C. R. Acad. Sci. Paris, Ser. I 353 (2015) 149–153

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Functional analysis/Algebraic geometry

Noncommutative affine spaces and Lie-complete rings

Espaces affines non commutatifs et anneaux de Lie complets

Anar Dosi

Middle East Technical University, Northern Cyprus Campus, Guzelyurt, KKTC, Mersin 10, Turkey

A R T I C L E   I N F O

Article history:
Received 11 April 2013
Accepted 23 October 2014
Available online 3 December 2014

Presented by the Editorial Board

A B S T R A C T

In this paper, we investigate the structure sheaves of an (infinite-dimensional) affine NC-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, affine Lie-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$, and their nilpotent perturbations $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h},q}$, respectively. We prove that the schemes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$ are identical if and only if $\mathbf{x}$ is a finite set of variables, that is, when we deal with finite-dimensional noncommutative affine spaces. For each (Zariski) open subset $U \subseteq X = \mathrm{Spec}(\mathbb{C}[\mathbf{x}])$, we obtain the precise descriptions of the algebras $\mathcal{O}_{\mathfrak{nc}}(U)$, $\mathcal{O}_{\mathfrak{nc},q}(U)$, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(U)$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(U)$ of noncommutative regular functions on U associated with the schemes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h},q}$ and $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$, respectively. The obtained result for $\mathcal{O}_{\mathfrak{nc}}(U)$ generalizes Kapranov's formula in the finite-dimensional case. Our approach to the matter is based on a noncommutative holomorphic functional calculus in Fréchet algebras.

R É S U M É

Dans cette note, nous étudions la structure des faisceaux des NC-espaces $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ et des Lie espaces $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$, affines (de dimension infinie), et de leur perturbations nilpotentes $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$ et $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h},q}$, respectivement. Nous montrons que les schémas $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ et $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$ sont identiques si et seulement si x est un ensemble fini de variables, c'est-à-dire lorsqu'on traite des espaces affines non commutatifs de dimension finie. Pour chaque ouvert (de Zariski) $U \subset X = \mathrm{Spec}(\mathbb{C}[\mathbf{x}])$, nous obtenons les descriptions précises des algèbres $\mathcal{O}_{\mathfrak{nc}}(U)$, $\mathcal{O}_{\mathfrak{nc},q}(U)$, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(U)$ et $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(U)$, de fonctions régulières non commutatives sur U, associées aux schémas $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h}}$ et $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}\mathfrak{h},q}$, respectivement. Ces résultats pour $\mathcal{O}_{\mathfrak{nc}}(U)$ généralisent la formule de Kapranov dans le cas où la dimension est finie. De plus, nous montrons que tout anneau Lie complet A est plongé dans $\Gamma(X, \mathcal{O}_A)$ comme sous-algèbre dense pour la topologie I_1-adique associée à l'idéal bilatère I_1 engendré par tous les commutateurs de A.

E-mail addresses: dosiev@metu.edu.tr, dosiev@yahoo.com.
URL: http://math.ncc.metu.edu.tr/content/members-dosiev.php.

http://dx.doi.org/10.1016/j.crma.2014.10.020

1. Introduction

The main idea of scheme-theoretic algebraic geometry is the duality correspondence between commutative rings and affine schemes [10,11]. Based on this duality, noncommutative affine schemes are defined as the dual of the category of associative rings [13,15]. The affine NC-schemes are defined as noncommutative nilpotent thickenings of commutative schemes due to Kapranov [12]. If A is a noncommutative associative algebra with its commutativization $A_c = A/\mathcal{I}([A, A])$, then the surjective homomorphism $A \to A_c$ allows us to embed the geometric object $X = \mathrm{Spec}(A_c)$ into an affine NC-scheme $(X, \mathcal{O}_A)$, which is a ringed space equipped with a noncommutative structure sheaf $\mathcal{O}_A$ of NC-complete algebras. Recall that an associative (complex) algebra A can be equipped with an NC-topology defined by the commutator filtration $(F^k(A))_k$, where $F^k(A) = \sum_m \sum_{i_1+\cdots+i_m=k} I_{i_1}\cdots I_{i_m}$ and $I_s = \mathcal{I}(A_{\mathfrak{lie}}^{(s+1)})$ is the two-sided ideal in A generated by the $(s+1)$-th member of the lower central series $A_{\mathfrak{lie}}^{(s+1)}$ of the related Lie algebra $A_{\mathfrak{lie}}$. The algebra A is called an NC-complete algebra if it is Hausdorff and complete with respect to the NC-topology in A. The formal spectrum $X = \mathrm{Spf}(A)$ of an NC-complete algebra A is reduced to $\mathrm{Spf}(A_c)$, and the structure sheaf $\mathcal{O}_A$ is defined as the sheaf of continuous sections of the covering space over X defined by the noncommutative topological localizations of A [12]. In particular, the affine NC-space $\mathbb{A}_{\mathfrak{nc}}^{\mathbf{x}}$ (over the complex field) is defined as the formal scheme $\mathrm{Spf}(\mathcal{O}_{\mathfrak{nc}}(\mathbf{x}))$ of the NC-completion $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x})$ of the free associative algebra $\mathbb{C}\langle\mathbf{x}\rangle$ in the independent variables $\mathbf{x} = (x_i)_{i\in\Xi}$, whose structure sheaf is denoted by $\mathcal{O}_{\mathfrak{nc}}$.

