for $\mu = \lambda = 1$ on an $h$-refined sequence of triangular meshes. The convergence results are presented in Fig. 1. In the left panel, we compare the quantities on the left-hand side of estimates (11) and (20). Although the order of convergence is the same, the original solution $\underline{\mathbf{u}}_h$ displays better accuracy in the energy-norm. This is essentially due to face unknowns, as confirmed in the right panel, where the square roots of the quantities $\sum_{T\in\mathcal{T}_h}\|\mathbf{u}_T - \widehat{\mathbf{u}}_T\|_T^2$ and $\sum_{T\in\mathcal{T}_h}\|\underline{\mathbf{c}}_T^k\underline{\mathbf{u}}_T - \widehat{\mathbf{u}}_T\|_T^2$ (both of which are discrete $L^2$-norms of the error) are plotted.

## References


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