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Algebraic geometry

Computing zeta functions on log smooth models

Calcul de fonctions zêta à partir de modèles log lisses

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ABSTRACT

We establish a formula for the volume Poincaré series of a log smooth scheme. This yields in particular a new expression and a smaller set of candidate poles for the motivic zeta function of a hypersurface singularity and of a degeneration of Calabi–Yau varieties.

RÉSUMÉ

Nous établissons une formule pour la série volume de Poincaré d'un schéma log lisse. Ceci nous fournit en particulier une nouvelle expression et un ensemble réduit de candidats pôles pour la fonction zêta motivique d'une singularité d'hypersurface et d'une dégénération de variétés de Calabi–Yau.

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1. Introduction

The motivic zeta function $Z_f(T)$ of a complex algebraic function $f: X \to \mathbb{A}^1_{\mathbb{C}}$ is one of the most emblematic objects of motivic integration. The function contains many invariants of singularities of f and its poles have attracted much interest for their conjectural relation with eigenvalues of the local monodromy action, as precisely stated by Denef and Loeser's monodromy conjecture (see for example [9, 5.2.3]).

In [8] Nicaise and Sebag introduced the volume Poincaré series $S(\mathfrak{X}, \omega; T)$ of a pair $(\mathfrak{X}, \omega)$ consisting of a generically smooth stft (for separated and topologically of finite type) formal scheme $\mathfrak{X}$ over a complete discrete valuation ring R and a volume form on its generic fiber $\mathfrak{X}_\eta$. Then they showed how to express $Z_f(T)$ as such a series, yielding a new interpretation of the motivic zeta function. This new way of considering Z_f allowed Halle and Nicaise in [4] to associate a motivic zeta function Z_X with a Calabi–Yau variety X over the fraction field of R.

We will show how to compute $S(\mathcal{X}, \omega; T)$ when $\mathcal{X}$ is a generically smooth log smooth R-scheme. By adding a suitable structure to schemes, logarithmic geometry allows one to handle so-called log smooth schemes as if they were smooth. A key feature of log smooth schemes is that their fans, as defined by Kato in [7], can be used to exhibit a desingularization of the scheme. In this process of desingularizing, fake poles are introduced into the expression of $S(\mathcal{X}, \omega; T)$, so that our formula, depending directly on the fan, substantially reduces the set of candidate poles.

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Let us finally mention that our formula is sufficiently general to recover a combinatorial expression of $Z_f(T)$ obtained by Guibert in [3] when f is a polynomial that is nondegenerate with respect to its Newton polyhedron.

Detailed proofs of these results will appear in the author's PhD thesis.

2. Log geometry

Every monoid will be assumed commutative. For a monoid M, we denote by $M^\times$ its group of invertible elements. A morphism of monoids $u: P \to Q$ is called *local* if $u^{-1}(Q^\times) = P^\times$.

Let X be a scheme. A *pre-log structure* on X consists of a sheaf of monoids $\mathcal{M}_X$ on X together with a morphism of sheaves $\alpha: \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$ to the multiplicative monoid of $\mathcal{O}_X$. A pre-log structure is called a *log structure* if, moreover, $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is an isomorphism. This is equivalent to saying that $\alpha_x: \mathcal{M}_{X,x} \to \mathcal{O}_{X,x}$ is local and induces an isomorphism $\mathcal{M}_{X,x}^\times \to \mathcal{O}_{X,x}^\times$ for every $x \in X$. Every pre-log structure $\alpha: \mathcal{M}_X \to \mathcal{O}_X$ induces canonically a log structure $\mathcal{M}_X^a$, called the *associated log structure*.

We present two examples that will be of great use to us.

Example 1. Let R be a discrete valuation ring and π a uniformizer. The morphism of monoids $\mathbb{N} \to (R, \cdot)$, $1 \mapsto \pi$ defines a pre-log structure on $\operatorname{Spec} R$. If $S = \operatorname{Spec} R$, we will denote by $S^\dagger$ the scheme S endowed with the associated log structure.

Example 2. Let $\mathcal{X} \to R$ be an sncd R-scheme, i.e. a regular scheme of finite type over R whose special fiber $\mathcal{X}_s = \sum_{i \in I} N_i E_i$ is a divisor with strict normal crossings. For $J \subseteq I$ we write

$$E_J = \bigcap_J E_j \quad \text{and} \quad E_J^\circ = E_J \setminus \bigcup_{i \notin J} E_i,$$

where $E_\emptyset = \mathcal{X}$. Around each point $x \in E_J$ we can find an affine open $\operatorname{Spec} A$ in $\mathcal{X}$ on which $\pi = u \prod_J x_j^{N_j}$ for u a unit and where $V(x_j) = E_j$. Then the morphism of monoids

$$\mathbb{N}^J \to A, \qquad e_j \mapsto x_j$$

defines a pre-log structure on $\mathcal{X}$.