The formal schemes can be constructed for Lie-complete rings either. Recall that a ring A is said to be a Lie-nilpotent ring if $A_{\mathfrak{lie}}$ is a nilpotent Lie ring. A Lie-complete ring A is defined as a complete filtered ring associated with a filtration $(J_\alpha)_\alpha$ whose quotients A/J_α are Lie-nilpotent rings. They admit topological localizations that are commutative modulo their topological nilradicals (see below Proposition 1.1). The free algebra $\mathbb{C}\langle\mathbf{x}\rangle$ admits various completions that are Lie-complete algebras. First consider the free Lie-nilpotent algebra $B_q(\mathbf{x}) = \mathbb{C}\langle\mathbf{x}\rangle/I_q$ of index q, which is the Hausdorff completion of $\mathbb{C}\langle\mathbf{x}\rangle$ with respect to the filtered topology of the (singleton) filtration (I_q). We have also its I_1-adic (or NC) completion $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})$, which is the Hausdorff completion of $\mathbb{C}\langle\mathbf{x}\rangle$ defined by one of the equivalent filtrations $(I_q + I_1^k)_k$ and $(I_q + F_k)_k$, where $F_k = F_k(\mathbb{C}\langle\mathbf{x}\rangle)$. Actually, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})$ is the NC-completion of $B_q(\mathbf{x})_{\mathfrak{h}} = \mathbb{C}\langle\mathbf{x}\rangle/\overline{I_q}^{\mathfrak{nc}}$, where $\overline{I_q}^{\mathfrak{nc}} = \bigcap_m (I_q + F_m)$ is the NC-closure of I_q in $\mathbb{C}\langle\mathbf{x}\rangle$. The completion of $\mathbb{C}\langle\mathbf{x}\rangle$ with respect to the filtration $(\overline{I_k}^{\mathfrak{nc}})_k$ is denoted by $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(\mathbf{x})$, whereas $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$ denotes the completion of $\mathbb{C}\langle\mathbf{x}\rangle$ associated with $(I_k)_k$. Since $\mathbb{C}\langle\mathbf{x}\rangle = \mathcal{U}(\mathfrak{L}(\mathbf{x}))$ is the universal enveloping algebra of the free Lie algebra $\mathfrak{L}(\mathbf{x})$ generated by $\mathbf{x}$, we have the two-sided ideal $\mathfrak{I}_q = \mathcal{I}(\mathfrak{L}(\mathbf{x})_{\mathfrak{lie}}^{(q+1)})$ in $\mathbb{C}\langle\mathbf{x}\rangle$. The Hausdorff completion $\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})$ of $\mathbb{C}\langle\mathbf{x}\rangle$ is defined by one of the equivalent filtrations $(\mathfrak{I}_q + I_1^k)_k$ and $(\mathfrak{I}_q + F_k)_k$. Note that it is just I_1-adic completion of $\mathcal{U}(\mathfrak{g}_q(\mathbf{x}))$, where $\mathfrak{g}_q(\mathbf{x}) = \mathfrak{L}(\mathbf{x})/\mathfrak{L}(\mathbf{x})_{\mathfrak{lie}}^{(q+1)}$ is the free nilpotent Lie algebra of index q generated by $\mathbf{x}$. Thus we have the Lie-complete algebras $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x})$, $\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})$, $B_q(\mathbf{x})$, $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(\mathbf{x})$. Note that $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x}) = \varprojlim\{\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})\} = \varprojlim\{\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}(\mathbf{x})\}$, $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x}) = \varprojlim\{B_q(\mathbf{x})\}$, and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}(\mathbf{x}) = \varprojlim\{B_q(\mathbf{x})_{\mathfrak{h}}\}$ up to the topological isomorphisms. The structure sheaves defined by these Lie-complete algebras are denoted by $\mathcal{O}_{\mathfrak{nc}}$, $\mathcal{O}_{\mathfrak{nc},q}$, B_q, $\mathcal{O}_{\mathfrak{lie}}$, $\mathcal{O}_{\mathfrak{lie}\mathfrak{h},q}$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}$, respectively, and they in turn generate the schemes $\mathbb{A}_{\mathfrak{nc}}^{\mathbf{x}}$, $\mathbb{A}_{\mathfrak{nc},q}^{\mathbf{x}}$, $\mathbb{A}_{\mathfrak{lie},q}^{\mathbf{x}}$, $\mathbb{A}_{\mathfrak{lie}}^{\mathbf{x}}$, $\mathbb{A}_{\mathfrak{lie}\mathfrak{h},q}^{\mathbf{x}}$ and $\mathbb{A}_{\mathfrak{lie}\mathfrak{h}}^{\mathbf{x}}$, called noncommutative affine spaces. Note that the (topological) commutativizations of these algebras are reduced to $\mathbb{C}[\mathbf{x}]$ and their formal spectra are reduced to $X = \mathrm{Spec}(\mathbb{C}[\mathbf{x}])$ equipped with the Zariski topology. The identity mapping over X generates the scheme morphisms $\mathbb{A}_{\mathfrak{lie}}^{\mathbf{x}} \to \mathbb{A}_{\mathfrak{lie}\mathfrak{h}}^{\mathbf{x}} \to \mathbb{A}_{\mathfrak{nc}}^{\mathbf{x}}$. In the finite-dimensional case, these morphisms are identical, that is, $\mathbb{A}_{\mathfrak{lie}}^{\mathbf{x}} = \mathbb{A}_{\mathfrak{lie}\mathfrak{h}}^{\mathbf{x}} = \mathbb{A}_{\mathfrak{nc}}^{\mathbf{x}}$ iff $\mathrm{Card}(\mathbf{x}) < \infty$. But in the infinite-dimensional case, we have different structure sheaves $\mathcal{O}_{\mathfrak{nc}}$ and $\mathcal{O}_{\mathfrak{lie}\mathfrak{h}}$ over X. This is a new phenomenon that appeared in the infinite-dimensional case having the affine Lie-space apart form the affine NC-space. Similar situation takes place for their q-versions.

