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Differential geometry

Rigidity in a conformal class of contact form on CR manifold

Rigidité dans une classe conforme de formes de contact sur une variété CR

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ABSTRACT

In this paper, we first prove that any two conformal contact forms on a compact CR manifold that have the same pseudo-Hermitian Ricci curvature must be different by a constant. In another direction, we prove a CR analogue of the conformal Schwarz lemma of Riemannian geometry.

RÉSUMÉ

Dans cet article, nous montrons d'abord que deux formes de contact conformes quelconques sur une variété compacte CR qui ont la même courbure de Ricci pseudo-hermitienne ne diffèrent que d'un facteur constant. Dans une autre direction, nous prouvons un analogue CR du lemme de Schwarz conforme de la géométrie riemannienne.

1. Introduction

In this paper, we are going to prove some rigidity results in CR geometry. First, we recall the following result of Xu in [8]:

Theorem 1.1. *Suppose (M, g) is a compact Riemannian manifold without boundary of dimension ≥ 2. If $\tilde{g} = e^{2u} g$ such that their Ricci curvatures satisfy $\mathrm{Ric}(\tilde{g}) = \mathrm{Ric}(g)$, then u is a constant.*

We will prove the CR analog of Theorem 1.1. More precisely, we prove the following:

Theorem 1.2. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n + 1$ with a given contact form θ. If $\tilde{\theta} = e^{2u}\theta$ is such that their pseudo-Hermitian Ricci curvatures satisfy $\mathrm{Ric}(\tilde{\theta}) = \mathrm{Ric}(\theta)$, then u is a constant.*

In another direction, we recall the following conformal Schwarz lemma, which was first proved by Yau [9]:

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Theorem 1.3. *Suppose (M, g) is a compact Riemannian manifold without boundary of dimension ≥ 2 whose scalar curvature satisfies $R_g \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$, and g_Y is the Yamabe metric conformally equivalent to g with scalar curvature $R_{g_Y} = -1$. Then we have*

$$\frac{g_Y}{|R_{\min}|} \leq g \leq \frac{g_Y}{|R_{\max}|}.$$

In [7], Suárez-Serrato and Tapie used the Yamabe-type flow to reprove Theorem 1.3. Using the CR Yamabe-type flow, we will prove the following CR analog of Theorem 1.3:

Theorem 1.4. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies $R_\theta \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$, and θ_Y is the contact form conformally equivalent to θ with Webster scalar curvature $R_{\theta_Y} = -1$. Then we have:*

$$\frac{\theta_Y}{|R_{\min}|} \leq \theta \leq \frac{\theta_Y}{|R_{\max}|}. \tag{1.1}$$

As a corollary, we have the following:

Corollary 1.5. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies $R_\theta \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$. Then we have:*

$$\mathrm{Vol}(M, \theta_Y)\Big|\min_M R_\theta\Big|^{-(n+1)} \leq \mathrm{Vol}(M, \theta) \leq \mathrm{Vol}(M, \theta_Y)\Big|\max_M R_\theta\Big|^{-(n+1)},$$

and each equality implies that R_θ is constant.

Corollary 1.6. *Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose CR Yamabe invariant satisfies $Y(M, \theta) < 0$. Then we have:*

$$\Big(\min_M R_\theta\Big)\mathrm{Vol}(M, \theta)^{\frac{1}{n+1}} \leq Y(M, \theta) \leq \Big(\max_M R_\theta\Big)\mathrm{Vol}(M, \theta)^{\frac{1}{n+1}},$$

and each equality implies that R_θ is constant.

The Riemannian version of Corollaries 1.5 and 1.6 was obtained in [7] and [5], respectively. See Corollary 16 in [7] and Lemma 1.6 in [5].

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We adopt the notation in [1].

Proof of Theorem 1.2. If $\tilde{\theta} = e^{2u}\theta$, then by the formula in p. 299 of [1] (see also [6]), their pseudo-Hermitian Ricci curvatures satisfy

$$\widetilde{R}_{\lambda\bar{\mu}} = R_{\lambda\bar{\mu}} - (n+2)(u_{\lambda\bar{\mu}} + u_{\bar{\mu}\lambda}) - \big(\Delta_\theta u + |\nabla_\theta u|^2_\theta\big)h_{\lambda\bar{\mu}}, \tag{2.1}$$

where $h_{\lambda\bar{\mu}}$ is the component of the Levi form (see p. 32 in [1]). Explicitly, let $\{T_\alpha : 1 \leq \alpha \leq n\}$ be a local frame of $T^{1,0}(M)$ on M, then

$$h_{\lambda\bar{\mu}} = L_\theta(T_\alpha, \overline{T_\mu})$$

where $L_\theta = -\sqrt{-1}\,d\theta$ is the Levi form with respect to θ. By assumption, $\mathrm{Ric}(\tilde{\theta}) = \mathrm{Ric}(\theta)$, (2.1) implies that

$$-(n+2)(u_{\lambda\bar{\mu}} + u_{\bar{\mu}\lambda}) - \big(\Delta_\theta u + |\nabla_\theta u|^2_\theta\big)h_{\lambda\bar{\mu}} = 0. \tag{2.2}$$