More background on log geometry can be found in [6].

Let $X \to S$ be a morphism of log schemes. We can make sense of the *module* $\Omega_{X/S}^{\log}$ *of* S*-log differentials* whose construction involves the classical module of differentials $\Omega_{X/S}$ and the log structures on X and S. One of the key notions of log geometry is *log smoothness* (see [7, 8.1]). It guarantees that the sheaf $\Omega_{X/S}^{\log}$ of log differentials is locally free, making it a good substitute for $\Omega_{X/S}$ when the morphism of schemes $X \to S$ is not smooth.

3. A formula for the volume Poincaré series

Let k be a field of characteristic zero. In this section, we set $R = k[\![\pi]\!]$, $S = \operatorname{Spec} R$, and we denote by $S^\dagger$ the log scheme $\operatorname{Spec} R$ endowed with the log structure defined in Example 1. All S-schemes will be assumed separated and will be of finite type. For $d \geq 1$, we consider the totally ramified extension $R(d) := R[T]/(T^d - \pi)$ and set $S(d) = \operatorname{Spec} R(d)$. Let $\mathcal{X}$ be a generically smooth S-scheme of pure relative dimension m and let ω be a volume form on the generic fiber $\mathcal{X}_\eta$, i.e., a nowhere vanishing differential form of degree m. We denote by $\widehat{\mathcal{X}}$ the π-adic completion of $\mathcal{X}$ and we write $\omega(d)$ for the inverse image of ω on the generic fiber of $\widehat{\mathcal{X}}(d) := \widehat{\mathcal{X}} \times_S S(d)$. Following [8, 7.2], the *volume Poincaré series* of the pair $(\mathcal{X}, \omega)$ is defined as

$$S(\mathcal{X}, \omega; T) := \sum_{d \geq 1} \left(\int_{\widehat{\mathcal{X}}(d)} |\omega(d)| \right) T^d \in \mathcal{M}_{\mathcal{X}_s}[\![T]\!],$$

where $\mathcal{M}_{\mathcal{X}_s}$ is the localization $K_0(\mathrm{Var}_{\mathcal{X}_s})[\mathbb{L}^{-1}]$ of the Grothendieck ring of $\mathcal{X}_s$-varieties and $\mathbb{L} := [\mathbb{A}^1_{\mathcal{X}_s}]$.

When $\mathcal{X}$ is sncd, we have by [8, 7.6] (with the notation of Example 2)

$$S(\mathcal{X}, \omega; T) = \mathbb{L}^{-m} \sum_{\emptyset \neq J \subseteq I} (\mathbb{L} - 1)^{|J|-1} \big[\widetilde{E_J^\circ}\big] \sum_{k_j \geq 1, j \in J} \mathbb{L}^{-\sum_J k_j \mu_j} T^{\sum_J k_j N_j} \in \mathcal{M}_{\mathcal{X}_s}[\![T]\!], \tag{1}$$

where μ_j is the order of ω along E_j (see [8, 6.8]) and $\widetilde{E_J^\circ}$ is a certain Galois cover of E_J°, as described in [8, §4].

If we endow $\mathcal{X}$ with the log structure described in Example 2, then $\mathcal{X}$ is log smooth over $S^\dagger$ and we can interpret all the elements appearing in the formula in terms of the log geometry of $\mathcal{X}$. This suggests the following formula.

Theorem 3.1. *Let $\mathcal{X}$ be a generically smooth log smooth scheme over $S^\dagger$ of pure relative dimension m. Let ω be a volume form on $\mathcal{X}_\eta$ and denote by F the fan of $\mathcal{X}$. Then*

$$S(\mathcal{X},\omega;T)=\mathbb{L}^{-m}\sum_{t\in F_s}(\mathbb{L}-1)^{r(t)-1}\big[\widetilde{U}(t)\big]\sum_{u\in M^{\vee,\mathrm{loc}}_{F,t}}\mathbb{L}^{-u(I_\omega)}\,T^{u(e_\pi)}\in\mathcal{M}_{\mathcal{X}_s}[\![T]\!]. \tag{2}$$

We now explain the different elements appearing in (2).