In the present note, we propose descriptions of the structure sheaves associated with these noncommutative affine spaces. Our approach to the matter is based on the formally-radical holomorphic functions $\mathcal{F}_{\mathfrak{g}}(U)$ in elements of a nilpotent Lie algebra $\mathfrak{g}$ developed in [2,3] (see also [1]). The Fréchet algebras of noncommutative holomorphic functions have been developed to implement Taylor's program on the noncommutative holomorphic functional calculus for operator families generating a nilpotent Lie algebra [3–7].

2. The structure sheaf of a Lie-complete ring

Let A be a filtered ring with its filtration $\mathfrak{a} = (J_\alpha)_\alpha$, $S \subseteq A\backslash\{0\}$ a (topologically) closed and multiplicatively closed subset in A satisfying the following *topological right* (similarly, *left*) *Ore conditions:*

(*TR*1) for $s \in S$ and $a \in A$ there exist nets $(t_\iota) \subseteq S$ and $(b_\iota) \subseteq A$ such that $\lim_\iota (sb_\iota - at_\iota) = 0$;

(*TR*2) if $sa \in J_\alpha$ with $s \in S$ and $a \in A$, then $at \in J_\alpha$ for some $t \in S$.

Then A admits the topological localization $A[S^{-1}]$ of right (respectively, left) fractions, which is a complete filtered ring with its continuous ring homomorphism $\varphi : A \to A[S^{-1}]$ such that $\varphi(S) \subseteq A[S^{-1}]^*$ (consists of units) and $\{\varphi(a)\varphi(s)^{-1} : a \in A, s \in S\}$ is dense in $A[S^{-1}]$. The filtered ring $A[S^{-1}]$ possesses the following universal property. If $\psi : A \to B$ is a continuous ring homomorphism into another complete filtered ring B such that $\psi(S) \subseteq B^*$, then there exists a unique continuous ring homomorphism $\sigma : A[S^{-1}] \to B$ such that $\sigma \cdot \varphi = \psi$.

Now let A be a Lie-complete ring with its topological nilradical $\mathfrak{Tnil}(A) = \{a \in A : \lim_n a^n = 0\}$, and $X = \mathrm{Spf}(A)$. Then $X = \mathrm{Spf}(A_c)$ is a topological space equipped with a Zariski topology, and $\mathfrak{Tnil}(A) = \bigcap\{\mathfrak{p} : \mathfrak{p} \in \mathrm{Spf}(A)\}$, where $A_c = A/\overline{I_1}$. If I_1 is open (in particular, $\mathfrak{Tnil}(A)$ is open), then $X = \mathrm{Spec}(A_c)$. Note that I_1 is open for an NC-complete ring A.