On the other hand, if we define the traceless Ricci tensor

$$B_{\lambda\bar{\mu}} = R_{\lambda\bar{\mu}} - \frac{R}{n}h_{\lambda\bar{\mu}}$$

where $R = R_{\lambda\bar{\mu}}h^{\lambda\bar{\mu}}$ is the Webster scalar curvature, then we have:

$$
\begin{aligned}
\widetilde{B}_{\lambda\bar{\mu}} &= \widetilde{R}_{\lambda\bar{\mu}} - \frac{\widetilde{R}}{n}\tilde{h}_{\lambda\bar{\mu}} \\
&= \widetilde{R}_{\lambda\bar{\mu}} - \frac{1}{n}\big(\widetilde{R}_{\lambda\bar{\mu}}\tilde{h}^{\lambda\bar{\mu}}\big)\tilde{h}_{\lambda\bar{\mu}} \\
&= R_{\lambda\bar{\mu}} - \frac{1}{n}\big(e^{-2u}R_{\lambda\bar{\mu}}h^{\lambda\bar{\mu}}\big)e^{2u}h_{\lambda\bar{\mu}} \\
&= R_{\lambda\bar{\mu}} - \frac{R}{n}h_{\lambda\bar{\mu}} = B_{\lambda\bar{\mu}},
\end{aligned} \tag{2.3}
$$

where we have used the fact that $\tilde{h}_{\lambda\bar{\mu}} = e^{2u}h_{\lambda\bar{\mu}}$ because the Levi forms satisfy $L_{\tilde{\theta}} = e^{2u}L_\theta$ (see for example (1.15) in p. 6 in [1]) and the assumption $\mathrm{Ric}(\tilde{\theta}) = \mathrm{Ric}(\theta)$. Note that if $\tilde{\theta} = e^{2u}\theta$, then by the formula in p. 299 of [1] we have:

$$
\widetilde{B}_{\lambda\bar{\mu}} = B_{\lambda\bar{\mu}} - (n+2)(u_{\lambda\bar{\mu}} + u_{\bar{\mu}\lambda}) + \frac{n+2}{n}(\Delta_\theta u)h_{\lambda\bar{\mu}} \tag{2.4}
$$

Combining (2.3) and (2.4), we obtain:

$$
-(n+2)(u_{\lambda\bar{\mu}} + u_{\bar{\mu}\lambda}) + \frac{n+2}{n}(\Delta_\theta u)h_{\lambda\bar{\mu}} = 0. \tag{2.5}
$$

It follows from (2.2) and (2.5) that

$$
(n+1)\Delta_\theta u + |\nabla_\theta u|_\theta^2 = 0.
$$

Integrating it over M, we have:

$$
\int_M |\nabla_\theta u|_\theta^2 \, dV_\theta = 0,
$$

which implies that u is constant. This proves Theorem 1.2. □

3. CR Yamabe-type flow

In this section, we prove Theorem 1.4. We consider

$$
\begin{aligned}
&\frac{\partial\theta_t}{\partial t} = \big(R_{\max}(\theta_t) - R_{\theta_t}\big)\theta_t, \\
&\theta_0 = \theta
\end{aligned} \tag{3.1}
$$

where $R_{\max}(\theta_t) = \max_M R_{\theta_t}$, which we call the curvature-normalized increasing CR Yamabe flow. If we write $\theta_t = u_t\theta$ for some positive function u_t, then (3.1) can be written as

$$
\frac{\partial u_t}{\partial t} = \big(R_{\max}(\theta_t) - R_{\theta_t}\big)u_t \ge 0.
$$

That is to say, the curvature-normalized increasing CR Yamabe flow increases the conformal factor of θ_t.