Let $\mathcal{X}$ be a log smooth $S^\dagger$-scheme. For every $x\in\mathcal{X}$ we denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{\mathcal{X},x}$ and by $\mathcal{M}^+_{\mathcal{X},x}\mathcal{O}_{\mathcal{X},x}$ the ideal of $\mathcal{O}_{\mathcal{X},x}$ generated by $\mathcal{M}_{\mathcal{X},x}\backslash\mathcal{M}^\times_{\mathcal{X},x}$. The set

$$F=F(\mathcal{X}):=\big\{x\in\mathcal{X}\;\big|\;\mathcal{M}^+_{\mathcal{X},x}\mathcal{O}_{\mathcal{X},x}=\mathfrak{m}_x\big\}$$

can be endowed with a fan structure in the sense of Kato [7, 9.3], which is closely related to the classical notion in toric geometry. For every point $t\in F$, we have a canonical morphism $\mathbb{N}\to M_{F,t}$ induced by the morphism of log structures, where $M_{F,t}:=\mathcal{M}_{\mathcal{X},t}/\mathcal{O}^\times_{\mathcal{X},t}$. We write e_π for the image of 1 under this morphism and set $F_s=\{t\in F\mid e_\pi\notin M^\times_{F,t}\}$.

The fan F determines a stratification of $\mathcal{X}$ by locally closed subschemes $U(t)$, where

$$U(t)=\big\{x\in\overline{\{t\}}\;\big|\;\mathcal{M}^+_{\mathcal{X},x}\mathcal{O}_{\mathcal{X},t}=\mathfrak{m}_t\big\}.$$

We denote by $\widetilde{U}(t)$ the inverse image of $U(t)$ in the fibered product $\mathcal{X}\times_{S^\dagger}S(d)^\dagger$ in the category of fine and saturated log schemes, with d sufficiently divisible. If $\mathcal{X}$ is sncd, then the ideal $\mathcal{M}^+_{\mathcal{X},x}\mathcal{O}_{\mathcal{X},x}$ is given by $(x_1,\dots,x_n)$, where the x_i are local equations for the components of $\mathcal{X}_s$ passing through x, so that the points of F are exactly the generic points of the E_J, for $J\subseteq I$. In particular, we see that the stratification $(U(t))_{t\in F}$ coincides with the stratification $(E^\circ_J)_{J\subseteq I}$. Also note that the condition $t\in F_s$ ensures that π is not invertible at t, so that the stratum $E^\circ_\emptyset=\mathcal{X}_\eta$ gets discarded. Finally, $\widetilde{U}(t)$ can be identified with the Galois cover $\widetilde{E^\circ_J}$.

Assume that the log smooth scheme $\mathcal{X}\to S^\dagger$ is generically smooth and of pure relative dimension m and let $\omega\in\Omega^m_{\mathcal{X}_\eta}(\mathcal{X}_\eta)$ be a volume form. We keep writing ω for its image in the sheaf $\Omega^{\log,m}_{\mathcal{X}_\eta/K}$ of m-log differentials where $\mathcal{X}_\eta$ is endowed with the log structure induced by $\mathcal{X}$. Let $\Omega^{\log}_{\mathcal{X}/S^\dagger}$ be the sheaf of log differentials of $\mathcal{X}$. The invertible sheaf $\Omega^{\log,m}_{\mathcal{X}/S^\dagger}$ together with the rational section $\omega\in\Omega^{\log,m}_{\mathcal{X}/S^\dagger}(\mathcal{X}_\eta)$ induces a Cartier divisor $\mathrm{div}(\omega)$ on $\mathcal{X}$. Since ω is a volume form, [7, 11.8] ensures that there is a unique fractional ideal I_ω of $F=F(\mathcal{X})$ such that $I_\omega\mathcal{O}_{\mathcal{X}}=\mathrm{div}(\omega)$.

Finally, for a point $t\in F$, $r(t)$ is the rank of the group $M^{\mathrm{gp}}_{F,t}$ generated by $M_{F,t}$. The set $M^{\vee,\mathrm{loc}}_{F,t}$ consists of all local morphisms of monoids $M_{F,t}\to\mathbb{N}$. If $r(t)=1$ then one can show that $M_{F,t}$ is canonically isomorphic to $\mathbb{N}$. This isomorphism, denoted by v_t, generates $M^{\vee,\mathrm{loc}}_{F,t}$ and is called the *valuation at* t.

When $\mathcal{X}$ is sncd, then each $M_{F,t}$ is isomorphic to $\mathbb{N}^{r(t)}$ so that

$$M^{\vee,\mathrm{loc}}_{F,t}\cong\bigoplus_{\tau\in S_t}\mathbb{N}_{\geq 1}v_\tau,$$

where S_t denotes the set of points τ of F with $r(\tau)=1$ that specialize to t. Those points are exactly the generic points of the components of $\mathcal{X}_s$ passing through t. Furthermore $v_\tau(I_\omega)$ equals the order of ω along the corresponding component of $\mathcal{X}_s$ and one can easily see that $v_\tau(e_\pi)$ is its multiplicity.

It is now clear that (1) and (2) coincide when $\mathcal{X}$ is sncd. The strategy of the proof of 3.1 is to show that the quantity (2) is invariant under subdivisons of fans (again, in the sense of Kato [7, 9.6], which is close in spirit to the classical notion).