Proposition 1.1. *Every closed and multiplicatively closed subset $S \subseteq A\backslash\{0\}$ of a Lie-complete ring A satisfies both topological left and right Ore conditions, $\mathfrak{Tnil}(A[S^{-1}])$ is a closed two-sided ideal in $A[S^{-1}]$, and $A[S^{-1}]$ is commutative modulo $\mathfrak{Tnil}(A[S^{-1}])$. If $\mathfrak{Tnil}(A)$ is open then $A[S^{-1}]^*$ is open for every topological localization of the ring A.*

If S is the closure of $\{s^n : n \in \mathbb{N}\}$ for a certain $s \in A\backslash\mathfrak{Tnil}(A)$, we use the notation $A_{(s)}$ instead of $A[S^{-1}]$. For each $e \in X$, we form stalks $A^e = \varinjlim\{A_{(x)} : x \notin e\}$. The disjoint union $\bigvee_{e\in X} A^e$ is made into a covering space of X in a standard way; define $\mathcal{O}_A$ as the sheaf of continuous sections called *the structure sheaf of the Lie-complete ring A*. As in the commutative case, $A_{(s)}$ is identified with a unital subring in $\mathcal{O}_A(X_s)$, where $X_s = \{e \in X : s \notin e\}$. A two-sided ideal I of A generates the closed two-sided ideal $I(s)$ of $A_{(s)}$, which is the closure of the set $\{a/s^n : a \in I, n \in \mathbb{Z}_+\}$. The filtration $\mathfrak{a}(s) = (J_\alpha(s))_\alpha$ defines the topology of $A_{(s)}$, whereas $\mathfrak{i}_1(s) = (I_1(s)^m)_m$ is the I_1-adic topology of $A_{(s)}$. *The weak I_1-adic topology of $A_{(s)}$* is defined by the filtration $\mathfrak{w}_1(s) = ((J_\alpha + I_1^m)(s))_{(\alpha,m)}$. Actually, $\mathfrak{w}_1(s) = \inf\{\mathfrak{a}(s), \mathfrak{i}_1(s)\}$. Further, we define $I(U)$ as the set of those sections $S \in \mathcal{O}_A(U)$ that are locally represented by elements of the ideal $I(s)$, where $U \subseteq X$ is an open subset. In particular, we have the filtrations $\mathfrak{a}(U) = (J_\alpha(U))_\alpha$, $\mathfrak{i}_1(U) = (I_1^m(U))_m$ and $\mathfrak{w}_1(U) = ((J_\alpha + I_1^m)(U))_{(\alpha,m)}$ of the ring $\mathcal{O}_A(U)$. The Hausdorff completions of $A_{(s)}$ and $\mathcal{O}_A(U)$ with respect to $\mathfrak{w}_1(s)$ and $\mathfrak{w}_1(U)$ are denoted by $\widehat{A}_{(s)}$ and $\widehat{\mathcal{O}}_A(U)$, respectively, and $\overline{A_{(s)}}$ denotes $\mathfrak{w}_1(X_s)$-closure of A in $\mathcal{O}_A(X_s)$.

Theorem 1.2. *Let A be a Lie-complete ring with open $\mathfrak{Tnil}(A)$. Then $\mathfrak{w}_1(X_s)|A_{(s)} = \mathfrak{w}_1(s)$ and $\overline{A_{(s)}} = \mathcal{O}_A(X_s)$. Thus $\widehat{A}_{(s)} = \widehat{\mathcal{O}}_A(X_s)$ up to a topological isomorphism. In particular, $\Gamma(X, \mathcal{O}_A) = \overline{A}$, and if the topology $\mathfrak{a}$ of A is discreet (in this case A is a Lie-nilpotent ring) then $\mathfrak{i}_1(X_s)|A_{(s)} = \mathfrak{i}_1(s)$.*

All Lie-completions of $\mathbb{C}\langle\mathbf{x}\rangle$ introduced above have open topological nilradicals. Below we derive the equality $\Gamma(X, \mathcal{O}_A) = A$ for most of them. In the general case, the I_1-adic topology of a Lie-nilpotent ring may not be Hausdorff. If $I = \bigcap_m I_1^m$, one may consider the related descending chain $(I^m)_m$ again. By transfinite induction, it will lead to a stabilization, that is, there will be a nil ideal, which may not be nilpotent. Note that there is an example of a nil-ring that has no nilpotent ideals [14]. Thus A may not be embedded into $\widehat{A}$.