Let θ be a contact form whose Webster scalar curvature satisfies:

$$
R_{\min} \le R_\theta \le R_{\max} < 0. \tag{3.2}
$$

Let $\tilde{\theta}_t$ be the solution to the (normalized) CR Yamabe flow:

$$
\begin{aligned}
&\frac{\partial\tilde{\theta}_t}{\partial t} = (r_{\tilde{\theta}_t} - R_{\tilde{\theta}_t})\tilde{\theta}_t, \\
&\tilde{\theta}_0 = \theta,
\end{aligned} \tag{3.3}
$$

where

$$
r_{\tilde{\theta}_t} = \frac{\int_M R_{\tilde{\theta}_t} dV_{\tilde{\theta}_t}}{\int_M dV_{\tilde{\theta}_t}}
$$

is the average scalar curvature of $(M, \tilde{\theta}_t)$. Since $R_{\max} < 0$, it follows from [10] that (3.3) has a unique solution, defined for all $t \ge 0$. Moreover, the contact form $\tilde{\theta}_t$ converges when $t \to \infty$ to a contact form that is conformal to θ and has constant Webster scalar curvature and the same volume as θ. See also [3].

Set

$$\phi(t) = \int_0^t \big(R_{\max}(\tilde{\theta}_\tau) - r_{\tilde{\theta}_\tau}\big)\,\mathrm{d}\tau \tag{3.4}$$

for all $t \geq 0$, and let $a : [0, \infty) \to [0, \infty)$ be the unique solution to

$$\begin{aligned} &a'(t) = \mathrm{e}^{-\phi(a(t))}, \\ &a(0) = 0. \end{aligned} \tag{3.5}$$

Hence, the map a is increasing and well defined as long as it stays finite. It follows from the exponential convergence of the contact form shown by Zhang in [10] that there exists $C, \epsilon > 0$ such that for all $t \geq 0$, we have:

$$\big|R_{\max}(\tilde{\theta}_t) - r_{\tilde{\theta}_t}\big| \leq C\mathrm{e}^{-\epsilon t}.$$

Therefore, a exists for all $t \geq 0$, and $\frac{a(s)}{s}$ converges to a positive limit a_∞ when $t \to \infty$.

Lemma 3.1. *If we define for all $t \geq 0$,*

$$\theta_t = \mathrm{e}^{\phi(a(t))}\tilde{\theta}_{a(t)}, \tag{3.6}$$

then θ_t satisfies (3.1).

Lemma 3.1 follows from differentiating (3.6) and applying (3.3)–(3.5). Therefore, θ_t given by (3.6) is a solution to (3.1). By the above argument, since the map is an increasing bijection on $[0, \infty)$, the uniqueness of the solution to (3.1) follows directly from the uniqueness of the solution to (3.3). We will prove that the Webster scalar curvature bounds are preserved along the flow.

Lemma 3.2. *For all $t \geq 0$, the Webster scalar curvature of θ_t satisfies:*

$$\frac{\partial R_{\theta_t}}{\partial t} = (n+1)\Delta_{\theta_t} R_{\theta_t} + R_{\theta_t}\big(R_{\theta_t} - R_{\max}(\theta_t)\big) \tag{3.7}$$

and

$$R_{\min} \leq R_{\theta_t} \leq R_{\max}. \tag{3.8}$$

Proof. Note that the Webster scalar curvature of the contact from $\tilde{\theta}_t$ satisfies (see (3.4) in [2]):

$$\frac{\partial R_{\tilde{\theta}_t}}{\partial t} = (n+1)\Delta_{\tilde{\theta}_t} R_{\tilde{\theta}_t} + R_{\tilde{\theta}_t}(R_{\tilde{\theta}_t} - r_{\tilde{\theta}_t}).$$

If follows from (3.4), (3.5), and (3.6) that

$$\begin{aligned} \frac{\partial R_{\theta_t}}{\partial t} &= \frac{\partial}{\partial t}\big(\mathrm{e}^{-\phi(a(t))} R_{\tilde{\theta}_{a(t)}}\big) \\ &= -\phi'\big(a(t)\big)a'(t)\,\mathrm{e}^{-\phi(a(t))} R_{\tilde{\theta}_{a(t)}} + a'(t)\,\mathrm{e}^{-\phi(a(t))}\frac{\partial R_{\tilde{\theta}_{a(t)}}}{\partial a(t)} \\ &= \mathrm{e}^{-2\phi(a(t))}\big[-\big(R_{\max}(\tilde{\theta}_{a(t)}) - r_{\tilde{\theta}_{a(t)}}\big)R_{\tilde{\theta}_{a(t)}} + (n+1)\Delta_{\tilde{\theta}_{a(t)}} R_{\tilde{\theta}_{a(t)}} + R_{\tilde{\theta}_{a(t)}}(R_{\tilde{\theta}_{a(t)}} - r_{\tilde{\theta}_{a(t)}})\big] \\ &= \mathrm{e}^{-2\phi(a(t))}\big[(n+1)\Delta_{\tilde{\theta}_{a(t)}} R_{\tilde{\theta}_{a(t)}} + R_{\tilde{\theta}_{a(t)}}\big(R_{\tilde{\theta}_{a(t)}} - R_{\max}(\tilde{\theta}_{a(t)})\big)\big] \\ &= (n+1)\,\Delta_{\theta_t} R_{\theta_t} + R_{\theta_t}\big(R_{\theta_t} - R_{\max}(\theta_t)\big) \end{aligned}$$