A subdivision $\varphi\colon F'\to F$ of the fan F of $\mathcal{X}$ determines in a canonical way a proper birational morphism $\varphi^*\mathcal{X}\to\mathcal{X}$. We can always find a subdivision of F such that $\varphi^*\mathcal{X}$ is an sncd scheme, hence fully desingularizing the scheme $\mathcal{X}$. This allows us to fall back on (1).

The major piece of work consists of the following proposition, which compels us to study more deeply the behavior of $\mathbb{N}$-monoids under base change.

Proposition 3.2. *Let $\mathcal{X}$ be a log smooth scheme over $S^\dagger$ and $\varphi\colon F'\to F$ a subdivision of its fan. For every $t\in F'$ the restriction of $\varphi^*\mathcal{X}\to\mathcal{X}$ to $\widetilde{U}(t)\to\widetilde{U}(\varphi(t))$ is a piecewise trivial fibration whose fiber is a torus of dimension $r(\varphi(t))-r(t)$. In particular*

$$(\mathbb{L}-1)^{r(t)-1}\big[\widetilde{U}(t)\big]=(\mathbb{L}-1)^{r(\varphi(t))-1}\big[\widetilde{U}\big(\varphi(t)\big)\big]\in K_0(\mathrm{Var}_{\mathcal{X}_s}),$$

where brackets denote classes in the Grothendieck ring of $\mathcal{X}_s$-varieties $K_0(\mathrm{Var}_{\mathcal{X}_s})$.

The rest of the proof makes use of basic properties of subdivisions to yield the claimed result.

An important advantage of our formula is that it reduces the set of candidate poles of $S(\mathcal{X},\omega;T)$.

Corollary 3.3. *Let $\mathcal{X}$ be a generically smooth log smooth scheme over $S^\dagger$ of pure relative dimension m, ω a volume form and F its fan. Then every pole of the function $S(\mathcal{X},\omega;\mathbb{L}^{-s})$ is of the form $s=-\nu_t(I_\omega)/\nu_t(e_\pi)$, for some point $t\in F_s$ with $r(t)=1$.*

We present two main contexts in which our formula can be applied.

• Let X be a Calabi–Yau variety of dimension m over $K=\operatorname{Frac} R$. Let $\omega\in\Omega^m_X(X)$ be a volume form on X. Then the *motivic zeta function* of (X,ω) is defined as

$$Z_{X,\omega}(T):=\mathbb{L}^m\sum_{d\geq 1}\left(\int_{X(d)}\left|\omega(d)\right|\right)T^d\in\mathcal{M}_k[\![T]\!]$$

(see [5, 6.4] when ω is distinguished). If $\mathcal{X}$ is a proper S-model of X, then by [8, 7.2]

$$Z_{X,\omega}(T)=\mathbb{L}^m S(\mathcal{X},\omega;T)\in\mathcal{M}_k[\![T]\!].$$

Hence Corollary 3.3 gives us a set of candidate poles for the zeta function $Z_{X,\omega}(T)$ when $\mathcal{X}$ is log smooth, and this set is much smaller than the one we would get from a desingularization of $\mathcal{X}$. Log smooth models of Calabi–Yau varieties over K appear naturally in the Gross–Siebert program on mirror symmetry (see for example [2]).

• Let X be a smooth irreducible scheme of finite type over k of dimension n and $f:X\to\mathbb{A}^1_k=\operatorname{Spec}k[\pi]$ a dominant morphism. Denote by X_s the zero locus of f and set $X^*=X\backslash X_s$. Shrinking X around X_s, we can assume that f is smooth on X^*. Then we can find a unique form $\alpha\in\Omega^{n-1}_{X^*/\mathbb{A}^1_k}(X^*)$ such that $\alpha\wedge \mathrm{d}f$ is the restriction of ϕ to X^*. The induced volume form is called the *Gelfand–Leray form* and is denoted by $\frac{\phi}{\mathrm{d}f}$. By [8, 9.10] the motivic zeta function of f (as defined in [1, 3.2.1]) can then be computed as

$$Z_f(T)=\mathbb{L}^{n-1}S\left(\mathcal{X},\frac{\phi}{\mathrm{d}f};\mathbb{L}^{-1}T\right)\in\mathcal{M}_{\mathcal{X}_s}[\![T]\!].$$

Hence if $\mathcal{Y}\to S^\dagger$ is a log smooth $S^\dagger$-scheme dominating $\mathcal{X}$ and such that $\widehat{\mathcal{Y}}_\eta\cong\widehat{\mathcal{X}}_\eta$, 3.1 gives a formula for $Z_f(T)$ in terms of the model $\mathcal{Y}$. As an application, we can recover a formula for Z_f given by Guibert in [3] when f is a polynomial that is nondegenerate with respect to its Newton polyhedron.

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