Corollary 1.3. *If A is an NC-complete ring with its NC-topology $\mathfrak{a}$, then $A_{(s)} = \mathcal{O}_A(X_s)$ and $\mathfrak{a}(s) = \mathfrak{w}_1(s) = \mathfrak{w}_1(X_s) = \mathfrak{a}(X_s)$. In particular, $\Gamma(X, \mathcal{O}_A) = A$.*

3. The subalgebras $R_q\langle\mathbf{y}\rangle$ and $\Lambda_q\langle\mathbf{y}\rangle$

Consider a Hall basis $\mathbf{y} = \bigvee_{i\in\mathbb{N}} \mathbf{y}_{(i)}$ for $\mathfrak{L}(\mathbf{x})^{(2)}$, where $\mathbf{y}_{(i)}$ consists of commutators in $\mathbf{x}$ of length i, and put $\mathbf{y}_q = \bigvee_{i=1}^q \mathbf{y}_{(i)}$ for each q. Put $\deg(y_u) = k$ for all $y_u \in \mathbf{y}_{(k)}$, $k \in \mathbb{N}$, and $\deg(y_{\gamma_1} \cdots y_{\gamma_k}) = \deg(y_{\gamma_1}) + \cdots + \deg(y_{\gamma_k})$ for a monomial in $\mathbf{y}$. If $\alpha : \mathbf{y} \to \mathbb{Z}_+$ is a function with finite support, we use the notation $\langle\alpha\rangle$ instead of $\deg(\mathbf{y}^\alpha)$. Note that $\langle\alpha\rangle = \sum_{k\geq 1} k \sum_{y_u\in\mathbf{y}_{(k)}} \alpha(y_u)$ is a weighted sum of values of α. Consider the subspace $R\langle\mathbf{y}\rangle$ in $\mathbb{C}\langle\mathbf{x}\rangle$ generated by all powers $\mathbf{y}^\alpha$, which is a unital subalgebra in $\mathbb{C}\langle\mathbf{x}\rangle$ generated by $\mathfrak{L}(\mathbf{x})^{(2)}$. Actually, it admits gradation $R\langle\mathbf{y}\rangle = \bigoplus_{n\geq 0} R^n\langle\mathbf{y}\rangle$, where $R^n\langle\mathbf{y}\rangle$ consists of all sums $\sum_{\langle\alpha\rangle=n} \lambda_\alpha \mathbf{y}^\alpha$. The range of $R\langle\mathbf{y}\rangle$ in $\mathcal{U}(\mathfrak{g}_q(\mathbf{x}))$ is denoted by $R_q\langle\mathbf{y}\rangle$. Thus $R_q\langle\mathbf{y}\rangle = \bigoplus_n R_q^n\langle\mathbf{y}\rangle$ consists of all sums $\sum_{\mathrm{supp}(\alpha)\subseteq\mathbf{y}_q} \lambda_\alpha \mathbf{y}^\alpha$. The range of $R\langle\mathbf{y}\rangle$ (or $R_q\langle\mathbf{y}\rangle$) in $B_q(\mathbf{x})$ is denoted by $\Lambda_q\langle\mathbf{y}\rangle$. Further, put $J_m = \bigoplus_{n\geq m} \mathbb{C}[\mathbf{x}]_n \subseteq \mathbb{C}\langle\mathbf{x}\rangle$ with $\mathbb{C}[\mathbf{x}]_n = \mathbb{C}[\mathbf{x}] \otimes R^n\langle\mathbf{y}\rangle$. Its range in $B_q(\mathbf{x})$ is denoted by $J_{m,q}$. Put $[\alpha] = \max\{m : \mathbf{y}_q^\alpha \in J_{m,q}\}$, which is the valuation of $\mathbf{y}_q^\alpha$ in $B_q(\mathbf{x})$ defined by the filtration $(J_{m,q})_m$. Thus $[\alpha] \geq \langle\alpha\rangle$ for all α, and $[\alpha] = \langle\alpha\rangle$ whenever $\langle\alpha\rangle < q$, for $I_q \subseteq J_q$. The monomials $\mathbf{y}_q^\alpha$ from $\bigcap_m J_{m,q}$ have infinite valuations. Choose a linearly independent subset $\mathfrak{b}_q^n$ of $\{\mathbf{y}_q^\alpha\}$ in $B_q(\mathbf{x})$ with valuations equal to n, $0 \leq n \leq \infty$, which is a basis for $J_{n,q}$ over $J_{n+1,q}$ whenever $n < \infty$, and put $\mathfrak{B}_q = \mathfrak{b}_q \cup \mathfrak{b}_q^\infty$, which is a basis for $\Lambda_q\langle\mathbf{y}\rangle$, where $\mathfrak{b}_q = \bigvee_{0\leq n<\infty} \mathfrak{b}_q^n$. The subspace in $\Lambda_q\langle\mathbf{y}\rangle$ generated by $\mathfrak{b}_q^n$ is denoted by $\Lambda_q^n\langle\mathbf{y}\rangle$. Thus $\Lambda_q\langle\mathbf{y}_q\rangle = \bigoplus_{0\leq n\leq\infty} \Lambda_q^n\langle\mathbf{y}\rangle$, and $\Lambda_q^n\langle\mathbf{y}\rangle = R_q^n\langle\mathbf{y}\rangle$, $n < q$ up to the canonical identification. The property $\langle\alpha\rangle = \deg(\mathbf{y}_q^\alpha) = [\alpha]$ for all nonzero $\mathbf{y}_q^\alpha$ in $B_q(\mathbf{x})$ is equivalent to the fact that the ideal I_q is graded, in the sense that $I_q = \bigoplus_{n\geq q}(\mathbb{C}[\mathbf{x}]_n \cap I_q)$. In this case, the filtration $(J_{m,q})_m$ is Hausdorff and $\overline{I_q}^{\mathfrak{nc}} = I_q$ (see Section 1). The ideal I_q is homogeneous for all q, $1 \leq q \leq 3$ as follows from the results of [8,9]. For $q \geq 4$ the problem remains open, and along with description of free Lie-nilpotent algebras of index $q \geq 3$, it is an interesting algebraic problem. For $q = 2$ the algebra $\Lambda_2\langle\mathbf{y}\rangle$ is reduced to the even differential forms $\Lambda_2\langle\mathrm{d}\mathbf{x}\rangle = \bigoplus_{n\geq 0} \Lambda^{2n}\langle\mathrm{d}\mathbf{x}\rangle$ due to the infinite-dimensional version of Feigin–Shoikhet construction [9]. In this case, $\mathbf{y}_1 = (y_{ij})$ with $y_{ij} = [x_i, x_j] = 2\,\mathrm{d}x_i \wedge \mathrm{d}x_j$ and $y_{ij}^2 = 0$ in $B_2(\mathbf{x})$.