which proves (3.7). To prove (3.8), we first note that

$$R_{\min} \leq R_{\theta_0} \leq R_{\max}. \tag{3.9}$$

in view of (3.2). To this end, for any fixed $\epsilon > 0$, we define a function $F_\epsilon : M \times [0, \infty) \to \mathbb{R}$ by

$$F_\epsilon(x, t) = R_{\theta_t}(x, t) - (1+t)\epsilon.$$

We claim that $F_\epsilon(x, t) < R_{\max}$ for all $(x, t) \in M \times [0, \infty)$. By contradiction, we suppose that

$$F_\epsilon(x_0, t_0) \geq R_{\max} \tag{3.10}$$

for some $(x_0, t_0) \in M \times [0, \infty)$. Note that $t_0 > 0$ because of (3.9). We may assume that t_0 is the smallest t that satisfies (3.10). Then we have:

$$F_\epsilon(x,t) < R_{\max} \quad \text{for } (x,t) \in M \times [0, t_0), \qquad F_\epsilon(x,t_0) \le R_{\max} \quad \text{for } x \in M, \qquad F_\epsilon(x_0,t_0) = R_{\max}. \tag{3.11}$$

Since M is compact, we may assume that $F_\epsilon(x_0,t_0) = \max_{x\in M} F(x,t_0)$. Hence, by (3.7), (3.10), and (3.11), we have at (x_0,t_0):

$$\begin{aligned} 0 \le \frac{\partial F_\epsilon}{\partial t} &= \frac{\partial R_{\theta_t}}{\partial t} - \epsilon \\ &= (n+1)\Delta_{\theta_t} R_{\theta_t} + R_{\theta_t}\big(R_{\theta_t} - R_{\max}(\theta_t)\big) - \epsilon \\ &\le R_{\theta_t}\big(R_{\theta_t} - R_{\max}(\theta_t)\big) - \epsilon \le -\epsilon, \end{aligned}$$

which contradicts the assumption that $\epsilon > 0$. This proves the claim that $F_\epsilon(x,t) < R_{\max}$ for all $(x,t) \in M \times [0,\infty)$. Letting $\epsilon \to 0$, we obtain $R_{\theta_t} \le R_{\max}$.

On the other hand, to prove $R_{\theta_t} \ge R_{\min}$, it suffices to consider the function $G_\epsilon : M \times [0,\infty) \to \mathbb{R}$ defined as

$$G_\epsilon(x,t) = R_{\theta_t}(x,t) + (1+t)\epsilon$$

where $\epsilon > 0$ is fixed. Following the same proof as above, one can prove that $G_\epsilon(x,t) > R_{\min}$ for all $(x,t) \in M \times [0,\infty)$. Letting $\epsilon \to 0$, we obtain $R_{\theta_t} \ge R_{\min}$. This proves the assertion. □

If follows from [10] that the solution $\tilde{\theta}_t$ to (3.3) converges to the unique contact form θ_∞ with constant Webster scalar curvature in the conformal class of θ with same volume as θ. Let us recall that the solution θ_t of (3.1) is given by

$$\theta_t = \mathrm{e}^{\phi(a(t))}\tilde{\theta}_{a(t)}$$

by Lemma 3.1, with a strictly increasing, $\frac{a(t)}{t}$ converging to a positive limit when $t \to \infty$ and $\phi(t)$ converges exponentially fast to a constant l when $t \to \infty$. This implies that θ_t converge to the metric

$$\theta_{\max} = \mathrm{e}^l \theta_\infty$$

which has constant Webster scalar curvature. Since for all $t \ge 0$, we have $R_{\min} \le R_{\theta_t} \le R_{\max}$ by Lemma 3.2, the Webster scalar curvature of $\theta_{\max}$ satisfies $R_{\min} \le R_{\theta_{\max}} \le R_{\max}$.