4. Affine NC-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc},q}$ of index q and affine NC-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$

Now consider NC-complete algebras $\mathcal{O}_{\mathfrak{nc},q}(\mathbf{x})$ and $\mathcal{O}_{\mathfrak{nc}}(\mathbf{x})$. The structure sheaf of the commutative algebra $\mathbb{C}[\mathbf{x}]$ is denoted by $\mathcal{O}$. The sheaf $\mathcal{O}$ in turn generates sheaves $\widetilde{\mathcal{F}}_q$ and $\widetilde{\mathcal{F}}$ of noncommutative algebras. As the sheaves of linear spaces they are defined in the following way:

$$\widetilde{\mathcal{F}}_q(U) = \prod_{n\in\mathbb{Z}_+} \mathcal{O}(U) \otimes R_q^n\langle\mathbf{y}_q\rangle \subseteq \mathcal{O}(U)\big[[\mathbf{y}_q]\big] \quad \text{and} \quad \widetilde{\mathcal{F}}(U) = \prod_{n\in\mathbb{Z}_+} \mathcal{O}(U) \otimes R^n\langle\mathbf{y}\rangle \subseteq \mathcal{O}(U)\big[[\mathbf{y}]\big]$$

for every quasicompact open subset $U \subseteq X$. Thus $\widetilde{\mathcal{F}}(U)$ consists of all formal series $f = \sum_{\alpha} f_{\alpha}\mathbf{y}^{\alpha}$ with finite sums $\sum_{\langle\alpha\rangle=n} f_{\alpha}\mathbf{y}^{\alpha}$ (for $\mathrm{Card}(\mathbf{x}) < \infty$, the latter condition on f is satisfied automatically). Similarly, $\widetilde{\mathcal{F}}_q(U)$ consists of all formal series $f = \sum_{\mathrm{supp}(\alpha)\subseteq \mathbf{y}_q} f_{\alpha}\mathbf{y}^{\alpha}$ with finite sums $\sum_{\langle\alpha\rangle=n} f_{\alpha}\mathbf{y}_q^{\alpha}$. The algebraic structures on $\widetilde{\mathcal{F}}(U)$ and $\widetilde{\mathcal{F}}_q(U)$ are associated with the ones of $\mathbb{C}\langle\mathbf{x}\rangle$ and $\mathcal{U}(\mathfrak{g}_q(\mathbf{x}))$, respectively. If $V \subseteq U$ are open subsets in X, then the restriction mapping $\mathcal{O}(U) \to \mathcal{O}(V)$ generates the $\widetilde{\mathcal{F}}_q$-diagonal linear mapping $\widetilde{\mathcal{F}}_q(U) \to \widetilde{\mathcal{F}}_q(V)$, $f \mapsto f|V = \sum_{\alpha}(f_{\alpha}|V)\mathbf{y}^{\alpha}$, which is a homomorphism.

Theorem 3.1. *There are sheaf isomorphisms $\mathcal{O}_{\mathfrak{nc},q} = \widetilde{\mathcal{F}}_q$, $\mathcal{O}_{\mathfrak{nc}} = \widetilde{\mathcal{F}}$, and $\widetilde{\mathcal{F}} = \varprojlim\{\widetilde{\mathcal{F}}_q\}$. In the case $\mathrm{Card}(\mathbf{x}) < \infty$, we obtain Kapranov's formula, $\mathcal{O}_{\mathfrak{nc}}(U) = \widetilde{\mathcal{F}}(U) = \mathcal{O}(U)[[\mathbf{y}]]$, for an open $U \subseteq X$.*

The Fréchet algebra version of this result has been obtained in [2,3].

5. Affine Lie-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie},q}$ of index q and affine Lie-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh},q}$

Consider the free Lie-nilpotent algebra $B_q(\mathbf{x})$, and Lie-complete algebra $\mathcal{O}_{\mathfrak{lieh},q}(\mathbf{x})$. The sheaf $\mathcal{O}$ generates the sheaves Ω_q and $\widetilde{\Omega}_q$ of noncommutative algebras in the following way

$$\Omega_q(U) = \mathcal{O}(U) \otimes \Lambda_q\langle\mathbf{y}\rangle \quad \text{and} \quad \widetilde{\Omega}_q(U) = \prod_{n\in\mathbb{Z}_+} \mathcal{O}(U) \otimes \Lambda_q^n\langle\mathbf{y}\rangle$$

for a quasicompact open $U \subseteq X$. Note that the algebra $\Omega_q(U)$ consists of all finite sums $f = \sum_{\alpha\in\mathfrak{B}_q} f_{\alpha}\mathbf{y}_q^{\alpha}$, whereas $\widetilde{\Omega}_q(U)$ consists of formal series $\sum_{\alpha\in\mathfrak{b}_q} f_{\alpha}\mathbf{y}_q^{\alpha}$ with finite homogeneous parts $\sum_{[\alpha]=n} f_{\alpha}\mathbf{y}_q^{\alpha}$.

Theorem 4.1. *There are sheaf isomorphisms $\mathcal{O}_{\mathfrak{lie},q} = \Omega_q$, $\mathcal{O}_{\mathfrak{lieh},q} = \widetilde{\Omega}_q$, and $\widetilde{\mathcal{F}} = \varprojlim\{\widetilde{\Omega}_q\}$. In particular, $\mathcal{O}_{\mathfrak{lie},2}(U) = \mathcal{O}(U) \otimes \Lambda_2\langle d\mathbf{x}\rangle$ is the algebra of all even differential forms over the algebra $\mathcal{O}(U)$ equipped with the multiplication that is uniquely defined (locally) by Fedosov-type multiplication, and $\mathcal{O}_{\mathfrak{lieh},2}(U) = \prod_{n\geq 0} \mathcal{O}(U) \otimes \Lambda^{2n}\langle d\mathbf{x}\rangle$ is the complete algebra of even forms.*

Based on Theorem 4.1, we obtain that $\mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}} = \varprojlim\{\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh},q}\}$ as the schemes.

6. Affine Lie-space $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}}$

Finally consider the Hausdorff–Lie completion $\mathcal{O}_{\mathfrak{lieh}}$ of $\mathbb{C}\langle\mathbf{x}\rangle$. The quotient mapping $\pi_q : R_q\langle\mathbf{y}\rangle \to \Lambda_q\langle\mathbf{y}\rangle$ generates the canonical homomorphism $\widetilde{\pi}_q : \widetilde{\mathcal{F}}_q(U) \to \widetilde{\Omega}_q(U)$ over an open $U \subseteq X$. An element $f \in \widetilde{\mathcal{F}}_q(U)$ is said to be a *q-Lie-vanishing series* if $\widetilde{\pi}_q(f) \in \bigoplus_{n\in\mathbb{Z}_+} \mathcal{O}(U) \otimes \Lambda_q^n\langle\mathbf{y}\rangle$. In the case of a finite $\mathbf{x}$, all elements from $\widetilde{\mathcal{F}}_q(U)$ are q-Lie-vanishing series. In particular, each function $f \in \widetilde{\mathcal{F}}_q(U)$ in a finite (noncommuting) variable (from $\mathbf{x}$) is a q-Lie-vanishing one. Take a countable subset $\{x_{i_k}, x_{j_k} : k \in \mathbb{N}\} \subseteq \mathbf{x}$ and put $y_k = [x_{i_k}, x_{j_k}] \in \mathbf{y}_{(1)}$. The series $f = \sum_{n=1}^{\infty} y_1 \cdots y_n \in \widetilde{\mathcal{F}}_2(U)$ is an example of non-2-Lie-vanishing series, whereas $g = \sum_{n=1}^{\infty} y_1 y_2^2 \cdots y_n^2 \in \widetilde{\mathcal{F}}_2(U)$ is an example of 2-Lie-vanishing series in infinite variables, for $\widetilde{\pi}_2(g) = y_1$. A formal series $f = \sum_{\alpha} f_{\alpha}\mathbf{y}^{\alpha}$ in $\widetilde{\mathcal{F}}(U)$ is said to be *a Lie-convergent series* if it is q-Lie-vanishing for every q. The subalgebra of all Lie-convergent series in $\widetilde{\mathcal{F}}(U)$ is denoted by $\widetilde{\mathcal{F}}_{\mathfrak{lie}}(U)$.