Combining all these, we have the following:

Theorem 3.3. *Let (M, θ) be a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies*

$$R_{\min} \le R_\theta \le R_{\max} < 0.$$

Then the curvature-normalized increasing CR Yamabe flow (3.1) with initial contact form θ has a solution $\theta_t = u_t\theta$ defined for all $t \ge 0$. Moreover, the conformal factor $t \mapsto u_t$ is non-decreasing in time t, and the flow θ_t converges as $t \to \infty$ to a contact form $\theta_{\max}$ in the conformal class of θ with constant Webster scalar curvature $R_{\theta_{\max}} \le R_{\max}$.

Now let us consider the curvature-normalized decreasing CR Yamabe flow on M with initial contact form θ, which is defined as

$$\begin{aligned} &\frac{\partial \theta_t}{\partial t} = \big(R_{\min}(\theta_t) - R_{\theta_t}\big)\theta_t, \\ &\theta_0 = \theta. \end{aligned} \tag{3.12}$$

where $R_{\min}(\theta_t) = \min_M R_{\theta_t}$. By using the analogous arguments to prove Theorem 3.3, we can prove the following:

Theorem 3.4. *Let (M, θ) be a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies*

$$R_{\min} \le R_\theta \le R_{\max} < 0.$$

Then the curvature-normalized decreasing CR Yamabe flow (3.12) with initial contact form θ has a solution θ_t defined for all $t \ge 0$. Moreover, Moreover, the conformal factor $t \mapsto u_t$ is non-increasing in time t, and the flow θ_t converges as $t \to \infty$ to a contact form $\theta_{\min}$ in the conformal class of θ with constant Webster scalar curvature $R_{\theta_{\min}} \ge R_{\min}$.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Suppose (M, θ) is a compact strongly pseudoconvex CR manifold of real dimension $2n+1$ whose Webster scalar curvature satisfies $R_\theta \in [R_{\min}, R_{\max}] \subset (-\infty, 0)$. Then it follows from Theorem 3.3 that the curvature-normalized increasing CR Yamabe flow with initial contact form θ increases the conformal factor, and converges to the contact form $\theta_{\max}$ with constant Webster scalar curvature $R_{\theta_{\max}} \leq R_{\max}$. By Theorem 7.1 in [4] (see also [3]), $\theta_{\max} = \frac{\theta_Y}{|R_{\theta_{\max}}|}$. This gives the upper bound in (1.1). Similarly, it follows from Theorem 3.4 that the curvature-normalized decreasing CR Yamabe flow with initial contact form θ decreases the conformal factor, and converges to the contact form $\theta_{\min}$ with constant Webster scalar curvature $R_{\theta_{\min}} \geq R_{\min}$. By Theorem 7.1 in [4] again, $\theta_{\min} = \frac{\theta_Y}{|R_{\theta_{\min}}|}$. This gives the lower bound in (1.1). This completes the proof of Theorem 1.4. $\square$

Using Theorem 1.4, we can prove Corollary 1.5 and 1.6.

Proof of Corollary 1.5. By integrating (1.1) over M, we obtain:

$$\mathrm{Vol}(M,\theta_Y)\Big|\min_M R_\theta\Big|^{-(n+1)} \leq \mathrm{Vol}(M,\theta) \leq \mathrm{Vol}(M,\theta_Y)\Big|\max_M R_\theta\Big|^{-(n+1)}.$$

If $\mathrm{Vol}(M,\theta) = \mathrm{Vol}(M,\theta_Y)|\min_M R_\theta|^{-(n+1)}$, then we have $\theta = \frac{\theta_Y}{|R_{\min}|}$, which implies that R_θ is constant. Similarly, if $\mathrm{Vol}(M,\theta) = \mathrm{Vol}(M,\theta_Y)|\max_M R_\theta|^{-(n+1)}$, then we have $\theta = \frac{\theta_Y}{|R_{\max}|}$, which again implies that R_θ is constant. $\square$

Proof of Corollary 1.6. By the solution to the CR Yamabe problem (see [4]), there exists a contact form θ_Y conformal to θ such that its Webster scalar curvature $R_{\theta_Y} = -1$ and the CR Yamabe invariant is attained by θ_Y, i.e.

$$Y(M,\theta) = \frac{\int_M R_{\theta_Y}\,dV_{\theta_Y}}{(\int_M dV_{\theta_Y})^{\frac{n}{n+1}}} = -\mathrm{Vol}(M,\theta_Y)^{\frac{1}{n+1}}. \tag{3.13}$$

Combining (3.13) with Corollary 1.5, we obtain

$$\Big(\min_M R_\theta\Big)\mathrm{Vol}(M,\theta)^{\frac{1}{n+1}} \leq Y(M,\theta) \leq \Big(\max_M R_\theta\Big)\mathrm{Vol}(M,\theta)^{\frac{1}{n+1}},$$

and each equalities implies that R_θ is constant. $\square$

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