Theorem 5.1. *There is a sheaf isomorphism $\mathcal{O}_{\mathfrak{lieh}} = \widetilde{\mathcal{F}}_{\mathfrak{lie}}$, and $\mathcal{O}_{\mathfrak{lieh}}$ is a subsheaf in $\mathcal{O}_{\mathfrak{nc}}$.*

We have also the formal scheme $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}} = \varprojlim\{\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie},q}\}$ generated by $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$. By Theorem 4.1, $\mathcal{O}_{\mathfrak{lie}} = \varprojlim\{\Omega_q\}$. Note that the identity mapping over X generates the scheme morphisms $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}} \to \mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}} \to \mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ thanks to Theorem 5.1. Moreover, $\mathbb{A}^{\mathbf{x}}_{\mathfrak{lie}} = \mathbb{A}^{\mathbf{x}}_{\mathfrak{lieh}} = \mathbb{A}^{\mathbf{x}}_{\mathfrak{nc}}$ iff $\mathrm{Card}(\mathbf{x}) < \infty$. The equality $\Gamma(X, \mathcal{O}_{\mathfrak{lie}}) = \mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$ remains unclear, except for the rest of the sheaves. Based on Theorem 1.2, we have just the density of $\mathcal{O}_{\mathfrak{lie}}(\mathbf{x})$ in $\Gamma(X, \mathcal{O}_{\mathfrak{lie}})$ with respect to the topology of the filtration $\mathfrak{w}_1(X)$.

Acknowledgements

I wish to thank Yu. V. Turovskii and X. Ma for useful discussions.

References

[1] A.A. Dosi, Noncommutative holomorphic functions in elements of a Lie algebra and noncommutative localizations, Izv. Math. RAN 73 (06) (2009) 77–100.

[2] A.A. Dosi, Formally-radical functions in elements of a nilpotent Lie algebra and noncommutative localizations, Algebra Colloq. 17 (Spec. 1) (2010) 749–788.

[3] A.A. Dosi, Taylor functional calculus for supernilpotent Lie algebra of operators, J. Oper. Theory 63 (1) (2010) 101–126.
[4] A.A. Dosi, Taylor spectrum and transversality for a Heisenberg algebra of operators, Mat. Sb. 201 (3) (2010) 39–62.
[5] A.A. Dosiev, Algebras of power series of elements of a Lie algebra and the Slodkowski spectra, J. Math. Sci. 124 (2) (2004) 4886–4908 (translated from Zapiski Nauchnykh Seminarov POMI).
[6] A.A. Dosiev, Cartan–Slodkowski spectra, splitting elements and noncommutative spectral mapping theorems, J. Funct. Anal. 230 (2) (2006) 446–493.
[7] A.A. Dosiev, Local left invertibility for operator tuples and noncommutative localizations, J. K-Theory 4 (1) (2009) 163–191.
[8] P. Etingof, J. Kim, X. Ma, On universal Lie nilpotent associative algebras, J. Algebra 321 (2) (2009) 697–703.
[9] B. Feigin, B. Shoiket, On $[A, A]/[A, [A, A]]$ and a W_n-action on the consecutive commutators of free associative algebra, Math. Res. Lett. 14 (5–6) (2007) 781–795.
[10] A. Grothendieck, Éléments de géométrie algébrique, I. Le langage des schémas, Publ. Math. Inst. Hautes Études Sci. 4 (1960) 5–228.
[11] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, vol. 52, Springer Verlag, 1977, p. 496.
[12] M. Kapranov, Noncommutative geometry based on commutator expansions, J. Reine Angew. Math. 505 (1998) 73–118.
[13] A.L. Rosenberg, Noncommutative Algebraic Geometry and Representation of Quantum Groups, Kluwer Academic Publishers, 1995.
[14] A. Smoktunowicz, A simple nil ring exists, Commun. Algebra 30 (1) (2002) 27–59.
[15] F. Van Oystaeyen, A.H. Verschoren, Non-Commutative Algebraic Geometry, Lecture Notes in Mathematics, vol. 887, Springer-Verlag, Berlin, 1981.